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# Characterization of solutions of nonsmooth variational problems and duality

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*Abstract*—In this paper, we introduce a new class of nonsmooth pseudo-invex and nonsmooth quasi-invex functions to non-smooth variational problems. By using these concepts, numbers of necessary and sufficient conditions are established for a nonsmooth variational problem wherein Clarke's generalized gradient is used. Also, weak, strong and converse duality are established.

*Keywords*—Variational problem; Nonsmooth pseudo-invex; Nonsmooth quasi-invex; Critical point; Duality

## I. Introduction

As is known to all, invexity is a generalization of convexity and can be used to extend the sufficiency of the Kuhn-Tucker conditions and duality theory of convex programs to more general optimization problems. This invexity idea was introduced by Hanson [1] for differential functions and was generalized to nonsmooth functions in [2],[3]. Invexity was also weakened in order that it can be served as a characterization of problems where every Kuhn-Tucker point is a global minimizer [4]. In [3], Reiland defined several types of invexity for locally Lipschitz functions and obtained some optimization results for nonsmooth mathematical programming problems.

In pioneering works, invextity was extended to variational problems by Mond, Chandra and Husain, see [5] for more details. There exists huge literature on necessary and sufficient conditions on calculus of variations (One can see [6],[7], and references therein).

In [8], the authors considered characterization of solutions and duality for variational problem:

$$\begin{array}{lll} \text{(CVP)} & \text{Minimize} & F(x)=\displaystyle\int_a^b f(t,x,\dot{x})\,\mathrm{d}t \\ & \text{subject to} & x(a)=\alpha,\ x(b)=\beta, \\ & & g(t,x,\dot{x})\le 0,\ t\in I. \end{array}$$

However, the functions in their papers were all continuously differential ones.

To the best knowledge of us, there exist few studies on nonsmooth variational problems. Motivated by the discussions above, the main purpose of this paper is to give some optimality conditions of the following problem:

$$\begin{array}{lll} \text{(NCVP)} & \text{Minimize} & F(x,\dot{x})=\displaystyle\int_a^b f(x,\dot{x})\,\mathrm{d}t \\ & \text{subject to} & x(a)=\alpha,\ x(b)=\beta,\ a.e., \\ & & g(x,\dot{x})\le 0, t\in I,\ a.e.. \end{array}$$

The organization of the rest of this paper is given as follows: In section 2, we give some preliminaries and definitions. In section 3, the concepts of nonsmooth invexity, nonsmooth pseudo-invexity and nonsmooth quasi-invexity are difined in terms of Clarke's generalized gradient. In section 4, we give some necessary and sufficient conditions for a Kuhn-Tucker (Fritz-John) critical point of the nonsmooth variational problem to be a minimum. In section 5, weak, strong and converse duality are established. I wish you the best of success.

## II. Preliminaries

Let us introduce the variational problem and definitions. Let $I=[a,b]$ be a real interval, and Let $f: R^n\times R^n\to R$, $g: R^n\times R^n\to R^m$ be globally Lipschitz. For notational convenience $f(x(t),\dot{x}(t))$ and $g(x(t),\dot{x}(t))$ will be written $f(x,\dot{x})$ and $g(x,\dot{x})$ respectively. Let $X=\Big\{(x,\dot{x})\in L_n^2[a,b]\times L_n^2[a,b] : \dot{x}(t):=\frac{\mathrm{d}}{\mathrm{d}t}x(t)\ a.e.,\ x(a)=\alpha,\ x(b)=\beta,\ t\in I\Big\}$ with the norm

$$\|x\|=\|x\|_\infty+\|Dx\|_\infty,$$

where $x: I\to R^n$ is an absolutely continuous function which can be expressed in the form

$$x(t)=\alpha+\int_a^t u(s)\,\mathrm{d}s$$

for some integrable function $u$ and $\alpha$ is a given boundary value; we then have $\dot{x}(t):=\frac{\mathrm{d}}{\mathrm{d}t}x(t)=u(t)$ a.e.. Note that $X$ is closed and convex. We now consider the nonsmooth constraint variational problem:

$$\begin{array}{lll} \text{(NCVP)} & \text{Minimize} & F(x,\dot{x})=\displaystyle\int_a^b f(x,\dot{x})\,\mathrm{d}t \\ & \text{subject to} & x(a)=\alpha,\ x(b)=\beta,\ a.e., \\ & & g(x,\dot{x})\le 0, t\in I,\ a.e.. \end{array}$$

We denote by K the set of feasible solutions of (NCVP), i.e.,

$$K=\Big\{(x,\dot{x})\in X : x(a)=\alpha,\ x(b)=\beta,\ g(x,\dot{x})\le 0,\ t\in I,\ a.e.\Big\}.$$

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In order to obtain our results, we need the following definitions:

*Definition 2.1:* $(x,\dot{x}) \in K$ is said to be an optimal solution or a minimum of (NCVP) if

$$F(\bar{x},\dot{\bar{x}}) \le F(x,\dot{x}),$$

for all $(x,\dot{x}) \in K$, or equivalent, there is no

$$(x,\dot{x}) \in K : F(x,\dot{x}) < F(\bar{x},\dot{\bar{x}}).$$

*Definition 2.2:* ([9]) A real valued function $\phi : X \to R$ is said to be locally Lipschitz at a point $(u,\dot{u}) \in X$ if there exists a number $K > 0$ such that

$$\|\phi(x,\dot{x}) - \phi(\bar{x},\dot{\bar{x}})\| \le K\|x-\bar{x}\|_\infty + K\|\dot{x}-\dot{\bar{x}}\|_\infty,$$

for all $(x,\dot{x}),(\bar{x},\dot{\bar{x}}) \in X$ in a neighborhood of $(u,\dot{u})$. A function $\phi : X \to R$ is said to be locally Lipschitz on $X$ if it is locally Lipschitz at each point of $X$. A function $\phi : X \to R$ is said to be globally Lipschitz on $X$ if there exists a number $K > 0$ such that

$$\|\phi(x,\dot{x}) - \phi(y,\dot{y})\| \le K\|x-y\|_\infty + K\|\dot{x}-\dot{y}\|_\infty,$$

for all $(x,\dot{x}),(y,\dot{y}) \in X$.

*Definition 2.3:* ([9]) Let $\phi : X \to R$ be a locally Lipschitz function, then $\phi^\circ(\mathcal{U};\mathcal{V})$ denotes Clarke's generalized directional derivative of $\phi$ at $\mathcal{U} = (u,\dot{u}) \in X$ in the direction $\mathcal{V} = (v,\dot{v}) \in X$ and is defined as

$$\phi^\circ(\mathcal{U};\mathcal{V}) = \limsup_{\substack{\mathcal{Y}\to\mathcal{U}\\ t\downarrow 0}} \frac{\phi(\mathcal{Y}+t\mathcal{V})-\phi(\mathcal{Y})}{t},$$

where, of course, $\mathcal{Y} = (y,\dot{y})$ is a vector in $X$ and $t$ is a positive scalar. $\partial\phi(\mathcal{U})$ denotes Clarke's generalized gradient of $\phi$ at $\mathcal{U}$, which is denoted as

$$\partial\phi(\mathcal{U}) = \Big\{\xi \in X : \phi^\circ(\mathcal{U};\mathcal{V}) \ge \langle \xi,\mathcal{V}\rangle,\ \textit{for all}\ \mathcal{V} = (v,\dot{v}) \in X\Big\}.$$

Let $g : X \to R^m$ be a vector valued function given by $g = (g_1,g_2,\ldots,g_m)$, where each $g_i (i = 1,\ldots,m)$ is a real valued function defined on $X$. Then $g$ is said to be a locally Lipschitz on $X$ if each $g_i (i = 1,\ldots,m)$ is locally Lipschitz on $X$. The generalized directional derivative of a locally Lipschitz function $g : X \to R^m$ at $\mathcal{U} \in X$ in the direction $\mathcal{V}$ is given by

$$g^\circ(\mathcal{U};\mathcal{V}) = \Big\{g_1^\circ(\mathcal{U};\mathcal{V}), g_2^\circ(\mathcal{U};\mathcal{V}),\ldots,g_m^\circ(\mathcal{U};\mathcal{V})\Big\}.$$

The generalized gradient of $g$ at $\mathcal{U}$ is the set

$$\partial g(\mathcal{U}) = \partial g_1(\mathcal{U}) \times \ldots \times \partial g_m(\mathcal{U}),$$

where $\partial g_i(\mathcal{U})$ is the generalized gradient of $g_i$ at $\mathcal{U}$ for $i = 1,2,\ldots,m$.

Every element $B = (B_1,B_2,\ldots,B_m) \in \partial g_i(\mathcal{U})$ is a continuous linear operator from $X$ to $X^m$ and $\langle B,\mathcal{U}\rangle = \big(\langle B_1,\mathcal{U}\rangle,\ldots,\langle B_m,\mathcal{U}\rangle\big)$ for all $\mathcal{U} \in X$.

The following lemmas are useful for the proof of our main results of this paper.

*Lemma 2.1:* ([6], [7])

(1) If $g_i : X \to R$ is a locally Lipschitz function, then for each $\mathcal{U} \in X$, $g_i^\circ(\mathcal{U},\mathcal{V}) = max\Big\{\langle \xi,\mathcal{V}\rangle : \xi \in \partial g_i(\mathcal{U})\Big\}$, for every $\mathcal{V} \in X,\ i = 1,2,\ldots,m$.

(2) Let $g_i (i = 1,\ldots,m)$ be a finite family of locally Lipschitz functions on $X$ and let $\lambda_i (i = 1,2,\ldots,m)$ be scalars. Then $\sum\limits_{i=1}^{m} g_i$ is also locally Lipschitz, and for every $\mathcal{U} \in X$,

$$\partial\big(\sum_{i=1}^{m}\lambda_i g_i\big)(\mathcal{U}) \subset \sum_{i=1}^{m}\lambda_i\partial g_i(\mathcal{U}).$$

*Lemma 2.2:* ([7]) If one of $h_1 : X \to R$, $h_2 : X \to R$ is Lipschitz near $(x,\dot{x})$, then

$$\partial_L(h_1+h_2)(x,\dot{x}) \subseteq \partial_L h_1(x,\dot{x}) + \partial_L h_2(x,\dot{x}),$$

where $\partial_L h_1(x,\dot{x}), \partial_L h_2(x,\dot{x}), \partial_L(h_1+h_2)(x,\dot{x})$ denotes the limiting subdifferential of $h_1, h_2, h_1{+}h_2$ at $(x,\dot{x})$, respectively (About limiting subdifferential, one can see P61 in [7] for more details).

*Lemma 2.3:* ([7]) The limiting subdifferential and the generalized gradient of $F(x,\dot{x}) = \int_a^b f(x,\dot{x})\,\mathrm{d}t$ coincide, and we have

$$\begin{aligned} &\partial_L F(x,\dot{x}) = \partial f(x,\dot{x}) \\ = &\Big\{\xi \in L_n^2[a,b] \times L_n^2[a,b] : \xi(t) \in \partial f\big(x(t),\dot{x}(t)\big)\ a.e.\Big\}, \end{aligned}$$

where $\xi = (\xi_1,\xi_2),\ \xi(t) = \big(\xi_1(t),\xi_2(t)\big)$.

*Lemma 2.4:* ([7]) Let $f(x,\dot{x})$ be a globally Lipschitz function on $X$, with Lipschitz constant $K$, then $F(x,\dot{x}) = \int_a^b f(x,\dot{x})\,\mathrm{d}t$ is well defined and finite on $X$, and globally Lipschitz with Lipschitz constant $K(b-a)^{\frac{1}{2}}$.

*Definition 2.4:* $(\bar{x},\dot{\bar{x}}) \in K$ is said to be a Fritz-John critical point if there exist $\tau \in R$ and $\lambda \in R^m$ such that

$$\begin{aligned} (0,0) \in \partial(\tau f)\big(\bar{x}(t),\dot{\bar{x}}(t)\big) + \lambda^T\partial g\big(\bar{x}(t),\dot{\bar{x}}(t)\big)\ a.e.,\\ \lambda_i G_i\big(\bar{x}(t),\dot{\bar{x}}(t)\big) = 0\,(i,\ldots,m)\ a.e.,\\ (\tau,\lambda) \ge 0,\ (\tau,\lambda) \ne 0. \end{aligned} \tag{1}$$

As is usual in optimization theory, if $\tau \ne 0$, we say that the problem is normal or regular [10] and the critical point is a Kuhn-Tucker critical point. Correspondingly, in that case, the condition (1) reduces to $\lambda \ge 0$, respectively.

*Remark 2.1:* We recall that some additional hypotheses are necessary to guarantee $\tau \ne 0$, for example, the generalized Slater constraint qualification.

## III. Invexity, pesuso-invexity and quasi-invexity of nonsmooth functions

Invexity was first introduced by Hanson [1] for differential functions and was generalized to nonsmooth functions in [3] and [11]:

*Definition 3.1:* $f : X \to R$ is said to be generalized invex at the point $(\bar{x},\dot{\bar{x}}) \in X$, if there exists $\eta : X \times X \to X$ such that, for every $(x,\dot{x}) \in X$ and $\xi \in \partial f(x,\dot{x})$,

$$f(x,\dot{x}) - f(\bar{x},\dot{\bar{x}}) \ge \langle \xi,\eta\rangle.$$

*Definition 3.2:* $f : X \to R$ is said to be nonsmooth invex at the point $(\bar{x},\dot{\bar{x}}) \in X$ if there exists $\eta : X \times X \to X$ such that, for every $(x,\dot{x}) \in X$,

$$f(x,\dot{x}) - f(\bar{x},\dot{\bar{x}}) \ge f^\circ\big((x,\dot{x});\eta\big).$$

*Lemma 3.1:* $f : X \to R$ is generalized invex at a point $(\bar{x}, \dot{\bar{x}}) \in X$ with respect to $\eta : X \times X \to X$ if and only if $f$ is nonsmooth invex at $(\bar{x}, \dot{\bar{x}}) \in X$ with respect to same $\eta$.

*Proof:* For the "only if " part, if $f$ is generalized invex at $(\bar{x}, \dot{\bar{x}})$, then there exists $\eta : X \times X \to X$ such that, for every $(x, \dot{x}) \in X$ and $\xi \in \partial f\big((\bar{x}, \dot{\bar{x}})\big)$,

$$f(x, \dot{x}) - f(\bar{x}, \dot{\bar{x}}) \geq \langle \xi, \eta \rangle.$$

Choose $\bar{\xi} \in \partial f(\bar{x}, \dot{\bar{x}})$ such that

$$\langle \bar{\xi}, \eta \rangle = \sup \left\{ \langle \xi, \eta \rangle : \xi \in \partial f(\bar{x}, \dot{\bar{x}}) \right\} = f^{\circ}\big((\bar{x}, \dot{\bar{x}}); \eta\big),$$

then

$$f(x, \dot{x}) - f(\bar{x}, \dot{\bar{x}}) \geq \langle \bar{\xi}, \eta \rangle = f^{\circ}\big((\bar{x}, \dot{\bar{x}}); \eta\big).$$

Therefore, $f$ is nonsmooth invex at $(\bar{x}, \dot{\bar{x}})$ with respect to same $\eta$.

Let us turn to the the "if " part. If $f$ is nonsmooth invex at $(\bar{x}, \dot{\bar{x}}) \in X$, then there exists $\eta : X \times X \to X$ such that, for every $(x, \dot{x}) \in X$,

$$f(x, \dot{x}) - f(\bar{x}, \dot{\bar{x}}) \geq f^{\circ}\big((\bar{x}, \dot{\bar{x}}); \eta\big),$$

where

$$f^{\circ}\big((\bar{x}, \dot{\bar{x}}); \eta\big) = \langle \bar{\xi}, \eta \rangle = \sup \Big\{ \langle \xi, \eta \rangle : \xi \in \partial f(\bar{x}, \dot{\bar{x}}) \Big\},$$

which implies

$$f(x, \dot{x}) - f(\bar{x}, \dot{\bar{x}}) \geq f^{\circ}\big((\bar{x}, \dot{\bar{x}}); \eta\big) \geq \langle \xi, \eta \rangle, \ \forall\, \xi \in \partial f(\bar{x}, \dot{\bar{x}}).$$

Therefore, $f$ is generalized invex at $(\bar{x}, \dot{\bar{x}})$ with respect to same $\eta \in X$. ■

Invextiy was weakened in order that it can be served as a necessary optimality condition ( One can see [5],[11],[12],[13]). We now introduce the various generalizations of nonsmooth invex functions.

*Definition 3.3:* $f : X \to R$ is said to be nonsmooth pseudo-invex at $(\bar{x}, \dot{\bar{x}}) \in X$ if there exists $\eta : X \times X \to X$ such that, for every $(x, \dot{x}) \in X$,

$$f(x, \dot{x}) - f(\bar{x}, \dot{\bar{x}}) < 0 \Rightarrow f^{\circ}\big((\bar{x}, \dot{\bar{x}}); \eta\big) < 0,$$

or equivalently, $f^{\circ}\big((\bar{x}, \dot{\bar{x}}); \eta\big) \geq 0 \Rightarrow f(x, \dot{x}) \geq f(\bar{x}, \dot{\bar{x}})$.

*Definition 3.4:* $f : X \to R$ is said to be nonsmooth quasi-invex at $(\bar{x}, \dot{\bar{x}}) \in X$, if there exists $\eta : X \times X \to X$ such that, for every $(x, \dot{x}) \in X$,

$$f(x, \dot{x}) - f(\bar{x}, \dot{\bar{x}}) \leq 0 \Rightarrow f^{\circ}\big((\bar{x}, \dot{\bar{x}}); \eta\big) \leq 0.$$

*Lemma 3.2:* If $f$ is nonsmooth invex at $(\bar{x}, \dot{\bar{x}}) \in X$ with respect to $\eta : X \times X \to X$, then it is nonsmooth pseudo-invex at $(\bar{x}, \dot{\bar{x}})$ with respect to same $\eta$.

*Proof:* If $f : X \to R$ is nonsmooth invex at $(\bar{x}, \dot{\bar{x}}) \in X$, then there exists $\eta : X \times X \to X$ such that, for every $(x, \dot{x}) \in X$,

$$f(x, \dot{x}) - f(\bar{x}, \dot{\bar{x}}) \geq f^{\circ}\big((\bar{x}, \dot{\bar{x}}); \eta\big). \tag{1}$$

Suppose that $f(x, \dot{x}) - f(\bar{x}, \dot{\bar{x}}) < 0$, directly from (1), we have

$$f^{\circ}\big((\bar{x}, \dot{\bar{x}}); \eta\big) < 0.$$

Then, $f$ is nonsmooth pseudo-invex at $(\bar{x}, \dot{\bar{x}}) \in X$ with respect to same $\eta$. ■

*Remark 3.1:* Converse of the above lemma is not true as can be seen from the following example.

*Example 3.1:* Let

$$f(x, \dot{x}) = \begin{cases} 0, & x(t) \geq 0, \\ \frac{x(t)}{2}, & x(t) < 0, \end{cases}$$

where $(x, \dot{x}) \in X$, $I = [0, 1]$, $\alpha = x(0)$, $n = 1$. Let $\eta = (\eta_1, \dot{\eta_1}) \in X$ be defined as $\eta_1 = (x - \bar{x})^3$. Then, at $\bar{x} = 0$, $\eta = (x^3, 3x^2\dot{x})$, $f^{\circ}\big((0, 0); \eta\big) = \begin{cases} 0, & x(t) \geq 0, \\ \frac{x^3(t)}{2}, & x(t) < 0. \end{cases}$

Then, $f(x, \dot{x}) < 0 \Rightarrow f^{\circ}\big((0, 0); \eta\big) < 0$, which implies $f$ is nonsmooth pseudo-invex at $(0, 0)$. But $f$ is not nonsmooth invex at $(0, 0) \in X$, because, for $x(t) = -t, t \in (0, 1)$,

$$f(x, \dot{x}) - f(0, 0) - f^{\circ}\big((0, 0); \eta\big) < 0.$$

*Lemma 3.3:* If $f$ is nonsmooth invex at $(\bar{x}, \dot{\bar{x}}) \in X$ with respect to $\eta : X \times X \to X$, then it is nonsmooth quasi-invex at $(\bar{x}, \dot{\bar{x}}) \in X$ with respect to same $\eta$.

*Proof:* If $f$ is nonsmooth invex at $(\bar{x}, \dot{\bar{x}}) \in X$, then there exists $\eta : X \times X \to X$ such that, for every $(x, \dot{x}) \in X$

$$f(x, \dot{x}) - f(\bar{x}, \dot{\bar{x}}) \geq f^{\circ}\big((\bar{x}, \dot{\bar{x}}); \eta\big). \tag{2}$$

Suppose that $f(x, \dot{x}) - f(\bar{x}, \dot{\bar{x}}) \leq 0$, then directly from (2), we have

$$f^{\circ}\big((\bar{x}, \dot{\bar{x}}); \eta\big) \leq 0,$$

which implies $f$ is nonsmooth quasi-invex at $(\bar{x}, \dot{\bar{x}})$ with respect to same $\eta$. ■

*Remark 3.2:* Converse of the above lemma is not true as can be seen form the following example.

*Example 3.2:* Let

$$f(x, \dot{x}) = \begin{cases} \frac{x(t)}{2}, & x(t) \leq 0, \\ x^2(t), & x(t) > 0, \end{cases}$$

where $(x, \dot{x}) \in X$, $I = [0, 1]$, $x(0) = x_0$, $n = 1$.

Let $\eta = (\eta_1, \dot{\eta_1}) \in X$ be defined as $\eta_1 = (x - \bar{x})^3$. Then, at $\bar{x} = 0$, $\eta = (x^3, 3x^2\dot{x})$,

$$f^{\circ}\big((0, 0); \eta\big) = \begin{cases} \frac{x^3(t)}{2}, & x(t) \leq 0, \\ 0, & x(t) > 0. \end{cases}$$

So, $f(x, \dot{x}) \leq 0 \Rightarrow f^{\circ}\big((0, 0); \eta\big) \leq 0$, which implies $f$ is nonsmooth quasi-invex at $(0, 0)$. But $f$ is not nonsmooth invex at $(0, 0) \in X$, because, for $x(t) = -t$, $t \in (0, 1)$,

$$f(x, \dot{x}) - f(0, 0) - f^{\circ}\big((0, 0); \eta\big) < 0.$$

*Remark 3.3:* If $f$ is differential, then nonsmooth invex, nonsmooth pseudo-invex and nonsmooth quasi-invex reduce to invex, pseudo-invex and quasi-invex defined by Hanson [1], respectively.

We recall the definitions of invexity for functional given in [8].

*Definition 3.5:* The functional $F(x, \dot{x}) = \int_a^b f(x, \dot{x})\, \mathrm{d}t$ is said to be nonsmooth invex at $(\bar{x}, \dot{\bar{x}}) \in X$, if there exists $\eta : X \times X \to X$ such that for every $(x, \dot{x}) \in X$ and $\xi \in \partial f(\bar{x}, \dot{\bar{x}})$,

$$F(x, \dot{x}) - F(\bar{x}, \dot{\bar{x}}) \geq \int_a^b \langle \xi, \eta \rangle\, \mathrm{d}t.$$

*Definition 3.6:* The functional $F(x,\dot{x}) = \int_a^b f(x,\dot{x})\,\mathrm{d}t$ is said to be nonsmooth pseudo-invex at $(\bar{x},\dot{\bar{x}}) \in X$, if there exists $\eta : X \times X \to X$ such that for every $(x,\dot{x}) \in X$ and $\xi \in \partial f(\bar{x},\dot{\bar{x}})$,

$$F(x,\dot{x}) - F(\bar{x},\dot{\bar{x}}) < 0 \Rightarrow \int_a^b \langle \xi, \eta \rangle \,\mathrm{d}t < 0,$$

or equivalently, $\int_a^b \langle \xi, \eta \rangle \,\mathrm{d}t \geq 0 \Rightarrow F(x,\dot{x}) \geq F(\bar{x},\dot{\bar{x}})$.

*Definition 3.7:* The functional $F(x,\dot{x}) = \int_a^b f(x,\dot{x})\,\mathrm{d}t$ is said to be nonsmooth quasi-invex at $(\bar{x},\dot{\bar{x}}) \in X$, if there exists $\eta : X \times X \to X$ such that for every $(x,\dot{x}) \in X$ and $\xi \in \partial f(\bar{x},\dot{\bar{x}})$,

$$F(x,\dot{x}) - F(\bar{x},\dot{\bar{x}}) \leq 0 \Rightarrow \int_a^b \langle \xi, \eta \rangle \,\mathrm{d}t \leq 0.$$

## IV. Optimality conditions

Let $F(x,\dot{x}) = \int_a^b f(x,\dot{x})\,\mathrm{d}t$ and $G(x,\dot{x}) = \int_a^b g(x,\dot{x})\,\mathrm{d}t$. In this section, we first prove that a minimum point is necessarily a KKT(Karush-Kuhn-Tucker) point of (NCVP) under nonsmooth invex and the Slater constraint qualification assumptions.

*Definition 4.1:* The problem (NCVP) is said to satisfy the Slater constraint qualification, if there exists $(\tilde{x},\dot{\tilde{x}}) \in K$ such that $g(\tilde{x},\dot{\tilde{x}}) < 0$.

*Theorem 4.1:* Let $F$ and $G_i(i = 1, 2, \ldots, m)$ be nonsmooth invex at $(\bar{x},\dot{\bar{x}}) \in K$ with respect to same $\eta : X \times X \to X$. Suppose that the Slater constraint qualification is satisfied, and if (NCVP) attains a minimum at $(\bar{x},\dot{\bar{x}})$, then it is a KKT point of (NCVP).

*Proof:* Since $(\bar{x},\dot{\bar{x}})$ is a minimum point of (NCVP), there is no $(x,\dot{x}) \in K$ such that

$$F(x,\dot{x}) - F(\bar{x},\dot{\bar{x}}) < 0.$$

Then, there is no solution $(x,\dot{x}) \in K$ of the system

$$\Big(\big(F(x,\dot{x}) - F(\bar{x},\dot{\bar{x}})\big), G(x,\dot{x})\Big) < 0.$$

Since $F$ and $G_i(i = 1, 2, \ldots, m)$ are all nonsmooth invex with respect to same $\eta$ (or we can say $(F, G_1, G_2, \ldots, G_m)$ is nonsmooth invex with respect to same $\eta$), it follows from the generalized alternative theorem ([14]) that there exist $\tau \in R_+$, $\tilde{\lambda} \in R_+^m$, with $(\tau, \tilde{\lambda}) \neq 0$, such that

$$\tau\big(F(x,\dot{x}) - F(\bar{x},\dot{\bar{x}})\big) + \tilde{\lambda}^T G(x,\dot{x}) \geq 0, \quad \forall (x,\dot{x}) \in K. \quad (1)$$

If possible $\tau = 0$, then $\tilde{\lambda} \neq 0$ and from (1),

$$\tilde{\lambda}^T G(x,\dot{x}) \geq 0, \ \forall (x,\dot{x}) \in K. \quad (2)$$

By generalized Slater constraint qualification, there exists $(\tilde{x},\dot{\tilde{x}}) \in K$ such that $g(\tilde{x},\dot{\tilde{x}}) < 0$, it follows that

$$\tilde{\lambda}^T G(\tilde{x},\dot{\tilde{x}}) < 0,$$

which is a contradiction to (2). Hence, $\tau \neq 0$.

Therefore, (1) is equivalent to

$$(F(x,\dot{x}) - F(\bar{x},\dot{\bar{x}})) + \bar{\lambda}^T G(x,\dot{x}) \geq 0, \quad \forall\, (x,\dot{x}) \in K, \quad (3)$$

where $\bar{\lambda} = \frac{\tilde{\lambda}}{\tau} = (\frac{\tilde{\lambda}_1}{\tau}, \ldots, \frac{\tilde{\lambda}_m}{\tau}) \geq 0$. Taking $(x,\dot{x}) = (\bar{x},\dot{\bar{x}})$ in (3), we get

$$\bar{\lambda}^T G(\bar{x},\dot{\bar{x}}) \geq 0, \ i.e., \ \sum_{i=1}^m \bar{\lambda}_i G_i(\bar{x},\dot{\bar{x}}) \geq 0.$$

Since $\bar{\lambda} \geq 0$ and $g(\bar{x},\dot{\bar{x}}) \leq 0$, we have

$$\bar{\lambda}_i G_i(\bar{x},\dot{\bar{x}}) \leq 0 \,(i = 1, \ldots, m).$$

Hence,

$$\bar{\lambda}_i G_i(\bar{x},\dot{\bar{x}}) = 0 \,(i = 1, \ldots, m). \quad (4)$$

Thus, from (3), (4), we have, for all $(x,\dot{x}) \in K$,

$$F(x,\dot{x}) + \bar{\lambda}^T G(x,\dot{x}) \geq F(\bar{x},\dot{\bar{x}}) + \bar{\lambda}^T G(\bar{x},\dot{\bar{x}}),$$

which implies that $(\bar{x},\dot{\bar{x}})$ is a minimum point of the problem

$$\min_{(x,\dot{x}) \in K} \big(F + \bar{\lambda}^T G\big)(x,\dot{x}).$$

Hence, by Lemma 2.2 and Lemma 3.5, we see that

$$(0,0) \in \partial_L (F + \bar{\lambda}^T G)(\bar{x},\dot{\bar{x}}) \subset \partial_L F(\bar{x},\dot{\bar{x}}) + \partial_L (\bar{\lambda}^T G)(\bar{x},\dot{\bar{x}}).$$

Combining Lemma 2.3 and Lemma 2.1, we deduce that

$$(0,0) \in \partial f(\bar{x},\dot{\bar{x}}) + \bar{\lambda}^T \partial g(\bar{x},\dot{\bar{x}}). \quad (5)$$

From (5), together with (4) and $\bar{\lambda} \geq 0$, we see that $(\bar{x},\dot{\bar{x}})$ is a KKT point of (NCVP). ■

In the same way, we can prove that a minimum point is necessarily a Fritz-John point of (NCVP) under nonsmooth invex assumption.

*Theorem 4.2:* Let $F$ and $G_i\,(i = 1, 2, \ldots, m)$ be nonsmooth invex at $(\bar{x},\dot{\bar{x}}) \in K$ with respect to same $\eta : X \times X \to X$. If (NCVP) attains a minimum at $(\bar{x},\dot{\bar{x}})$, then $(\bar{x},\dot{\bar{x}})$ is a Fritz-John point of (NCVP).

Now we give sufficient optimality conditions in the form of the following theorem:

*Theorem 4.3:* Let $F$ and $G_i(i = 1, 2, \ldots, m)$ be nonsmooth invex at $(\bar{x},\dot{\bar{x}}) \in K$ with respect to same $\eta : X \times X \to X$ and suppose that $(\bar{x},\dot{\bar{x}}) \in K$ is a KKT point. Then $(\bar{x},\dot{\bar{x}})$ is a minimum point of (NCVP).

*Proof:* Since $(\bar{x},\dot{\bar{x}})$ is a KKT point, there exists $\bar{\lambda} \geq 0$ such that

$$(0,0) \in \partial f(\bar{x},\dot{\bar{x}}) + \partial\big(\bar{\lambda}^T g\big)(\bar{x},\dot{\bar{x}}) \ a.e.,$$

$$\bar{\lambda}_i G_i(\bar{x},\dot{\bar{x}}) = 0 \,(i = 1, \ldots, m) \ a.e..$$

So, there exist $\xi^* \in \partial f(\bar{x},\dot{\bar{x}})$, $\zeta^* \in \partial\big(\bar{\lambda}^T g\big)(\bar{x},\dot{\bar{x}})$ such that

$$\xi^* + \zeta^* = 0. \quad (6)$$

Let if possible $(\bar{x},\dot{\bar{x}})$ be not a minimum of (NCVP). Then, there exists $(x,\dot{x}) \in K$, such that

$$F(x,\dot{x}) - F(\bar{x},\dot{\bar{x}}) < 0. \quad (7)$$

Since $F$ and $G_i\,(i = 1, 2, \ldots, m)$ are nonsmooth invex at $(\bar{x},\dot{\bar{x}}) \in K$ with respect to same $\eta \in X$, we have

$$F(x,\dot{x}) - F(\bar{x},\dot{\bar{x}}) \geq \int_a^b \langle A, \eta \rangle \,\mathrm{d}t, \quad \forall\ A \in \partial f(\bar{x},\dot{\bar{x}}), \quad (8)$$

and

$$G(x,\dot{x}) - G(\bar{x},\dot{\bar{x}}) \geq \int_a^b \langle B, \eta \rangle \, \mathrm{d}t, \quad \forall\, B \in \partial g(\bar{x},\dot{\bar{x}}). \tag{9}$$

By (7) and (8), we get

$$\int_a^b \langle A, \eta \rangle \, \mathrm{d}t < 0, \quad \forall\, A \in \partial f(\bar{x},\dot{\bar{x}}).$$

In particular,

$$\int_a^b \langle \xi^*, \eta \rangle \, \mathrm{d}t < 0, \quad \xi^* \in \partial f(\bar{x},\dot{\bar{x}}).$$

Now using (6), we see that

$$\int_a^b \langle -\zeta^*, \eta \rangle \, \mathrm{d}t < 0.$$

As $\bar{\lambda} \geq 0$ and $\zeta^* \in \partial\big(\bar{\lambda}^T g\big)(\bar{x},\dot{\bar{x}})$, we have

$$\zeta^* = \bar{\lambda}^T B^*, \quad \text{for some } B^* \in \partial g(\bar{x},\dot{\bar{x}}).$$

Therefore,

$$\int_a^b \langle -B^* \bar{\lambda}, \eta \rangle \, \mathrm{d}t < 0, \quad B^* \in \partial g(\bar{x},\dot{\bar{x}}). \tag{10}$$

Since $\bar{\lambda} \geq 0$, it follows from (9)

$$\bar{\lambda}^T \big( G(x,\dot{x}) - G(\bar{x},\dot{\bar{x}}) - \int_a^b \langle B, \eta \rangle \, \mathrm{d}t \big) \geq 0, \quad \forall\, B \in \partial g(\bar{x},\dot{\bar{x}}),$$

which, on using $\bar{\lambda}_i G_i(\bar{x},\dot{\bar{x}}) = 0\,(i = 1,\ldots,m)$ and $\bar{\lambda}^T G(x,\dot{x}) \leq 0$, implies that

$$-\int_a^b \langle \bar{\lambda}^T B, \eta \rangle \, \mathrm{d}t \geq 0, \quad \forall\, B \in \partial g(\bar{x},\dot{\bar{x}}).$$

This is a contradiction to (10). Hence, $(\bar{x},\dot{\bar{x}})$ is a minimum of (NCVP). ■

We have proved that under nonsmooth invex assumption the Kuhn-Tucker condition is a necessary and sufficient one for a feasible point to be a minimum. Next, we shall prove the sufficiency of the Kuhn-Tucker optimality conditions under nonsmooth pseudo-invex and nonsmooth quasi-invex assumptions of $F$ and $G$, respectively.

For notational convenience, we denote $M = \{i : i = 1,\ldots,m\}$ and $I = \{i \in M : \lambda_i \neq 0\}$.

*Theorem 4.4:* Let $F$ be nonsmooth pseudo-invex and $G_i(i = 1,2,\ldots,m)$ be nonsmooth quasi-invex at $(\bar{x},\dot{\bar{x}}) \in K$ with respect to same $\eta : X \times X \to X$ and suppose that $(\bar{x},\dot{\bar{x}})$ is a KKT point of (NCVP). Then $(\bar{x},\dot{\bar{x}})$ is a minimum of (NCVP).

*Proof:* Since $(\bar{x},\dot{\bar{x}})$ is a KKT point, there exists $\bar{\lambda} \geq 0$ such that

$$(0,0) \in \partial f(x,\dot{x}) + \partial(\bar{\lambda}^T g)(\bar{x},\dot{\bar{x}}) \ a.e.,$$

$$\bar{\lambda}_i G_i(\bar{x},\dot{\bar{x}}) = 0\,(i = 1,\ldots,m) \ a.e..$$

Therefore, there exists $\xi^* \in \partial f(x,\dot{x})$, $\zeta^* \in \partial(\bar{\lambda}^T g)(\bar{x},\dot{\bar{x}})$ such that

$$\xi^* + \zeta^* = 0. \tag{11}$$

Let if possible $(\bar{x},\dot{\bar{x}})$ be not a minimum of (NCVP). Then, there exists $(x,\dot{x}) \in K$, such that

$$F(x,\dot{x}) - F(\bar{x},\dot{\bar{x}}) < 0.$$

Since $F$ is nonsmooth pseudo-invex at $(\bar{x},\dot{\bar{x}}) \in K$, we get

$$\int_a^b \langle A, \eta \rangle \, \mathrm{d}t < 0, \quad \text{for all } A \in \partial f(\bar{x},\dot{\bar{x}}), \tag{12}$$

Since $g(x,\dot{x}) \leq 0$, we have

$$\bar{\lambda}_i G_i(\bar{x},\dot{\bar{x}}) = 0\,(i = 1,\ldots,m).$$

Since $\bar{\lambda}_i G_i(\bar{x},\dot{\bar{x}}) = 0\,(i = 1,\ldots,m)$, it follows that

$$\bar{\lambda}_i \big( G_i(x,\dot{x}) - G_i(\bar{x},\dot{\bar{x}}) \big) \leq 0\,(i = 1,\ldots,m). \tag{13}$$

Now we claim that

$$\bar{\lambda}_i \int_a^b \langle B_i, \eta \rangle \, \mathrm{d}t \leq 0, \quad \forall B_i \in \partial g_i(\bar{x},\dot{\bar{x}}),\ i = 1,\ldots,m. \tag{14}$$

If $\bar{\lambda} = 0$, then the above inequality holds trivially. If $\bar{\lambda} \neq 0$, then from (13), we obtain

$$G_i(x,\dot{x}) - G_i(\bar{x},\dot{\bar{x}}) \leq 0\,(i \in I).$$

Since $G_i(i = 1,2,\ldots,m)$ is nonsmooth quasi-invex at $(\bar{x},\dot{\bar{x}})$ we have

$$\int_a^b \langle B_i, \eta \rangle \, \mathrm{d}t \leq 0, \quad \forall B_i \in \partial g_i(\bar{x},\dot{\bar{x}}),\ i \in I.$$

which implies (14) holds. In particular,

$$\int_a^b \langle \zeta_i^*, \eta \rangle \, \mathrm{d}t \leq 0,$$

where $\bar{\lambda}_i \geq 0$, $\zeta_i^* \in \partial(\bar{\lambda}_i g_i)(\bar{x},\dot{\bar{x}}),\ i \in I$.

Combining with (11), we get

$$\int_a^b \langle \xi^*, \eta \rangle \, \mathrm{d}t \geq 0, \quad \text{for some } \xi^* \in \partial f(\bar{x},\dot{\bar{x}}),$$

which is a contradiction to (12). Hence, $(\bar{x},\dot{\bar{x}})$ is a minimum of (NCVP). ■

We have proved that nonsmooth pseudo-invex together with nonsmooth quasi-invex is a sufficient condition, and now, we shall prove it is a necessary condition.

*Theorem 4.5:* If all Kuhn-Tucker critical points are minimums for (NCVP), then $F$ is nonsmooth pseudo-invex and $G_i(i = 1,2,\ldots,m)$ is nonsmooth quasi-invex.

*Proof:* Let $(x,\dot{x}), (\bar{x},\dot{\bar{x}}) \in K$, $(\bar{x},\dot{\bar{x}},\bar{\lambda})$ verifies (2.2) (2.3), with $\tau = 1$, such that

$$\begin{cases} F(x,\dot{x}) - F(\bar{x},\dot{\bar{x}}) < 0, \\ G(x,\dot{x}) - G(\bar{x},\dot{\bar{x}}) \leq 0. \end{cases}$$

We have to find $\eta((x,\dot{x}),(\bar{x},\dot{\bar{x}})) \in X$, such that

$$\int_a^b \langle A, \eta \rangle \, \mathrm{d}t < 0, \quad \int_a^b \langle \lambda^T B, \eta \rangle \, \mathrm{d}t \leq 0, \tag{15}$$

where $A \in \partial f(\bar{x},\dot{\bar{x}}),\ B \in \partial g(\bar{x},\dot{\bar{x}})$. On the contrary, suppose (15) has no solution for all $A \in \partial f(\bar{x},\dot{\bar{x}}),\ B \in \partial g(\bar{x},\dot{\bar{x}})$, then from the alternative theorem [15], there exist $\omega_1, \omega_2 \in X$, $\omega_1 \geq 0$, $\omega_2 \geq 0$, with $(\omega_1, \omega_2) \neq 0$ such that

$$\int_a^b \langle A, \omega_1\rangle \,\mathrm{d}t + \int_a^b \langle \lambda^T B, \omega_2)\rangle \,\mathrm{d}t = 0, \qquad (16)$$

then $(0,0)$ is necessarily a solution to $(16)$ with respect to $\omega_1$, $\omega_2$. Thus,

$$(0,0) \in \partial f(\bar{x}, \dot{\bar{x}}) + \partial(\bar{\lambda}^T g)(\bar{x}, \dot{\bar{x}}).$$

That is to say $(\bar{x}, \dot{\bar{x}})$ is a KKT point, then, by the assumption, $(\bar{x}, \dot{\bar{x}})$ is a minimum point, which stands in contradiction to $F(x,\dot{x}) - F(\bar{x}, \dot{\bar{x}}) < 0$. So, there exists $\eta((x,\dot{x}), (\bar{x}, \dot{\bar{x}})) \in X$ such that

$$\begin{cases} \int_a^b \langle A, \eta\rangle \,\mathrm{d}t < 0, \\ \int_a^b \langle \lambda^T B, \eta\rangle \,\mathrm{d}t \leq 0, \end{cases}$$

for all $A \in \partial f(\bar{x}, \dot{\bar{x}}),\ B \in \partial g(\bar{x}, \dot{\bar{x}})$, and then, $F$ is nonsmooth pseudo-invex and $G$ is nonsmooth quasi-invex at $(\bar{x}, \dot{\bar{x}})$ with respect to same $\eta$. ■

Therefore, in Theorem 4.4, 4.5, we have proved that nonsmooth pseudo-invex and nonsmooth quasi-invex of $F$ and $G$ respectively are both sufficient and necessary in order that a Kuhn-Tucker critical point is a minimum of (NCVP).

## V. Duality

We now establish duality between (NCVP) and the next dual problem (NCVD1), which is a modified Mond-Weir dual problem formulated by Bector, Chandra and Husain, see [10] for more details.

$$\text{(NCVD1)} \qquad \text{Maximize} \quad \int_a^b f(u,\dot{u}) \,\mathrm{d}t$$
$$\begin{aligned} \text{subject to} \quad & (0,0) \in \partial f\big(u,\dot{u}\big) + \partial(\lambda^T g)(u,\dot{u}) \ a.e., \\ & \bar{\lambda}_i G_i(\bar{x}, \dot{\bar{x}}) = 0 \, (i = 1, \ldots, m) \ a.e., \\ & \lambda \geq 0, \\ & (u,\dot{u}) \in K \ . \end{aligned}$$

Let $H$ be the feasible set of (NCVP).

*Theorem 5.1:* (Weak duality) Let $(x,\dot{x})$ be feasible for (NCVP) and $(u,\dot{u},\lambda)$ be feasible for (NCVD1). If $F$ and $G$ are nonsmooth invex at $(u,\dot{u})$ with respect to same $\eta : X \times X \to X$ and $\lambda \geq 0$, then

$$\int_a^b f(x,\dot{x}) \,\mathrm{d}t \geq \int_a^b f(u,\dot{u}) \,\mathrm{d}t.$$

*Proof:* Since $(u,\dot{u},\lambda)$ is feasible for (NCVD1), we get that there exist $\xi^* \in \partial f(u,\dot{u})$ and $\zeta^* \in \partial(\lambda^T g)(u,\dot{u})$ such that

$$\xi^* + \zeta^* = 0. \qquad (1)$$

Let if possible

$$\int_a^b f(x,\dot{x}) \,\mathrm{d}t < \int_a^b f(u,\dot{u}) \,\mathrm{d}t. \qquad (2)$$

Since $F$ and $G$ are nonsmooth invex at $(u,\dot{u})$ with respect to same $\eta : X \times X \to X$, we have

$$F(x,\dot{x}) - F(u,\dot{u}) \geq \int_a^b \langle A, \eta\rangle \,\mathrm{d}t, \quad \forall\ A \in \partial f(u,\dot{u}), \qquad (3)$$

and

$$G(x,\dot{x}) - G(u,\dot{u}) \geq \int_a^b \langle B, \eta\rangle \,\mathrm{d}t, \quad \forall\ B \in \partial g(u,\dot{u}). \qquad (4)$$

Combining $(2)$ and $(3)$, we get

$$\int_a^b \langle A, \eta\rangle \,\mathrm{d}t < 0, \quad \text{for all } A \in \partial f(u,\dot{u}).$$

In particular,

$$\int_a^b \langle \xi^*, \eta\rangle \,\mathrm{d}t < 0, \quad \text{for some } \xi^* \in \partial f(u,\dot{u}).$$

By using $(1)$, we get

$$\int_a^b \langle -\zeta^*, \eta\rangle \,\mathrm{d}t < 0.$$

Since $\zeta^* \in \partial\big(\lambda^T g\big)(u,\dot{u})$ and $\lambda \geq 0$, we see that

$$\zeta^* = \lambda^T B^*, \quad \text{for some } B^* \in \partial g(u,\dot{u}).$$

Thus,

$$\int_a^b \langle -\lambda^T B^*, \eta\rangle \,\mathrm{d}t < 0, \quad \text{for some } B^* \in \partial g(u,\dot{u}). \qquad (5)$$

Now as $\lambda \geq 0$, from $(4)$, we have

$$\lambda^T \big(G(x,\dot{x}) - G(u,\dot{u}) - \int_a^b \langle B, \eta\rangle \,\mathrm{d}t\big) \geq 0. \qquad (6)$$

Since $(x,\dot{x}) \in K$ and $\big(u,\dot{u},\lambda\big) \in H$,

$$\lambda^T G(x,\dot{x}) \leq 0 = \lambda^T G(u,\dot{u}),$$

therefore, $(6)$ gives that

$$-\int_a^b \langle B, \eta\rangle \,\mathrm{d}t \geq 0, \ \ \forall\ B \in \partial g(u,\dot{u}),$$

which is a contradiction to $(5)$. Hence,

$$\int_a^b f(x,\dot{x}) \,\mathrm{d}t \geq \int_a^b f(u,\dot{u}) \,\mathrm{d}t.$$

■

*Theorem 5.2:* (Weak duality) Let $(x,\dot{x})$ be feasible for (NCVP) and $(u,\dot{u},\lambda)$ be feasible for (NCVD1). If $F$ is nonsmooth pesudo-invex and $G$ is nonsmooth quasi-invex at $(u,\dot{u})$ with respect to same $\eta : X \times X \to X$ and $\lambda \geq 0$, then

$$\int_a^b f(x,\dot{x}) \,\mathrm{d}t \geq \int_a^b f(u,\dot{u}) \,\mathrm{d}t.$$

*Proof:* Since $(u,\dot{u},\lambda)$ is feasible for (NCVD1), there exist $\xi^* \in \partial f(u,\dot{u})$ and $\zeta^* \in \partial(\lambda^T g)(u,\dot{u})$ such that

$$\xi^* + \zeta^* = (0,0). \qquad (7)$$

Let if possible $\int_a^b f(x,\dot{x})\,\mathrm{d}t < \int_a^b f(u,\dot{u})\,\mathrm{d}t$. Since $F$ is nonsmooth pesudo-invex at $(u,\dot{u})$, we get

$$\int_a^b \langle A,\eta\rangle\,\mathrm{d}t < 0,\ \ \forall\ A \in \partial f(u,\dot{u}).$$

In particular ,

$$\int_a^b \langle \xi^*,\eta\rangle\,\mathrm{d}t < 0, \quad \text{where } \xi^* \in \partial f(u,\dot{u}).$$

By using (7), we get

$$\int_a^b \langle -\zeta^*,\eta\rangle\,\mathrm{d}t < 0.$$

Since $\zeta^* \in \partial(\lambda^T g)(u,\dot{u})$, $\lambda \geq 0$, we see that

$$\zeta^* = \lambda^T B^*, \quad \text{for some } B^* \in \partial g(u,\dot{u}).$$

Thus,

$$\int_a^b \langle -\lambda^T B^*,\eta\rangle\,\mathrm{d}t < 0, \quad \text{where } B^* \in \partial g(u,\dot{u}). \tag{8}$$

Also, $(x,\dot{x})$ is feasible for (NCVP) and $(u,\dot{u},\lambda)$ is feasible for (NCVD1), therefore,

$$\lambda_i G_i(x,\dot{x}) \leq 0 = \lambda_i G_i(u,\dot{u})\,(i=1,\ldots,m). \tag{9}$$

Now we claim that

$$\lambda^T \int_a^b \langle B,\eta\rangle\,\mathrm{d}t \leq 0, \tag{10}$$

If $\lambda = 0$, then above inequality holds trivially. Suppose $\lambda \neq 0$, then from (9) we get

$$G_i(x,\dot{x}) - G_i(u,\dot{u}) \leq 0\,(i \in I).$$

Since $G_i$ is nonsmooth quasi-invex at $(u,\dot{u})$, we get

$$\int_a^b \langle B_i,\eta\rangle\,\mathrm{d}t \leq 0\,(i \in I),$$

which implies (10) holds. This is a contradiction to (8). Hence,

$$\int_a^b f(x,\dot{x})\,\mathrm{d}t \geq \int_a^b f(u,\dot{u})\,\mathrm{d}t.$$

The proof is completed. ■

*Theorem 5.3:* (Strong duality) Let $(\bar{x},\dot{\bar{x}})$ be a minimum point of (NCVP) and the Slater constraint qualification is satisfied. If $F$ and $G$ are nonsmooth invex at $(\bar{x},\dot{\bar{x}})$ with respect to same $\eta : X \times X \to X$, then there exist $\bar{\lambda} \geq 0$ such that $(\bar{x},\dot{\bar{x}},\bar{\lambda})$ is a feasible point of (NCVD1). Furthermore, if the conditions of Weak duality Theorem 5.1 hold for all feasible $(x,\dot{x})$ for (NCVP) and feasible $(u,\dot{u},\lambda)$ for (NCVD1), then $(\bar{x},\dot{\bar{x}},\bar{\lambda})$ is a maximum of (NCVD1) and the value of the objective functions are equal.

*Proof:* Since $(\bar{x},\dot{\bar{x}})$ is a minimum point of (NCVP) and the Slater qualification is satisfied, by Theorem 4.1, we know that there exists $\bar{\lambda} \geq 0$ with $\tau = 1$ such that

$$(0,0) \in \partial f(\bar{x},\dot{\bar{x}}) + \partial(\bar{\lambda}^T g)(\bar{x},\dot{\bar{x}})\,a.e,$$

and

$$\bar{\lambda}_i G_i(\bar{x},\dot{\bar{x}}) = 0,$$

which implies that $(\bar{x},\dot{\bar{x}},\bar{\lambda})$ is feasible for (NCVD1). Let if possible $(\bar{x},\dot{\bar{x}},\bar{\lambda})$ be not a maximum of (NCVD1), then there exists $(u,\dot{u},\lambda)$ feasible for (NCVD1) such that

$$\int_a^b f(u,\dot{u})\,\mathrm{d}t > \int_a^b f(\bar{x},\dot{\bar{x}})\,\mathrm{d}t,$$

which is a contradiction to Weak duality Theorem 5.1. Hence, $\left(\bar{x},\dot{\bar{x}},\bar{\lambda}\right)$ is a maximum of (NCVD1). The values of the objective functions are trivially equal. ■

*Theorem 5.4:* (Converse duality) Let $\left(\bar{u},\dot{\bar{u}},\bar{\lambda}\right)$ be a maximum of (NCVD1), and $(x,\dot{x}) \in K$. If $F$ is nonsmooth pseudo-invex and $G$ is nonsmooth quasi-invex at $(\bar{u},\dot{\bar{u}})$ with respect to same $\eta : X \times X \to X$, then $(\bar{u},\dot{\bar{u}})$ is a minimum point of (NCVP) and the value of the objective functions are equal.

*Proof:* Since $F$ is nonsmooth pseudo-invex and $G$ is nonsmooth quasi-invex at $(\bar{u},\dot{\bar{u}})$ with respect to same $\eta$, and it follows from Theorem 5.2 that

$$\int_a^b f(\bar{u},\dot{\bar{u}})\,\mathrm{d}t \leq \int_a^b f(x,\dot{x})\,\mathrm{d}t, \quad \text{for all } (x,\dot{x}) \in K.$$

And since $(\bar{u},\dot{\bar{u}}) \in K$, we have $(\bar{u},\dot{\bar{u}})$ is a minimum of (NCVP), and their objective function values are trivially equal. ■

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## References

[1] M. Hanson, *On sufficiency of the Kuhn-Tucker conditions*, J. Math. Anal. Appl., 80 (1981) 545-550.
[2] B. Craven, *Nondifferential optimization by nonsmooth approximations*, Optimization, 17 (1986) 3-17.
[3] T. Reiland, *Nonsmooth invexity*, Bull. Austral. Math. Soc., 42 (1990) 437-446.
[4] D. Martin, *The essence of invexity*, J. Optim. Th. Appl., 47 (1986) 65-76.
[5] B. Mond, S. Chandra, I. Husain, *Duality for variational problems with invexity*, J. Math. Anal. Appl., 134 (1988) 322-328.
[6] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley, New York, 1983.
[7] F. H. Clarke, *Nonsmooth Analysis and Control Theory*, Berlin: Springer-Verlag, 1998.
[8] M. Arana-Jiménez, R. Osuna-Gómez, G. Ruiz-Garzón, M. Rojas-Medar, *On variational problems: Characterization of solutions and duality*, J. Math. Anal. Appl., 311 (2005) 1-12.
[9] S. K. Suneja, S. Khurana, Vani, *Generalized nonsmooth invexity over cones in vector optimization*, European Journal of Operational Research, 186 (2008) 28-40.
[10] C. Bector, S. Chandra, I. Husain, *Generalized concavity and duality in continuous programming*, Util. Math., 25 (1984) 171-190.
[11] N. Yen, P. Sach, *On locally Lipschitz vector valued invex functions*, Bull. Austral. Math. Soc., 47 (1993) 259-271.
[12] M. Hanson, B. Mond, *Necessary and sufficient conditions in constrainted optimization*, Math. Programming, 37 (1987) 51-58.
[13] M. Hanson, *Invexity and the Kuhn-Tucker theorem*, J. Math. Anal. Appl., 236 (1999) 594-604.
[14] B. Craven, X. Yang, *A nonsmooth version of alternative theorem and nonsmooth multiobjective programming*, Utilitas Mathematica, 40 (1991) 117-128.
[15] M. Bazaraa, C. Shetty, *Nonlinear Programming: Theory and Algorithms*, Wiley, New York, 1979.