

Comparative Study on Recent Integer DCTs

Sakol Udomsiri And Masahiro Iwahashi

Abstract—This paper presents comparative study on recent integer DCTs and a new method to construct a low sensitive structure of integer DCT for colored input signals. The method refers to sensitivity of multiplier coefficients to finite word length as an indicator of how word length truncation effects on quality of output signal. The sensitivity is also theoretically evaluated as a function of auto-correlation and covariance matrix of input signal. The structure of integer DCT algorithm is optimized by combination of lower sensitive lifting structure types of IRT. It is evaluated by the sensitivity of multiplier coefficients to finite word length expression in a function of covariance matrix of input signal. Effectiveness of the optimum combination of IRT in integer DCT algorithm is confirmed by quality improvement comparing with existing case. As a result, the optimum combination of IRT in each integer DCT algorithm evidently improves output signal quality and it is still compatible with the existing one.

Keywords—DCT, sensitivity, lossless, wordlength.

I. INTRODUCTION

REFER to JPEG[1] and MPEG[2] standards for image and video coding, discrete cosine transform (DCT) is the basis for many image and video coding algorithms. It is one of the most fundamental operations in discrete signal processing. However, the conventional DCT algorithm [3] mapping integer-valued input signal to floating-point coefficient is real-valued transform. It is seen a counter to the trend of low-cost system requirement and also it is difficult in applying to lossless coding due to finite-length representation. Therefore, many integer transforms have been proposed [4,5,6,7,8,9,10,11,16,17] such as integer DCT algorithms [4,5,6,7,8,16,17] and the integer wavelet algorithms [9,10,11,17]. All of integer transform algorithms preserve their basic mathematical properties and also compatible with the conventional transform algorithms. Beside low memory and power consumption requirement, they are also applied for lossless and lossy coding in unified algorithm.

Integer DCT algorithm, one of integer transforms, is evidently interesting because it is simply applied more proficiency with low power consumption. Recently, various integer DCT algorithms [4,5,6,7,8,16,17] are greatly innovated by many researchers, and also several methods are proposed to optimize those algorithms. For examples, in previous report, optimum word length assignment of multiplier coefficients

[12,13] is one of qualified improvement to reduce more hardware complexity and power consumption and still to preserve decoded signal quality. The low sensitive structuring method proposed by Dang et al.[14] evaluates the best combination of integer rotation transform (IRT) in integer DCT algorithm in which it has the least sensitivity of multiplier coefficients to finite word length expression. However, the accuracy of this method depends on finite word length expression in the experiment. If word length expression of multiplier coefficients is adjusted, the least sensitivity of multiplier coefficients in any combination of IRTs is also changed. This paper presents a new method of low sensitive structure of integer DCT algorithm for colored input signals. The method refers to sensitivity of multiplier coefficients to finite word length as an indicator of how word length truncation affects quality of output signal. While word length expression of all multiplier coefficients is supremely truncated into finite word length expression, the optimum structure of integer DCT algorithm is theoretically evaluated by only a function of sensitivity of multiplier coefficients to finite word length expression. The lower sensitive lifting structure type of IRT is applied to optimum combination of IRT in each integer DCT structure. Effectiveness of the optimum combination of integer DCT algorithm is confirmed by quality improvement comparing with existing case.

II. INTEGER DCT ALGORITHMS

A. Integer DCT Algorithms

In this report, we refer to five algorithms of integer DCT type II such as the BinDCT-IIC,-IIL,-IIS, IntDCT-II, LDCT-II [15] and one algorithm of integer DCT type IV as the BinDCT-IV [15]. All of algorithms preserve their basic mathematic properties and refer to the sparse matrix factorization of the real-valued original DCT type II and IV, respectively. The signal flow graphs of 8-point 1-D forward transform of each integer DCT algorithm are illustrated in Figs. 1. A dotted line denotes negative sign. Each algorithm consists of permutation (P) matrices with element ± 1 and combination of 2-point integer rotation transforms (IRT). The permutation (P) matrices do not contain any multiplier coefficients but just rearrange input or output of transform algorithm. Input signal X is given by $X = [x(0), x(1), \dots, x(7)]^T$, and transformed signal is given by $Y = [y(0), y(1), \dots, y(7)]^T$. Colored input signal modeled as AR(1) model with zero-mean and auto-correlation ρ near to unity is applied to be input signal X and transformed into output signal Y . For image coding

S. Udomsiri, Department Of Electrical Engineering, Nagaoka University Of Technology, 1603-1 Kamitomioka, Nagaoka, Niigata, 940-2188, Japan (E-Mail: Sakol_Ud@Tech.Nagaokaut.Ac.Jp).

M. Iwahashi, Department Of Electrical Engineering, Nagaoka University Of Technology, 1603-1 Kamitomioka, Nagaoka, Niigata, 940-2188, Nagaoka, Japan (E-Mail: Iwahashi@Nagaokaut.Ac.Jp).

application, the 2-D integer DCT algorithm can be simply operated by separately applying these 1-D integer DCT algorithms in horizontal and vertical scan, respectively. These integer DCT structures are optimized by selecting lower sensitive lifting structure type of IRT to finite word length expression and discussed in Section III.

B. Integer Rotation Transform (IRT)

The combination of 2-point IRTs is important part in integer DCT algorithm. It is very helpful to integer-to-integer mapping for integer transforms. For this case, the structure of IRT referring to Given-Jacobi rotations [15] is originally comprised of four multiplier coefficients in butterfly structure. However, various lifting structure types of IRT are recently proposed [15] and they are significant in practical applications. For this case, we concentrate on only two lifting structure types of IRT shown in Fig. 2 as type A and B. They are useful to reduce the number of multiplier coefficients from four to three and all of multiplier coefficients are mapped into integer for lossless coding by rounding operation in implementation.

$X_q = [x_q(0) \ x_q(1)]^T$ and $Y_q = [y_q(0) \ y_q(1)]^T$ denote input signal and rotated output vector of IRT, respectively. Variable q denotes a sequence of IRTs in integer DCT algorithm, $q \in \{IR_0, IR_1, \dots, IR_{Q-1}\}$ and Q denotes the number of IRTs using in each integer DCT algorithm.

The factorized matrices of both type A and B of lifting structure are given by

Type A

$$H_{Aq} = \begin{pmatrix} 1 & 0 \\ h_{Aq(0)} & 1 \end{pmatrix} \begin{pmatrix} 1 & h_{Aq(1)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ h_{Aq(2)} & 1 \end{pmatrix} \quad (1)$$

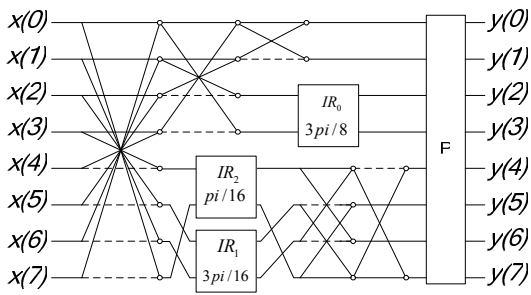
Type B

$$H_{Bq} = \begin{pmatrix} 1 & h_{Bq(0)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ h_{Bq(1)} & 1 \end{pmatrix} \begin{pmatrix} 1 & h_{Bq(2)} \\ 0 & 1 \end{pmatrix} \quad (2)$$

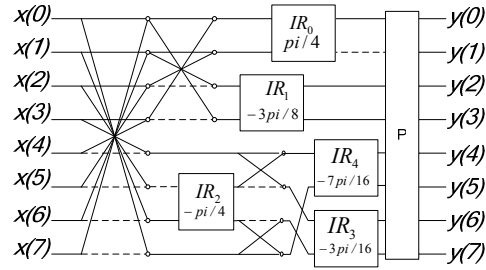
where,

$$h_{Aq(0)} = h_{Aq(2)} = (1 - \cos \theta_q) \sin^{-1} \theta_q, \quad h_{Aq(1)} = -\sin \theta_q$$

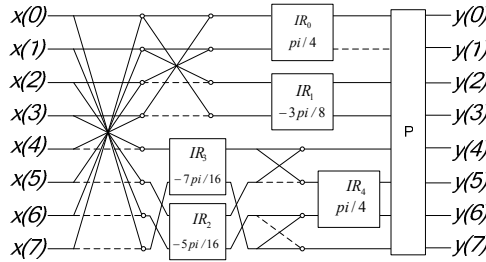
$$h_{Bq(0)} = h_{Bq(2)} = -(1 - \cos \theta_q) \sin^{-1} \theta_q, \quad h_{Bq(1)} = \sin \theta_q.$$



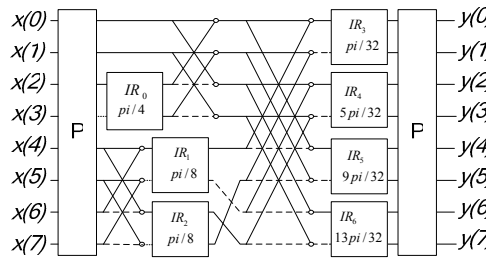
(a) BinDCT-III algorithm



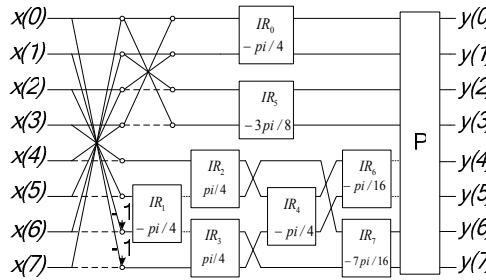
(b) BinDCT-IIC algorithm



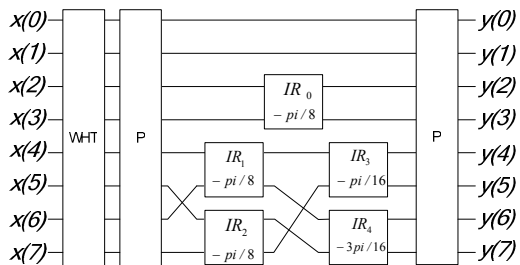
(c) BinDCT-IIS algorithm



(d) BinDCT-IV algorithm



(e) LDCT-II algorithm



(f) IntDCT-II algorithm

Fig. 1 Integer DCT algorithms evaluated in this report.

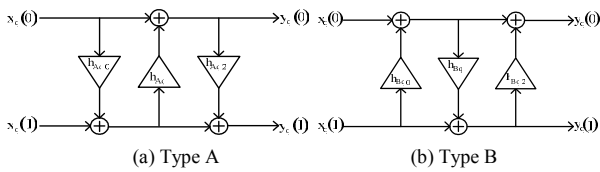


Fig. 2 Two types of integer rotation transforms (IRTs)

III. THE OPTIMUM STRUCTURE OF INTEGER DCT ALGORITHM

A. The Lower Sensitive Lifting Structure Type of IRT.

Sensitivity of a multiplier coefficient in lifting structure of IRT is a function of covariance matrix of its input signal and given by

$$S_{mq(k)} = \sqrt{\frac{1}{2} \text{trace} \left[\frac{\partial H_{mq}}{\partial h_{mq(k)}} R_{XXq} \frac{\partial H_{mq}^T}{\partial h_{mq(k)}} \right]} \quad (3)$$

where, $S_{mq(k)}$ and $\Delta h_{mq(k)}$ denote the sensitivity and truncation error of a multiplier coefficient $k \in \{0,1,2\}$ in lifting structure type $m \in \{A,B\}$ of IRT, respectively. Rotation matrix H_{mq} is replaced by either rotation transform matrix of lifting structure type A or B. In this case, the colored input signal of integer DCT algorithm is modeled as AR(1) model with the auto-correlation ρ near to unity. The covariance matrix of input signal of each lifting IRT in integer DCT can be also easily evaluated as the function of the auto-correlation ρ of AR(1) model as the colored input signal of integer DCT algorithm.

Input signal of a lifting structure of IRT is defined as $X_q = [x_q(0) \ x_q(1)]^T$. The covariance matrix R_{XXq} of input signal of IRT_q , $R_{XXq} = E\{X_q X_q^T\}$, can be easily obtained for each block of transformation by

$$R_{XXq} = \sigma_{in}^2 \begin{bmatrix} f_{q0}(\rho) & f_{q2}(\rho) \\ f_{q2}(\rho) & f_{q1}(\rho) \end{bmatrix} \quad (4)$$

where, $f_{q0}(\rho)$, $f_{q1}(\rho)$, and $f_{q2}(\rho)$ are the function of auto-correlation ρ and σ_{in}^2 denotes variance of input signal of integer DCT algorithm.

The lower sensitive type of lifting structure of IRT in combination can be comparatively evaluated in aspect of the quality (PSNR) improvement under condition of supreme truncation error of word length assignment of all multiplier coefficients. It is obtained by means of quality (PSNR) improvement comparing between two lifting structure types. The worst case of PSNR of each IRT is given by

$$PSNR = 10 \log_{10} \frac{255^2}{\sigma_{\Delta Y_{mq}(\text{total})}^2} \quad [dB] \quad (5)$$

where, $\sigma_{\Delta Y_{mq}(\text{total})}^2$ denotes variance of the total error of output signal. It is obtained by the summation of variance of the error due to finite word length of each multiplier coefficient in lifting structure type A or B as

$$\sigma_{\Delta Y_{mq}(\text{total})}^2 = \sum_{k=0}^{K-1} \sigma_{\Delta Y_{mq(k)}}^2 \quad (6)$$

Variable K denotes the number of multiplier coefficients in each lifting structure type. For this case, both type A and B of lifting structure of IRT consist of 3 multiplier coefficients ($K=3$) and variance of the error due to finite word length of each multiplier coefficient is given by $\sigma_{\Delta Y_{mq(k)}}^2 = S_{mq(k)}^2 \cdot \Delta h_{mq(k)}^2$. Therefore,

$$\sigma_{\Delta Y_{mq}(\text{total})}^2 = \sum_{k=0}^2 S_{mq(k)}^2 \cdot \Delta h_{mq(k)}^2 \quad (7)$$

If word length expression of all multiplier coefficients is supremely truncated to finite word length expression, the comparison of sensitivity to finite word length expression between lifting structure type A and B is evaluated in function of covariance matrix of its input signal. Since the probability of truncation error $\Delta h_{mq(k)}$ is defined as a uniform histogram in range of $0 \leq \Delta h_{mq(k)} \leq 2^{-(W_k+1)}$, variance of total error in condition of supreme word length truncation error $\Delta h_{mq(k)} = 2^{-(W_k+1)}$ is given by

$$\sigma_{\Delta Y_{mq}(\text{total})}^2 = \sum_{k=0}^2 S_{mq(k)}^2 \cdot 2^{-2(W_k+1)} \quad (8)$$

Substituting Eq.(16) into Eq.(13), then the worst case of PSNR is given by

$$PSNR = 10 \log_{10} 255^2 - 10 \log_{10} \sum_{k=1}^2 S_{mq(k)}^2 \cdot 2^{-2(W_k+1)} \quad (9)$$

The comparative quality improvement between type A and B of lifting IRT can be evaluated by the difference of PSNR of type A and B as

$$\begin{aligned} \Delta PSNR &= PSNR_{(Type B)} - PSNR_{(Type A)} \\ &= 10 \log_{10} \frac{\sigma_{\Delta Y_{Aq}(\text{total})}^2}{\sigma_{\Delta Y_{Bq}(\text{total})}^2} \end{aligned} \quad (10)$$

where, $\sigma_{\Delta Y_{Aq}(\text{total})}^2$ and $\sigma_{\Delta Y_{Bq}(\text{total})}^2$ are defined as variance of the total error of lifting structure type A and B of IRT, respectively. While word length expression is supremely

truncated, variance of the total error of type A and B is defined by

$$\begin{aligned}\sigma_{\Delta Y_{Aq}(total)}^2 &= \sum_{k=0}^2 S_{Aq(k)}^2 \cdot 2^{-2(W_k+1)} \\ \sigma_{\Delta Y_{Bq}(total)}^2 &= \sum_{k=0}^2 S_{Bq(k)}^2 \cdot 2^{-2(W_k+1)}\end{aligned}\quad (11)$$

The PSNR improvement ($\Delta PSNR$) comparing between PSNR of type A and B of lifting structure of IRT is evaluated in term of the summation of sensitivity of multiplier coefficients to finite word length expression by

$$\Delta PSNR = 10 \log_{10} \frac{\sum_{k=0}^2 S_{Aq(k)}^2}{\sum_{k=0}^2 S_{Bq(k)}^2} \quad (12)$$

If the summation of sensitivity of multiplier coefficients of type A is less than that of type B, the PSNR improvement is a negative value. It means that multiplier coefficients of type A are lower sensitive to finite word length expression and higher PSNR improvement. It is suitable to be replaced in a location of IRT in integer DCT algorithm. On the contrary, if PSNR improvement is evaluated as a positive value, type B of lifting IRT is better to be selected.

The integer DCT algorithms [15] such as BinDCT-IIC, -IIL, -IIS, -IV, IntDCT-II, and LDCT-II shown in Fig. 1(a) to 1(f) originally comprise only all type A of lifting structure of IRT in combination. All of integer DCT algorithms are optimized by selecting either lower sensitive type A or B of lifting structure of IRT and replacing it in each location of IRT in these integer DCT structures. The lower sensitive multiplier coefficients of type A and B to finite word length is comparatively evaluated in aspect of PSNR improvement referring to (12). The sensitivity of multiplier coefficients of each IRT is evaluated as a function of covariance matrix of its input signal. And also, the covariance matrix of input signal of each IRT is evaluated as a function of auto-correlation ρ of input signal of integer DCT algorithm.

IV. PERFORMANCE EVALUATION

A. Uniform Word Length Allocation

In case of a uniform word length allocation, we assign the same word length of all multiplier coefficients. The quality of reconstructed signal in worst case is considered and the comparative quality improvement is the difference of PSNR between the proposed and existing integer DCT algorithm by

$$\Delta PSNR = -10 \log \frac{\sum_{k=0}^{K-1} S_{(k)}^2(optimum)}{\sum_{k=0}^{K-1} S_{(k)}^2(existing)} \quad (13)$$

It means that the quality improvement of reconstructed signal for the uniform word length allocation is a function of summation of sensitivity of each multiplier coefficient in existing and proposed integer DCT algorithm.

B. Optimum Word Length Allocation

Referring to the optimum word length assignment [13], this method is proved by minimizing total energy $\sigma_{total}^2 = \sum_{k=1}^K \sigma_k^2$ under a given average word length \bar{W} and solved by linear equations with the Lagrange's method [13]. Word length of each multiplier can be obtained by

$$W_k = \bar{W} + \log_2 \frac{S_k}{\bar{S}}, \quad k \in \{0, 1, \dots, K-1\} \quad (14)$$

where,

$$\bar{W} = \frac{1}{K} \sum_{k=0}^{K-1} W_k, \quad \bar{S} = \prod_{k=0}^{K-1} \sqrt[K]{S_{(k)}}$$

W_k and \bar{W} denote word length of each multiplier coefficient h_k and average word length, respectively. The PSNR improvement in case of the optimum word length assignment is defined by

$$\Delta PSNR = -10 \log \frac{\bar{S}_{(optimum)}^2}{\bar{S}_{(existing)}^2}. \quad (15)$$

C. Lossless Coding

Comparative performance in aspect of lossless coding is shown in Fig. 4. Some standard images are used as input signal. It is found that the first order entropy rate of LDCT-II is the best and that of BinDCT-IV and IntDCT-II is the worst. A line of entropy rate in case of BinDCT-IV is not compatible to integer DCT type II so its shape is not similar to the others.

D. Compatibility with the Conventional DCT

Figure 5 and 6 show the rate distortion curve of integer DCT type II and IV with the conventional DCT type II and IV, respectively. We apply the conventional floating-point DCT at one side of transformation and another side is applied by the integer DCT algorithm with rounding operation. It is found that the PSNR of each integer DCT is nearly similar to the case of the conventional DCT at low bit rate but it is not compatible at high bit rate because of rounding effect.

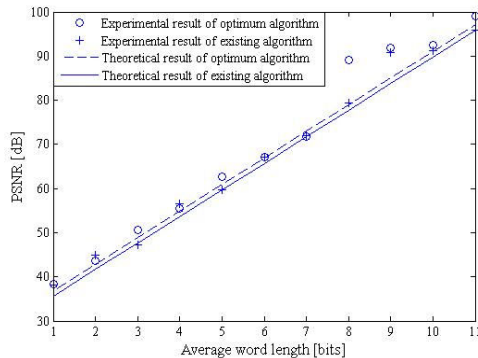
E. Computational Load

For hardware implementation, computational load of each integer DCT algorithm is comparatively considered by the number of multiplier coefficients. It is found that the BinDCT-IIL has the least number of multiplier coefficients in which it is advantageous in aspect of hardware complexity. Meanwhile, the LDCT-II has the most hardware complexity.

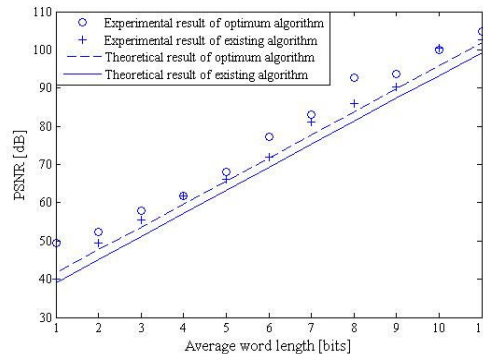
Comparative performance of six integers DCT algorithms is confirmed by the normalized scale of each criterion in Fig. 7.

TABLE I
THE OPTIMUM INTEGER DCT ALGORITHMS

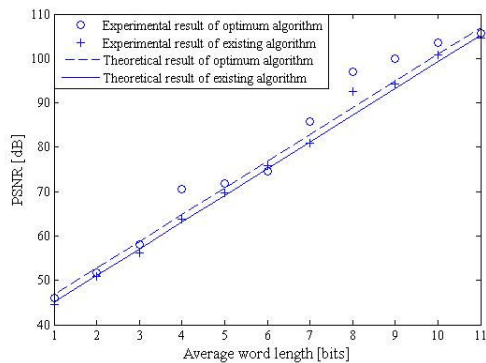
IRT	BinDCT-II C		BinDCT-II L		BinDCT-II S		BinDCT-IV		IntDCT-II		LDCT-II	
	PSNR	Optimum	PSNR	Optimum	PSNR	Optimum	PSNR	Optimum	PSNR	Optimum	PSNR	Optimum
	Improvement	type of IRT	Improvement	type of IRT	Improvement	type of IRT	Improvement	type of IRT	Improvement	type of IRT	Improvement	type of IRT
IRT0	0.074	B	-0.763	A	0.074	B	0.072	B	1.828	B	0.074	B
IRT1	-0.763	A	-0.214	A	-0.763	A	0.693	B	1.683	B	0.039	B
IRT2	-0.073	A	-0.589	A	-0.626	A	0.006	B	1.846	B	0.003	B
IRT3	-0.399	A	-	-	-0.434	A	1.999	B	0.613	B	-0.057	A
IRT4	-1.223	A	-	-	-0.137	A	0.894	B	-1.492	A	0.016	B
IRT5	-	-	-	-	-	-	1.054	B	-	-	-0.763	A
IRT6	-	-	-	-	-	-	-3.960	A	-	-	2.745	B
IRT7	-	-	-	-	-	-	-	-	-	-	-0.788	A



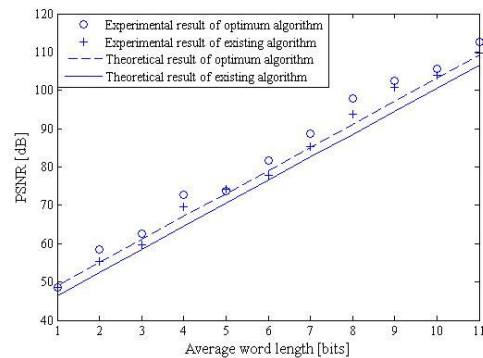
(a). PSNR improvement of BinDCT-IV by uniform word length allocation



(c). PSNR improvement of BinDCT-IV by optimum word length allocation.



(b). PSNR improvement of IntDCT-II by uniform word length allocation.



(d). PSNR improvement of IntDCT-II by optimum word length allocation.

Fig. 3 Comparative PSNR improvement between uniform and optimum word length allocation.

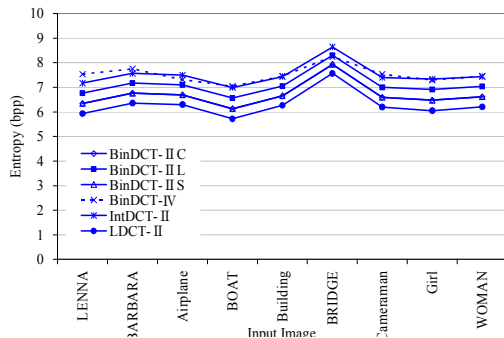


Fig. 4 Total entropy rate (bpp) for lossless coding.

TABLE II
COMPARATIVE PSNR IMPROVEMENT

Type	PSNR Improvement[dB]	
	Uniform length Allocation	Optimum Word length Allocation
BinDCT-II C	0.082	0.674
BinDCT-II L	0.000	0.000
BinDCT-II S	0.083	0.193
BinDCT-IV	1.259	2.525
IntDCT-II	1.668	2.580
LDCT-II	0.071	0.619

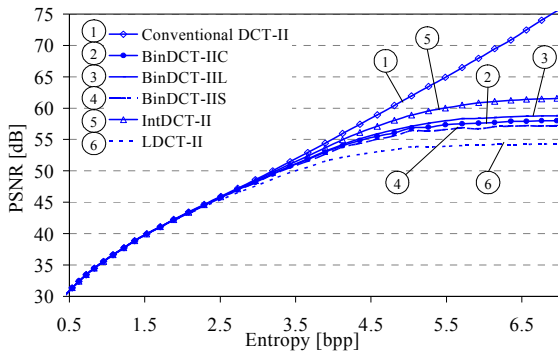


Fig. 5 Rate distortion curve of a decoded image "LENNA" by integer DCT type II.

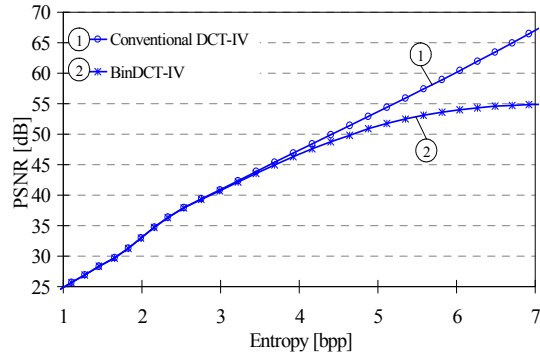
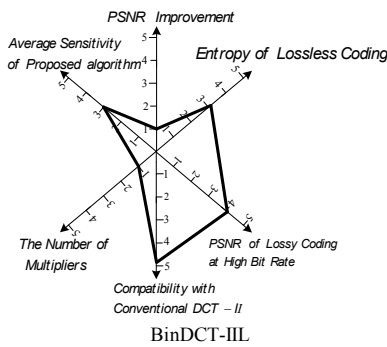
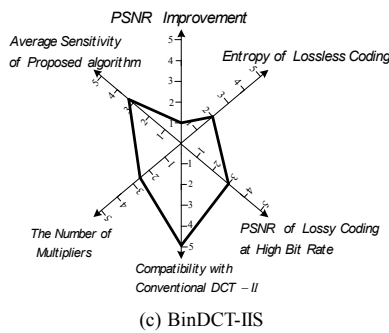


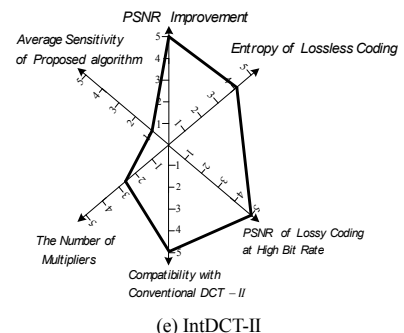
Fig. 6 Rate distortion curve of a standard image "LENNA" by integer DCT type IV.



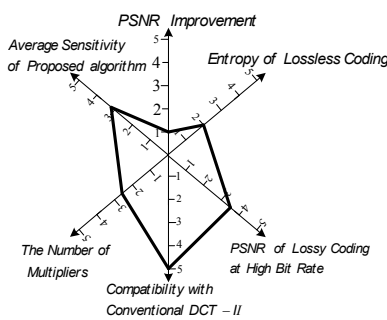
(a) BinDCT-IIC



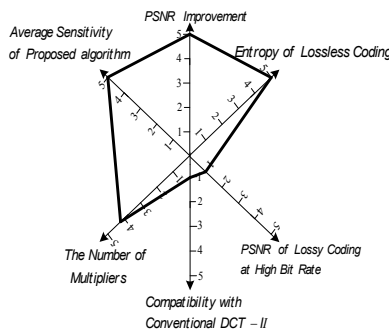
(c) BinDCT-IIS



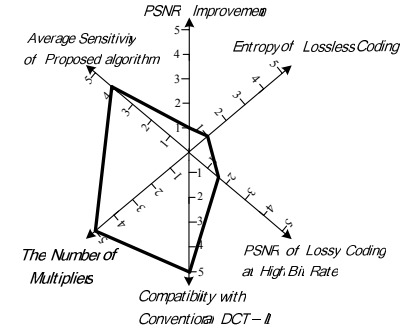
(e) IntDCT-II



(a) BinDCT-IIC



(d) BinDCT-IV



(f) LDCT-II

Fig. 7 Normalized results for comparing six types of integer DCT

V. CONCLUSIONS

The optimum structure of integer DCT algorithms are effectively confirmed by quality (PSNR) improvement comparing with the decoded signal quality of existing algorithms. It is found that the decoded signal quality of the optimum structure of integer DCT algorithms is obviously superior to the signal quality of existing algorithms. For a uniform word length assignment, the quality improvement of the IntDCT-II is the highest followed by the BinDCT-IV, BinDCT-IIS, BinDCT-IIC, LDCT-II, and BinDCT-IIL,

respectively. For an optimum word length assignment, the PSNR improvement of the IntDCT-II is still the highest and up to 2.58 dB followed by the BinDCT-IV, BinDCT-IIC, BinDCT-IIS, LDCT-II, and BinDCT-IIL, respectively. In lossless mode, the total entropy rate of the LDCT-II algorithms is the least bit rate and all of integer DCT algorithms are affirmatively compatible to the conventional DCT in lossy mode.

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