International Journal of Information, Control and Computer Sciences
ISSN: 2517-9942
Vol:7, No:1, 2013

Characterizations of Star-Shaped, *L*-Convex, and Convex Polygons

Thomas Shermer, and Godfried T. Toussaint

Abstract**—**A *chord* of a simple polygon P is a line segment $[xy]$ that intersects the boundary of P only at both endpoints x and y. A chord of P is called an *interior* chord provided the interior of $[xy]$ lies in the interior of P. P is *weakly visible* from $[xy]$ if for every point v in P there exists a point w in $[xy]$ such that $[vw]$ lies in P. In this paper star-shaped, L-convex, and convex polygons are characterized in terms of weak visibility properties from internal chords and star-shaped subsets of P. A new Krasnoselskii-type characterization of *isothetic* star-shaped polygons is also presented.

Keywords**—**Convex polygons, L-convex polygons, star-shaped polygons, chords, weak visibility, discrete and computational geometry.

I. Introduction

THIS paper is concerned mainly with new characterizations of star-shaped and L-convex polygons. However, as a corollary a new characterization of convex polygons is also obtained. For any integer $n \geq 3$, a *polygon* in the Euclidean plane E^2 is defined as the figure $P = [x_1, x_2, ..., x_n]$ formed by n points $x_1, x_2, ... , x_n$ in E^2 and n line segments $[x_i\, x_{i+1}]$, $i = 1, 2, ..., n$-1, and $[x_n\, x_1]$. The points x_i are called the *vertices* of the polygon and the line segments are termed its *edges*. The vertices of P are assumed to be in *general position*, *i.e*., no three vertices are collinear. For an accessible introduction to polygons and their classification see the paper by Grunbaum [13].

Definition: A polygon P is called a *simple* polygon provided that no point of the plane belongs to more than two edges of P and the only points of the plane that belong to precisely two edges are the vertices of P.

A simple polygon P has a well defined boundary denoted by $bd(P)$, an interior denoted by $int(P)$, and an exterior denoted by $ext(P)$. By convention, the interior of a polygon is included when referring to P. The vertices of a simple polygon are of two types: *convex* and *concave*. In the mathematics literature the terminologies *reentrant vertex* and *local non-convexity point* are often used instead of *concave vertex*, whereas in the computational geometry literature the word *reflex* vertex is preferred. However, in this paper the more natural term *concave* is used. For a given vertex x_j let $y = \lambda x_{j-1} + (1-\lambda)x_j$ and $z = \mu x_{j+1} + (1-\mu)x_j$. For all sufficiently small positive values of μ and λ $int[yz]$ lies either totally in $int(P)$ or totally in $ext(P)$; in the former case x_j is a *convex* vertex whereas in the latter case it is a *concave* vertex.

Definition: A simple polygon P is called *star-shaped* if there exists a point $x \in P$ such that for all points $y \in P$, $[xy] \in P$. The collection of all such points x is called the *kernel* of P.

Definition: (Grunbaum [13]) A simple polygon P is called *convex* provided that all its vertices are convex.

This is a special case of a well-known theorem due to Tietze [29], which states that if S is a closed connected set in a Euclidean space, all of whose points are points of *local convexity,* then S is convex. A point $x \in S$ is a point of local convexity of S if there exists a neighborhood N of x such that $N \cap S$ is convex; otherwise x is called a point of local non-convexity of S.

The convexity of polygons is frequently characterized in terms of the connectivity of the intersection-sets of lines that intersect the polygon Fary [11]. More recently Pinelis characterized the convexity of cyclic polygons in terms of the central angles of the polygon [23]. Closer in spirit to the results presented here, Nagel characterizes convex polygons in terms of the orientation-dependent chord length distributions [22].

Characterizations of objects such as convex and star-shaped polygons as well as more general sets are of interest for at least two reasons. Mathematicians are motivated by the desire to obtain a deeper understanding of geometric objects such as polygons. Different characterizations of an object provide different views of the object and thus further this understanding [11], [16], [27], [35], [36]. The study of convex sets is relevant to a variety of disciplines in science and technology [17]. For example, computer scientists are interested in designing algorithms for recognizing whether polygons are convex, L-convex, star-shaped, etc., in a variety of contexts driven by applications in pattern recognition and computer vision problems. Different characterizations yield alternative algorithms, with varying computational complexities, for solving such problems [2, 3], [12], [25], [30].

A simple polygon P is also said to be convex if every pair of points x, $y \in P$ can be joined by a line segment $[xy] \in P$ [11]. This well known characterization of convex polygons is equivalent to the demand that all three of the segments determined by each triplet of pairwise distinct points in P lie totally in P. One can relax this criterion and still obtain a characterization of convex polygons. A simple polygon P is convex if, and only if, it contains two of the three segments determined by each triple of its points [20]. Further weakening the criterion to the new demand that only one of the three segments be contained in P does not lead to convexity but to

T. Shermer is with the School of Computing Science at Simon Fraser University in Burnaby, British Columbia, Canada (e-mail: shermer@cs.sfu.ca).

G. T. Toussaint is with the Computer Science Department at New York University Abu Dhabi in Abu Dhabi, United Arab Emirates (e-mail: gt42@nyu.edu).

the notion of P_3-convexity [32]. Valentine [32] has shown that a P_3-convex polygon can be represented as the union of three or fewer convex polygons.

Convex polygons have also been characterized in terms of nearest point properties [35], as illustrated by the *Theorem of Bunt-Motzkin* [37] which states that a simple polygon P is convex if, and only if, for every point p P there is exactly one point of P nearest to p.

Furthermore, there has been interest in characterizing convex polygons in terms of *unimodality* properties. There are several possibilities for definitions of the notion of unimodality depending on the distance functions employed. For example, one can define for a vertex z of P, a function $f(z)$ which is the Euclidean distance between z and each vertex of P in the order in which the vertices occur in P. If $f(z)$ is unimodal then z is called a *unimodal vertex*. It has been incorrectly assumed in several published papers that a polygon is convex if all its vertices are unimodal in this sense. Furthermore algorithms for computing geometric properties of convex polygons based on this assumption have also been published. However, counter examples to the claim [2] and to such algorithms [3] have since appeared. Just as the Euclidean distance between pairs of vertices is used to create $f(z)$ one may instead consider vertex-edge or edge-vertex pairs and measure the separation by means of the perpendicular distance between the vertex and the line collinear with the edge in question. In this way for an edge e of P one can define a function $g(e)$ which is the perpendicular distance from the line collinear with e to every vertex of P in the order in which the vertices appear in P. If $g(e)$ is unimodal then e called a *unimodal edge*. In [30] it is shown that if all the edges of P are unimodal in this sense then P is convex.

A type of unimodality that is very different to that discussed above was considered by Dharmadhikari and Jogdeo [8]. Let P be a simple polygon in R^2. Given a non-zero vector $u \in R^2$ and $k \in R$, denote by $L(u, k)$ the line determined by the dot product $u \cdot x = k$. Let $f_u(k)$ denote the measure of $P \cap L(u, k)$. If P is a convex polygon then $f_u(k)$ as a function of k, is first non-decreasing and then non-increasing for every value of u. A non-negative function f on R is said to be *unimodal* if there exits a $v \in R$ such that f is non-decreasing on $(-\infty, v]$ and non-increasing on $[v, \infty)$. Furthermore such a number v need not be unique. Consider now the following condition:

Condition A: For every fixed non-zero $u \in R^2$, the function $f_u(k)$ is unimodal in k.

It is natural to ask whether condition A is sufficient for a simple polygon P to be convex. The answer to this question is negative. Consider the following example from [8], and refer to Fig. 1. Let $ABCD$ and $A'B'C'D'$ be parallelograms which are mirror images of each other and are such that the lines through CD and $C'D'$ meet outside the polygon at some point z. It can easily be verified by inspection that this polygon satisfies condition A but it is not even star-shaped; no point in P sees both C and C'. On the other hand in [8] it is shown that if P is such that for every fixed $u \in R^2$, the function $f_u(k)$ is *continuous* on the interior of its support then P is convex.

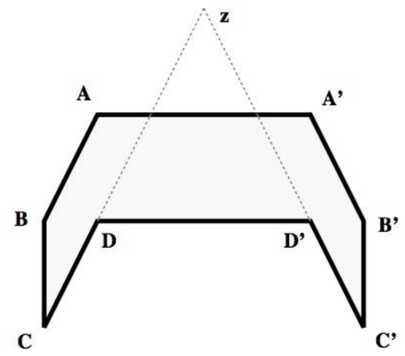

Fig. 1 Condition A is not sufficient to ensure either convexity

The earliest characterization of star-shaped polygons is due to Krasnoselskii [18], and this result is known as Krasnoselskii's theorem [37]. This theorem states that if for every set of three points x, y, and $z \in P$ there exists a point $w \in P$ (possibly dependent on x, y, and z, such that the three segments $[wx]$, $[wy]$, $[wz]$ all lie in P then P is star-shaped. If a polygon is *isothetic* then it has been shown that the number *three* can be reduced to *two* [31]. A polygon P is called *isothetic* provided that all its edges are parallel to either the x or the y axis. For additional Krasnoselskii-type characterizations of star-shaped polygons the reader is referred to [4], [21], [24], and [33]. A different characterization is provided in [26], where it is shown that a polygon P is star-shaped if, and only if, the intersection of all the maximal convex sub-polygons of P is non-empty.

II. Weak Visibility Characterizations

In this section new characterizations of star-shaped and convex polygons are presented based on the notion of weak visibility. A polygon P is said to be *weakly visible* [1] from a subset S of P if for every point $x \in P$ there exists a point $y \in S$ such that the line segment $[xy] \in P$. A *chord* of a polygon P is a line segment $[xy]$ that intersects the boundary $bd(P)$ only at x and y. If the interior of $[xy]$ lies in the interior of P then the chord is termed an *interior chord*. If the interior of $[xy]$ lies in the exterior of P then the chord is said to be an *exterior chord*.

Theorem 2.1: A simple polygon P is star-shaped if, and only if, there exists a point $x \in P$ such that P is weakly visible from every internal chord traversing x.

Proof: (only if part) Choose x to be any point in the kernel of P. Since P is star-shaped from x it follows that it is weakly visible from every internal chord of P that traverses x.

(if part) Assume P contains a point x such that P is weakly visible from every internal chord traversing x. Then P must be star-shaped from x. If this were not so it would imply the existence of a point y in P that is not visible from x. Now construct a line through both x and y and let a, b denote the first points of intersection of L with $bd(P)$ as L is traversed in both directions starting at x. Let $L' \in L$ denote the segment $[ab]$. Since y, a, and b all lie on L and y is not visible from x it follows that y is not visible from any point on L'. Therefore P is not weakly visible from chord L' which is a contradiction.

Q.E.D.

Note that as a corollary a new characterization of convex polygons is obtained. The theorem actually proves that if a point x exists such that P is weakly visible from every internal chord traversing x then P is star-shaped from x. Thus if this property holds true for every point $x \in P$, it follows that P is star-shaped from every point in P, and is convex. Thus the following result follows.

Corollary 2.1: A simple polygon P is convex if, and only if, it is weakly visible from every internal chord of P.

Consider now the case of isothetic polygons, *i.e.*, polygons with all their edges parallel to the coordinate axes. Such a polygon with its four types of *tabs* is illustrated in Fig. 2.

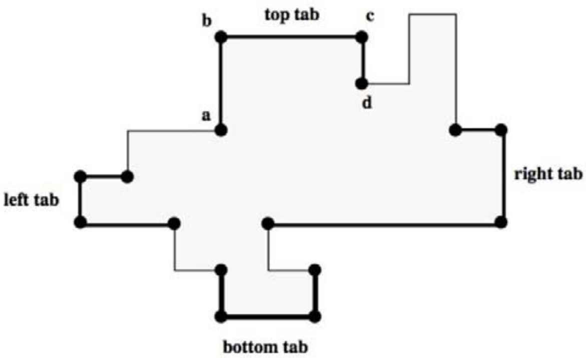

Fig. 2 An *isothetic* polygon and its four types of *tabs*

A *tab* is a set of two adjacent convex vertices along with the three edges of P incident on these two vertices. There are four types of tabs. For example, in Fig. 2, [*a, b, c, d*] is a *top tab*. Recall that Theorem 2.1 states that an arbitrary simple polygon P is star-shaped if, and only if, there exists a point $x \in P$ such that P is weakly visible from *every* internal chord traversing x. The word *every* is highlighted to indicate that for all the infinite number of unoriented directions q there exists such a chord. The term *unoriented direction q* refers to an equivalence class of parallel lines that make an angle of q with respect to some agreed upon fixed axis. Also observe that in an arbitrary simple polygon P each of its edges can occur in any one of an infinite number of un-oriented directions. Return now to the case of isothetic polygons. It is natural to conjecture the following result analogous to theorem 2.1.

Conjecture: A simple *isothetic* polygon P is star-shaped if, and only if, there exists a point $x \in P$ such that P is weakly visible from both the *horizontal* and *vertical* internal chords traversing x. As it turns out however one can prove a stronger result for isothetic polygons in the form of Theorem 2.2.

Theorem 2.2: A simple *isothetic* polygon P is star-shaped if, and only if, P is weakly visible from *both* some *horizontal* and some *vertical* internal chord of P.

Proof: (only if part) Let x be a point in the kernel of P. Clearly P is weakly visible from any internal chord traversing point x. Therefore P is weakly visible from both some horizontal and some vertical internal chord of P, namely the horizontal and vertical chords traversing point x.

(if part) Let P be weakly visible from some vertical internal chord [tb] where t and b are the upper and lower endpoints, respectively, of the chord. It follows that t must occur on a top tab and b on a bottom tab for otherwise there would exist at least one vertex of P not visible from [tb]. Now [tb] decomposes P into two polygons P_1 and P_2. Furthermore, polygons P_1 and P_2 cannot themselves contain any top or bottom tabs other than those determined by [tb] or they would contain vertices not visible from [tb]. Therefore P must contain only one top tab and only one bottom tab and [tb] must connect these two tabs. Similar arguments show that P must contain precisely one left tab and one right tab and that [lr], the horizontal chord from which P is weakly visible must have its end-points l, r on the unique left and right tabs, respectively. Furthermore, the right tab of P must lie to the right of [tb] and the left tab of P must lie to the left of [tb]. For assume this not to be the case and, without loss of generality, let P contain a right tab to the left of [tb]. This would imply that P_1 or P_2 contains a top or bottom tab other than those determined by [tb] which in turn would contradict the fact that P is weakly visible from [tb]. Therefore each horizontal chord that weakly sees P will intersect each such vertical chord. It still remains to show that P is star-shaped. Let z be the intersection point of [tb] and [lr]. It will be shown that P is star-shaped from z. Assume P is not star-shaped from z. This implies that there exists a point w on $bd(P)$ that is not visible from z. Without loss of generality assume that w lies on that part of $bd(P)$ between r and b as P is traversed in a clockwise manner, and denote this portion of P by Chain[r, ..., b]. Let P_{zrb} = Chain[r, ..., b] ∪ [b, z] ∪ [z, r] and let VP[P_{zrb}, z] be the visibility region of P_{zrb} from z. The region VP[P_{zrb}, z] cuts off regions of P_{zrb} which are hidden from z and lie either to the left or to the right of the cutting visibility rays emanating from z. Clearly w must lie either in a left or a right such hidden region. In the former case w is not visible from [lr], and in the latter case w is not visible from [tb]. In both cases a contradiction results, and therefore P must be star-shaped from z. Q.E.D.

It has been shown in Theorem 2.2 that the point x in the conjecture could be disposed of and that it was sufficient to impose weak visibility from some horizontal and some vertical chord in order to characterize isothetic star-shaped polygons. This opens a similar question for the original non-isothetic simple polygons. In other words, is it true that an arbitrary simple polygon P is star-shaped if, and only if, for every un-oriented direction q there exists an internal chord of P with direction q from which P is weakly visible? The answer to this question is negative and a counterexample due to ElGindy [10] is illustrated in Fig. 3. Apart from three thin spikes at a, b, and c the polygon in Fig. 3 has its remaining vertices on a circle with center z. Furthermore the spikes are so thin and so placed that their extended visibility lines form a triangle that encloses z. It is clear by observation that for every unoriented direction there exists an internal chord of P with direction q from which P is weakly visible. Consider any chord of P with direction q, and passing through z. If q is not one of the directions in the set determined by the three spikes, then P is weakly visible from this chord. On the other hand if q is contained in one such set, say that of spike a, then translate the chord in a direction orthogonal to q until it intersects the visibility cone of spike a.

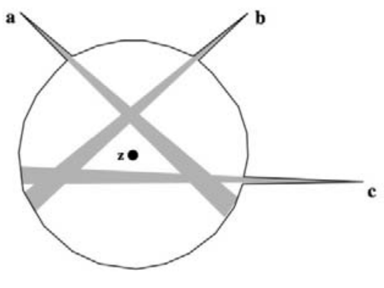

Fig. 3 ElGindy's counterexample

However, note that P is not star-shaped. Therefore Theorem 2.2, which concerns polygons with edges parallel to two directions, does not have its counterpart in the case of polygons with edges parallel to an infinite number of directions. An obvious question arises. Does Theorem 2.2 have a counterpart for a finite fixed number of directions. In other words, if P is such that all its edges are parallel to k fixed un-oriented directions, where k is some fixed positive integer, is it true that P is star-shaped if, and only if, P is weakly visible from some chord in each of the k directions? The answer to this question is also negative and a counterexample for the case of three directions is shown in Fig. 4. This polygon is weakly visible from each of the three dotted lines parallel to the three directions constraining the edges of P and yet it is not star-shaped.

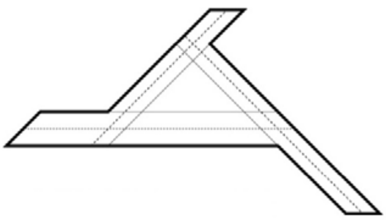

Fig. 4 A non-star-shaped polygon with chords in the three directions from which the polygon is weakly visible

III. Krasnoselskii-Type Characterizations

In this section two characterizations of isothetic star-shaped polygons are presented that resemble Krasnoselskii's theorem [18] for arbitrary simple polygons in their combinatorial flavor. First the original theorems for arbitrary and isothetic polygons are stated.

Theorem 3.1: (Krasnoselskii [18]) If every *three* points on the boundary of a simple polygon P are visible from some common point in P then there exists a point in P from which the entire boundary of P is visible.

Theorem 3.2: (Toussaint & ElGindy [31]) If every *two* points on the boundary of an *isothetic* simple polygon P are visible from some common point in P then there exists a point in P from which the entire boundary of P is visible.

In order to proceed straightforwardly some definitions for arbitrary simple polygons are introduced.

Definition: The two closed rays of a line L which have only a point $x \in L$ in common are called *complementary* rays. If $R(x)$ is a ray with endpoint x, its complementary ray is denoted by $R'(x)$.

Definition: A set of rays is said to be *concurrent* if there exists a point of the plane that intersects each and every ray in the set.

Definition: A ray $R(x)$ with endpoint $x \in bd(P)$ is an *external ray of support to int(P)* if $R(x) \cap int(P) = \emptyset$.

Definition: If $x \in bd(P)$ then $K(x)$ is the union of all the external rays of support to $int(P)$ at x. The set $K(x)$ is called an *external cone of support*. The union of all the complementary rays $R'(x)$ where $R(x) \in K(x)$ is denoted by $K'(x)$.

Valentine [33] proved the following result.

Theorem 3.3: (Valentine [33]) Let P be a non-convex simple polygon. Suppose that for each set of *three* (not necessarily distinct) concave vertices x_1, x_2, x_3 of P, there exist three external rays of support at x_1, x_2, x_3 respectively to $int(P)$ whose corresponding complementary rays are concurrent and meet in P. Then P is star-shaped.

The main result of this section will now be proved.

Theorem 3.4: Let P be a non-convex simple *isothetic* polygon. Suppose that for each set of *two* (not necessarily distinct) *concave* vertices x_1, x_2 of P, there exist two external rays of support at x_1 and x_2, to $int(P)$ whose corresponding complementary rays are concurrent and meet in P. Then P is star-shaped.

Proof: Let x_1 and x_2 be two concave vertices of P. By hypothesis there exist two external rays of support at x_1, x_2, respectively, to int(P) whose corresponding complementary rays are concurrent and meet in P at some point z. By construction it follows that $[x_1z] \in K'(x_1)$ and $[x_2z] \in K'(x_2)$. Therefore $z \in [K'(x_1) \cap K'(x_2)]$. Since P is isothetic $K'(x_1)$ and $K'(x_2)$ can each be expressed as the intersection of the half-planes (containing z) determined by $bd[K'(x_1)]$, and $bd[K'(x_2)]$. Denote the vertical and horizontal such half-planes by $V(x_i)$ and $H(x_i)$, respectively, where $i = 1, 2$. Clearly the pair $V(x_1)$ and $V(x_2)$ must contain a non-zero intersection. Since the above arguments are true for all pairs of concave vertices of P it follows that all pairs of vertical half-planes intersect. From Helly's theorem [14] it follows that all such vertical half-planes contain a non-zero intersection. Similar arguments hold for the horizontal half-planes. Therefore the intersection of $K'(x_i)$ over all concave vertices x_i must be non-zero. Denote this intersection by K^*. Next it is shown that the intersection of K^* with P is non-empty. First note that P cannot contain more than one of each of the four types of tabs. For if this were not so, one type would contain at least two tabs and this would imply that there exist at least two concave vertices with external rays of support whose corresponding complementary rays are not concurrent, thus contradicting the hypothesis. Let K^{**} denote the intersection of K^* with the four interior half-planes determined by each of the four tabs. Each such half-plane contains P and is bounded by the line collinear with the two convex vertices making up the corresponding tab. It is a straightforward matter to show that if K^* is bounded, this intersection operation will not change K^*, and if it is unbounded then K^{**} will be bounded but non-zero. Note also that the above discussion implies that each convex vertex of P not belonging to a tab must be such that both of its adjacent vertices are concave. This in turn implies that K^{**} is the intersection of the interior half-planes determined by all the edges of P and therefore P has a non-zero kernel. Therefore P is star-shaped. Q.E.D.

IV. *L*-Convex Polygons

In 1949 Horn and Valentine introduced the definition of a *link-distance* between two points a, b in P [15]. Since then mathematicians have investigated several properties of this distance measure [5, 6], [34], whereas computer scientists have investigated its computational aspects [19, 28]. The link-distance is defined as the smallest number of links (i.e., straight line segments) in a polygonal path connecting a and b within P. This distance is a useful metric for spatial path planning in robotics when straight motion is easy to accomplish but turns are expensive. Alternately, it is the ideal metric for modeling robots that use telescopic-joint manipulators to pick and place objects in a work-space modeled as a simple polygonal region.

A polygon P is said to be L_2-convex (or simply L-convex) if every pair of points a, b in P are *link-distance* two apart. More generally P is said to be L_k-convex if every pair of points a, b in P have link-distance k between them. L_2-convex polygons have received some attention in the computational geometry literature. In particular, ElGindy, Avis and Toussaint [9] have shown that if a polygon is known to be L_2-convex it can be triangulated with a very simple algorithm in linear time. No such practical efficiency is known for arbitrary simple polygons, although a rather complicated linear-time algorithm was discovered by Chazelle [7]. ElGindy, Avis and Toussaint [9] also showed that testing a simple polygon with n vertices for L_2-convexity can be done in $O(n^2)$ time. Castiglioni et al. [6] investigate a special class of L-convex polygons called L-convex polyominoes from a tomographical point of view, and characterize them by means of horizontal and vertical projections. Horn and Valentine [15] proved that if P is L-convex then for every point x in P there exists a chord that traverses x, say $L(x)$, such that P is weakly visible from $L(x)$. Since the converse also holds true this is in fact a characterization of L-convex polygons. An interesting question arises when one relaxes the chord $L(x)$ traversing x to allow more general regions such as *star-shaped* regions.

Horn and Valentine [15] characterized L-convex polygons in terms of a covering of P expressed by the following theorem.

Theorem 4.1: (Horn & Valentine [15]) A simple polygon P is L-convex if, and only if, P can be expressed as the sum of convex subsets of P, every two of which have a point in common.

Here an alternate characterization of L-convex polygons is obtained in terms of weak visibility. In the sequel let $S^*(x)$ denote a star-shaped subset of P containing x from which P is weakly visible.

Theorem 4.2: A simple polygon P is L-convex if, and only if, P has the property that for every point x in P there exists a subset S^* of P such that: (1) x is contained in S^*, (2) S^* is star-shaped from x, and (3) P is *weakly-visible* from S^*.

Proof: [only if] If P is L-convex it has the property that for every point x in P there exists a traversing chord $L(x)$ from which P is weakly visible [15]. Clearly $L(x)$ satisfies the three conditions of the theorem.

[if] Let x and y be any two points in P. The weak visibility of P from $S^*(x)$ implies that there exists a point z in $S^*(x)$ visible from y. From the star-shapedness of $S^*(x)$ from x it follows that x and z are visible. Therefore x and y have link-distance two. Since x and y were chosen arbitrarily it follows that P is L-convex. Q.E.D.

V. A New Class of Polygons

It is interesting to consider a further generalization of Theorem 4.2 by removing condition (2) requiring that S^* be star-shaped from x. Then a new class of polygons is obtained.

Definition: A simple polygon P is said to be *P^*-convex* provided that every point x in P is contained in a star-shaped subset of P, from which P is weakly visible.

An L-convex polygon is P^*-convex. However, the converse is no longer true as illustrated in Fig. 5. The polygon in Fig. 5 is not L-convex because the link-distance between vertices 2 and 5 is three. On the other hand the polygon is P^*-convex. To see this let S_{12} denote the union of S_1 and S_2, and let S_{23} denote the union of S_2 and S_3. Every point x in P must lie in either region S_{12} or S_{23}, both regions are star-shaped from vertices 4 and 1, respectively, and P is weakly visible from each such region.

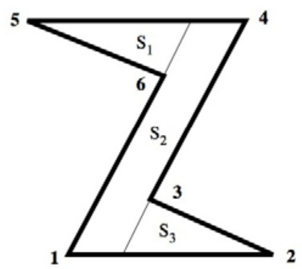

Fig. 5 A polygon that is not L-convex but is P^*-convex

As a consequence, if a polygon is P^*-convex it must be L_3-convex. To see this choose any two points p, q in a polygon that is P^*-convex and let $S^*(p)$ be the star-shaped region in P that contains p as guaranteed by the definition. Let q' be a point in $S^*(p)$ that is visible from q as guaranteed by the definition of $S^*(p)$. Finally, let k be a point contained in the kernel of $S^*(p)$. Then it follows that the path p, k, q', q lies in

P and is of link-distance three. Since the choice of p and q was arbitrary it implies that P is L_3-convex. On the other hand, an L_3-convex polygon is not necessarily P^*-convex, as illustrated in Fig. 6. Consider the point p. There is no star-shaped region $S^*(p)$ from which P is weakly visible. For S^* to contain p the kernel of $S^*(p)$ must lie in triangle psq. If this kernel lies below $[ss']$ then q' is not visible from $S^*(p)$. On the other hand if the kernel lies above $[ss']$ and close enough to r so that q' is visible from $S^*(p)$ then r' becomes invisible from $S^*(p)$. Therefore following result is established.

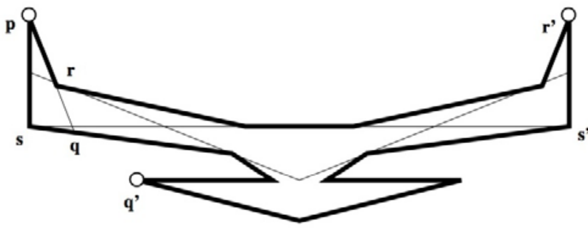

Fig. 6 An L_3-convex polygon that is not P^*-convex

Theorem 5.1: P^*-convex polygons subsume L_2-convex polygons and are a subclass of L_3-convex polygons.

Fig. 7 illustrates the various relationships that exist between the different classes of polygons.

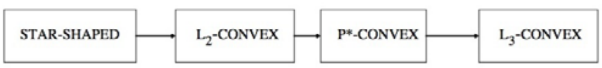

Fig. 7 The hierarchy of polygons induced by the properties of star-shapedness, L_2-convexity, P^*-convexity, and L_3-convexity

VI. CONCLUSION

It would be interesting to explore the type of characterizations introduced here, based on weak-visibility from internal chords and star-shaped sets, in the case of restricted versions of convex bodies, such as shapes of constant width, as well as for convex, L-convex and star-shaped polyhedra in three and higher dimensions.

ACKNOWLEDGMENT

This research was supported by a grant from the Provost's Office of New York University Abu Dhabi, through the Faculty of Science, in Abu Dhabi, The United Arab Emirates, as well as by a research grant from the Natural Sciences and Engineering Research Council of Canada (NSERC), administered through the School of Computer Science of McGill University in Montreal, Canada.

REFERENCES

[1] D. Avis. and G. T. Toussaint, “An optimal algorithm for determining the visibility of a polygon from an edge,” *IEEE Transactions on Computers,* vol. C-30, No. 12, pp. 910-914, December 1981.

[2] D. Avis, G. T. Toussaint, and B. K. Bhattacharya, “On the multimodality of distances in convex polygons,” *Computers & Mathematics With Applications*, vol. 8, No. 2, pp. 153-156, 1982.

[3] B. K. Bhattacharya and G. T. Toussaint, “A counterexample to a diameter algorithm for convex polygons,” *IEEE Transactions on Pattern Analysis and Machine Intelligence,* vol. PAMI- No. 3, pp. 306-309, May 1982.

[4] M. Breen, “Krasnoselskii-type theorems,” in *Discrete Geometry and Convexity*, Eds., J. E. Goodman *et al*., New York Academy of Sciences, 1985, pp.142-146.

[5] C. K. Bruckner and J. B. Bruckner, “On L_n-sets, the Hausdorff metric, and connectedness,” *Proceedings of the American Mathematical Society,* vol. 13, pp. 765-767, 1962.

[6] G. Castiglioni, A. Frosini, A. Restivo, and S. Rinaldi, "Tomographical aspects of *L*-convex polyominoes," *Pure Mathematics and Applications: Algebra and Theoretical Computer Science*, (PU.M.A.), vol. 18, No. 3-4, pp. 239-256, 2007.

[7] B. Chazelle, "Triangulating a simple polygon in linear time", *Discrete & Computational Geometry*, vol. 6, pp. 485–524, 1991.

[8] S. W. Dharmadhikari, and K. Jogdeo, “A characterization of convexity and central symmetry for planar polygonal sets,” *Israel Journal of Mathematics*, vol. 15, pp. 356-366, 1973.

[9] H. ElGindy, D. Avis, and G. T. Toussaint, “Applications of a two-dimensional hidden-line algorithm to other geometric problems,” *Computing*, vol. 31, pp. 191-202, 1983.

[10] H. ElGindy, private communication, School of Computer Science and Engineering, University of New South Wales, Sydney, Australia, elgindyh@cse.unsw.edu.au

[11] I. Fary, “A characterization of convex bodies,” *American Mathematical Monthly*, vol. 69, pp. 25-31, 1962.

[12] L. P. Gewali, "Recognizing s-star polygons," *Pattern Recognition*, vol. 28, no. 7, pp. 1019-1032, July 1995.

[13] B. Grunbaum, “Polygons,” in *The Geometry of Metric and Linear Spaces*, A. Dold & B. Eckman, eds., Springer-Verlag, New York, 1965, pp.147-184.

[14] E. Helly, “Uber Mengen konvexer Korper mit gemeinshaftlichen Punkten,” *Jber. Deutsch. Math. Verein*. vol. 32, 175-176, 1923.

[15] A. Horn and F. A. Valentine, “Some properties of *L*-sets in the plane,” *Duke Mathematics Journal*, vol. 16, pp. 131-140, 1949.

[16] V. L. Klee, “A characterization of convex sets,” *American Mathematical Monthly*, vol. 56, pp. 247-249, 1949.

[17] V. L. Klee, "What is a convex set?" *The American Mathematical Monthly*, vol. 78, no. 6, pp. 616-631, June-July 1971.

[18] M. A. Krasnoselskii, “Sur un critere qu’un domaine soit etoile, *Math. Sb.* vol. 61, No. 19, 1946.

[19] W. Lenhart, R. Pollack, J. Sack, R. Seidel, M. Sharir, S. Suri, G. T. Toussaint, S. Whitesides and C. Yap, “Computing the link center of a simple polygon,” *Discrete & Computational Geometry,* vol. 3, 1988, pp. 281-293.

[20] J. M. Marr, and W. L. Stamey, “A three-point property,” *American Mathematical Monthly*, vol. 69, pp. 22-25, 1962.

[21] J. Molnar, "Uber Sternpolygone," *Publ. Math. Debrecen,* vol. 5, pp. 241-245, 1958.

[22] W. Nagel, "Orientation-dependent chord length distributions characterize convex polygons," *Journal of Applied Probability,* vol. 30, pp. 730–736, 1993.

[23] I. Pinelis, "A Characterization of the convexity of cyclic polygons in terms of the central angles," *Journal of Geometry*, vol. 87, pp. 106–119, 2007.

[24] E. E. Robkin, *Characterizations of star-shaped sets*, Ph.D. thesis, UCLA, 1965.

[25] T. Shermer, "On recognizing unions of two convex polygons and related problems," *Pattern Recognition Letters*, vol. 14, no. 9, pp. 737-745, 1993.

[26] C. R. Smith, "A characterization of star-shaped sets," *The American Mathematical Monthly*, vol. 75, no. 4, p. 386, April 1968.

[27] E. G. Strauss, and F. A. Valentine, "A characterization of finite dimensional convex sets," *American Journal of Mathematics,* vol. 74, pp. 683-686, 1952.

[28] S. Suri, “Minimum link paths in polygons and applications,” Ph.D. Thesis, The Johns Hopkins University, Department of Computer Science, August 1987.

[29] H. Tietze, "Uber Konvexheit im kleinen und im grossen und uber gewisse den Punkten einer Menge zugeordnete Dimensionzahlen," *Math. Z.,* vol. 28, pp. 697-707, 1928.

[30] G. T. Toussaint, "Complexity, convexity, and unimodality," *International Journal of Computer and Information Sciences*, vol. 13, No. 3, pp. 197-217, June 1984.

[31] G. T. Toussaint and H. ElGindy, "Traditional galleries are star-shaped if every two paintings are visible from some common point," Tech. Rept. SOCS-81.10, School of Computer Science, McGill University, March 1981.
[32] F. A. Valentine, "A three point convexity property," *Pacific Journal of Mathematics*, vol. 7, pp. 1227-1235, 1957.
[33] F. A. Valentine, "Local convexity and star-shaped sets," *Israel Journal of Mathematics*, vol. 3, pp. 39-42, March 1965.
[34] F. A. Valentine, "Local convexity and L_n-sets," *Proceedings of the American Mathematical Society*, vol. 16, pp. 1305-1310, 1965.
[35] F. A. Valentine, *Convex Sets*, McGraw-Hill, New York, 1964.
[36] F. A. Valentine, "Characterizations of convex sets by local support properties," *Proc. Amer. Math. Soc.,* vol. 11, pp. 112-116, 1960.
[37] I. M. Yaglom, and V. G. Boltyanskii, *Convex Figures*, Holt, Rinehart and Winston, New York, 1961.