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# Certain Conditions for Strongly Starlike and Strongly Convex Functions

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*Abstract*—In the present paper, we investigate a differential subordination involving multiplier transformation related to a sector in the open unit disk $\mathbb{E} = \{z : |z| < 1\}$. As special cases to our main result, certain sufficient conditions for strongly starlike and strongly convex functions are obtained.

*Keywords*—Analytic function, Multiplier transformation, Strongly starlike function, Strongly convex function.

## I. Introduction

Let $\mathcal{H}$ be the class of functions analytic in the open unit disk $\mathbb{E} = \{z : |z| < 1\}$ and for $a \in \mathbb{C}$ (set of complex numbers) and $n \in \mathbb{N} = \{1, 2, \cdots\}$, let $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of functions $f$ of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots .$$

Let $\mathcal{A}$ be the class of functions $f$, analytic in $\mathbb{E}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$.

A function $f \in \mathcal{A}$ is said to be strongly starlike of order $\alpha$, $0 < \alpha \leq 1$, if

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\alpha\pi}{2},$$

equivalently

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha}.$$

A function $f \in \mathcal{A}$ is said to be strongly convex of order $\alpha$, $0 < \alpha \leq 1$, if

$$\left|\arg \left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\alpha\pi}{2},$$

equivalently

$$1 + \frac{zf''(z)}{f'(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha}.$$

For two analytic functions $f$ and $g$ in the open unit disk $\mathbb{E}$, we say that $f$ is subordinate to $g$ in $\mathbb{E}$ and write as $f \prec g$ if there exists a Schwarz function $w$ analytic in $\mathbb{E}$ with $w(0) = 0$ and $|w(z)| < 1,\ z \in \mathbb{E}$ such that $f(z) = g(w(z))$.

In case the function $g$ is univalent, the above subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{E}) \subset g(\mathbb{E})$.

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Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k,\ p \in \mathbb{N},$$

which are analytic in the open unit disk $\mathbb{E} = \{z : |z| < 1\}$. We note that $\mathcal{A}_1 = \mathcal{A}$.

For $f \in \mathcal{A}_p$, we define the multiplier transformation $I_p(n, \lambda)$ as

$$I_p(n, \lambda)[f](z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{k+\lambda}{p+\lambda}\right)^n a_k z^k,\ (\lambda \geq 0, n \in \mathbb{Z}).$$

The operator $I_p(n, \lambda)$ has been recently studied by Aghalary et al. [1]. Earlier, the operator $I_1(n, \lambda)$ was investigated by Cho and Kim [2] and Cho and Srivastava [3], whereas the operator $I_1(n, 1)$ was studied by Uralegaddi and Somanatha [9]. $I_1(n, 0)$ is the well-known Sălăgean [8] derivative operator $D^n$, defined as:

$$D^n[f](z) = z + \sum_{k=2}^{\infty} k^n a_k z^k,\ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$$

where $f \in \mathcal{A}$.

In 1989, the operator $I_1(n, 0)$ has been studied by Owa, Shen and Obradovič [7]. Recently, Li and Owa [4] studied the operator $I_1(n, 0)$. Many significant results regarding the operator $I_p(n, \lambda)$ have been obtained by different authors.

In the present paper, we study a differential subordination involving multiplier transformation in a sector. As special cases to our main result, we derive some simple sufficient conditions for members of the class $\mathcal{A}$ to be strongly starlike and strongly convex functions.

## II. Preliminaries

We shall need the following lemma to prove the main result.

*Lemma 2.1:* ([5]). Let $\mu > 0$ be a real number and let $\beta_0 = \beta_0(\mu, n), n \in \mathbb{N}$ be the root of the equation $\beta\pi = \frac{3\pi}{2} - \arctan(n\mu\beta)$.

Let

$$\alpha = \alpha(\beta, \mu, n) = \beta + \frac{2}{\pi} \arctan(n\mu\beta),\ 0 < \beta \leq \beta_0. \quad (1)$$

If $P \in \mathcal{H}[1, n]$, then $P(z) + \mu z P'(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha}$ implies $P(z) \prec \left(\frac{1+z}{1-z}\right)^{\beta}$.

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## III. Main Result

*Theorem 3.1:* If $f \in \mathcal{A}_p$ satisfies

$$(1-\gamma)\frac{I_p(n,\lambda)[f](z)}{z^p} + \gamma\ \frac{I_p(n+1,\lambda)[f](z)}{z^p} \prec \left(\frac{1+z}{1-z}\right)^{\alpha}, \tag{2}$$

then

$$\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)} \prec \left(\frac{1+z}{1-z}\right)^{\delta},$$

where $\alpha = \alpha\left(\frac{\gamma}{p+\lambda}, \delta\right)$ satisfies the equation

$$2\arctan\left[\frac{\gamma}{p+\lambda}(\delta-\alpha)\right] + \pi(\delta - 2\alpha) = 0, \tag{3}$$

and $\gamma$ and $\delta$ are real numbers such that $\gamma \geq 1,\ 0 < \delta \leq 1$.

*Proof:* Let us define

$$\frac{I_p(n,\lambda)[f](z)}{z^p} = u(z). \tag{4}$$

Differentiate (4) logarithmically, we obtain

$$\frac{zI_p'(n,\lambda)[f](z)}{I_p(n,\lambda)[f](z)} - p = \frac{zu'(z)}{u(z)}. \tag{5}$$

In view of the equality

$$zI_p'(n,\lambda)[f](z) = (p+\lambda)I_p(n+1,\lambda)[f](z) - \lambda I_p(n,\lambda)[f](z),$$

(5) reduces to

$$\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)} = 1 + \frac{zu'(z)}{(p+\lambda)u(z)}.$$

A little calculation yields

$$u(z) + \frac{\gamma}{p+\lambda}zu'(z) = (1-\gamma)\frac{I_p(n,\lambda)[f](z)}{z^p} + \gamma\ \frac{I_p(n+1,\lambda)[f](z)}{z^p}.$$

Therefore, in view of (2), we have

$$u(z) + \frac{\gamma}{p+\lambda}zu'(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha}. \tag{6}$$

We note that for $\alpha + \beta = \delta$ and $\mu = \frac{\gamma}{p+\lambda}$, the condition (3) corresponds to the condition (1) of Lemma 2.1. Therefore, in view of Lemma 2.1, we have

$$u(z) \prec \left(\frac{1+z}{1-z}\right)^{\beta} \tag{7}$$

where $\beta$ satisfies the condition (1) of Lemma 2.1.

Let us, now, write $u(z) + \frac{\gamma}{p+\lambda}zu'(z) = v(z)$ and therefore, we have

$$\frac{I_p(n+1,\lambda)[f](z)}{z^p} = \left(1 - \frac{1}{\gamma}\right)u(z) + \frac{1}{\gamma}\ v(z).$$

Obviously, $\frac{I_p(n+1,\lambda)[f](z)}{z^p}$ is a convex combination of $u(z)$ and $v(z)$.

In view of condition (1) of Lemma 2.1, we conclude that $\alpha > \beta$, thus, from (6) and (7), we have

$$\frac{I_p(n+1,\lambda)[f](z)}{z^p} \prec \left(\frac{1+z}{1-z}\right)^{\alpha}. \tag{8}$$

Write $w(z) = \frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)}$, obviously $w \in \mathcal{H}[1,1]$ and we can rewrite $w$ as

$$w(z) = \frac{I_p(n+1,\lambda)[f](z)/z^p}{u(z)}.$$

From (7) and (8),we obtain

$$|\arg w(z)| \leq \left|\arg \frac{I_p(n+1,\lambda)[f](z)}{z^p}\right| + |\arg u(z)|$$
$$< \alpha\frac{\pi}{2} + \beta\frac{\pi}{2} = (\alpha+\beta)\frac{\pi}{2} = \delta\frac{\pi}{2}.$$

Hence, we have

$$\frac{I_p(n+1,\lambda)[f](z)}{I_p(n,\lambda)[f](z)} \prec \left(\frac{1+z}{1-z}\right)^{\delta},\ z \in \mathbb{E}.$$

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## IV. Applications to Univalent Functions

In this section, using Theorem 3.1, we derive certain sufficient conditions for strongly starlike and strongly convex functions.

On writing $p = 1$ and $\lambda = 0$ in Theorem 3.1. We have the following result.

*Corollary 4.1:* If $f \in \mathcal{A}$ satisfies

$$(1-\gamma)\frac{D^n[f](z)}{z} + \gamma\ \frac{D^{n+1}[f](z)}{z} \prec \left(\frac{1+z}{1-z}\right)^{\alpha},$$

then

$$\frac{D^{n+1}[f](z)}{D^n[f](z)} \prec \left(\frac{1+z}{1-z}\right)^{\delta},$$

where $\alpha = \alpha(\gamma, \delta)$ satisfies the equation

$$2\arctan[\gamma(\delta-\alpha)] + \pi(\delta - 2\alpha) = 0,$$

and $\gamma$ and $\delta$ are real numbers with $\gamma \geq 1,\ 0 < \delta \leq 1$.

When we select $p = 1, n = 0$ and $\lambda = 0$ in Theorem 3.1. We obtain the following result of Oros [6].

*Corollary 4.2:* If $f \in \mathcal{A}$ satisfies

$$(1-\gamma)\frac{f(z)}{z} + \gamma\ f'(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha},$$

then

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\delta},$$

where $\alpha = \alpha(\gamma, \delta)$ satisfies the equation

$$2\arctan[\gamma(\delta-\alpha)] + \pi(\delta - 2\alpha) = 0,$$

and $\gamma$ and $\delta$ are real numbers with $\gamma \geq 1,\ 0 < \delta \leq 1$.

By taking $p = 1$, $n = 1$ and $\lambda = 0$ in Theorem 3.1. We obtain the following result.

*Corollary 4.3:* If $f \in \mathcal{A}$ satisfies

$$f'(z) + \gamma\ zf''(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha},$$

then

$$1 + \frac{zf''(z)}{f'(z)} \prec \left(\frac{1+z}{1-z}\right)^{\delta},$$

where $\alpha = \alpha(\gamma, \delta)$ satisfies the equation

$$2\arctan[\gamma(\delta - \alpha)] + \pi(\delta - 2\alpha) = 0,$$

and $\gamma$ and $\delta$ are real numbers with $\gamma \geq 1,\ 0 < \delta \leq 1$.

By setting $p = 1$, $n = 0$ and $\lambda = 1$ in Theorem 3.1. We have the following result.

*Corollary 4.4:* If $f \in \mathcal{A}$ satisfies

$$\left(1 - \frac{\gamma}{2}\right)\frac{f(z)}{z} + \frac{\gamma}{2}f'(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha},$$

then

$$\frac{1}{2}\left(1 + \frac{zf'(z)}{f(z)}\right) \prec \left(\frac{1+z}{1-z}\right)^{\delta},$$

where $\alpha = \alpha\left(\frac{\gamma}{2}, \delta\right)$ satisfies the equation

$$2\arctan\left[\frac{\gamma}{2}(\delta - \alpha)\right] + \pi(\delta - 2\alpha) = 0,$$

and $\gamma$ and $\delta$ are real numbers such that $\gamma \geq 1,\ 0 < \delta \leq 1$.

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