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Cash Flow Optimization on Synthetic CDOs

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***Abstract*—Collateralized Debt Obligations are not as widely used nowadays as they were before 2007 Subprime crisis. Nonetheless there remains an enthralling challenge to optimize cash flows associated with synthetic CDOs. A Gaussian-based model is used here in which default correlation and unconditional probabilities of default are highlighted. Then numerous simulations are performed based on this model for different scenarios in order to evaluate the associated cash flows given a specific number of defaults at different periods of time. Cash flows are not solely calculated on a single bought or sold tranche but rather on a combination of bought and sold tranches. With some assumptions, the simplex algorithm gives a way to find the maximum cash flow according to correlation of defaults and maturities. The used Gaussian model is not realistic in crisis situations. Besides present system does not handle buying or selling a portion of a tranche but only the whole tranche. However the work provides the investor with relevant elements on how to know what and when to buy and sell.

***Keywords*—Synthetic Collateralized Debt Obligation (CDO), Credit Default Swap (CDS), Cash Flow Optimization, Probability of Default, Default Correlation, Strategies, Simulation, Simplex.

I. Introduction

COLLATERALIZED Debt Obligations have been playing a substantial role since 2007 economic crisis. The whole banking world collapsed when the number of defaults, due to unpaid mortgages, impacted CDO tranches and caused protection sellers to pay their default legs. Thus, it is important to know how the number of defaults influences the resulting cash flow.

The main difficulties to be considered for doing so are 1) choosing the model to be retained for estimating probabilities of default, 2) finding correlations between them and 3) acquiring relevant data and physical hardware limitations. These concerns should however not be a problem to make a point as the main focus here concerns the methodology and the approach rather than the accuracy of presented data.

Amongst the various existing financial instruments, CDOs are exotic derivatives [7], which were created in order to perform securitization, i.e. to transform illiquid assets such as real estate (mortgages), significant blocks of stock etc. into easily tradable products. So, CDOs tend to gather illiquid assets and to normalize the way they are built and handled, which enables comparison between products of different kinds. Synthetic CDOs, the ones being dealt with, are described as CDOs made out of Credit Default Swaps (CDSs). Fig. 1 briefly describes the relation between protection buyer and protection seller when synthetic CDOs are used to manage defaults. Synthetic CDOs can be considered as a contract between stakeholders transferring part of the risks from one to the other as individuals would do when subscribing to an insurance contract.

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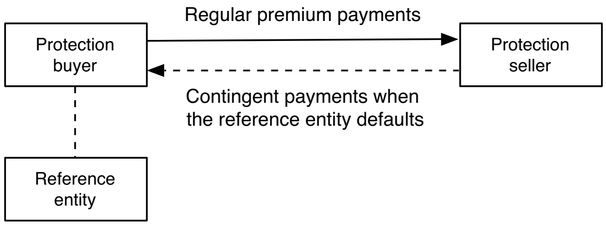

Fig. 1: Relation between Protection Buyer and Protection Seller.

If CDOs have the major advantage of being relatively liquid, they also have the disadvantage of hiding the underlying assets. Therefore, they are split into tranches rated into different categories according to the amount of risk they include. As an example, Fig. 2 displays the standardized iTraxx CDOs.

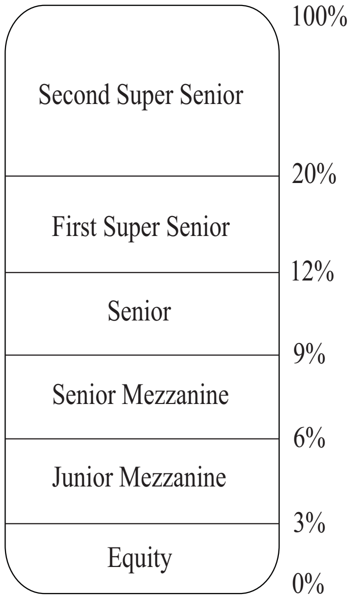

Fig. 2: CDO tranches with iTraxx Standardization. Top Tranches are safest while Bottom ones are riskiest.

Numerous models exist to represent default probabilities such as the Gaussian copula model [8], the bottom-up affine jump-diffusion model, a local intensity model [1] or the top-down bivariate spread-loss model [4], [6]. Because theoretical aspect of this model is complex, the aim of this paper is to pick one of them and consider it as if it were correct.

A. Pricing CDOs

A CDO pricer is implemented in [9] based on the general structure of credit risk developed in [12].

$$K_{n,t,T} = \frac{ln(D_n) - ln(A_{n,t}) - (\mu_n - \frac{\sigma_n^2}{2})(T-t)}{\sigma\sqrt{T-t}} \tag{1}$$

Where

- D_n the face value
- $A_{n,t}$ the sum of equity and debt values
- μ_n the total expected return
- σ_n the asset relative instantaneous volatility
- T the maturity
- $p_{n,t,T}$ is the probability of default at maturity T evaluated at time $t < T$
- $p_{n,t,T} = \Phi(K_{n,t,T})$
  with $\Phi(u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} exp[-\frac{1}{2}z^2]dz$

CDO quotes that had already been priced using this model have been taken for granted in present study [3], [17].

In order to describe the probabilities of default, the single factor asymptotic Vasicek model is applied, but this leads directly to formulate several hypotheses concerning an infinite amount of credits or a correlation between identical defaults. Thus, when applying the model to CDO pricing [9], the various hypotheses restrain the study to an artificial type of CDO but nevertheless useful for testing the validity of such financial product over time. The main used parameters are:

- The lower and higher strike price in relation with the chosen CDO tranche
- The payment frequency
- The default probabilities
- The default correlation
- The losses associated to each default

The type of considered CDO (synthetic or cash) alongside tranche choice of the same CDO can influence significantly the parameters as for example the default correlation and associated losses. With an extended version of the Vasicek model the default probability can be formulated as:

$$P(L_t = x \mid Y_t) = \binom{N}{x} p(Y_t)^x (1 - p(Y_t))^{N-x} \tag{2}$$

Where:

- L_t designates the number of defaults
- Y_t is the common factor
- N is the total number of CDSs

It is important to correctly calibrate parameters and to define the context in which they are used [9], [17].The pricer based on Vasicek's method enables to approach the relation existing between several parameters and the capacity to get rid of some of them.

Hedging strategies consist in fitting operator positions with regard to market evolution [11]. In other words, he is buying or selling parts of his portfolio, whose prices are regularly and frequently re-assessed in order to prevent losses. Even though it is not the focus of this paper, it is a step that cannot be neglected for the same reasons that one cannot avoid the implementation of a model: they are prerequisites to base work hypothesis upon [4].

The goal of present study is to analyze CDO cash flows. Their mechanism are taken from [4], [5], [6]. Corresponding formulae are the starting point of this paper in simplified form with changes in parameters.

General formulae to calculate cash flows are given by [4]

(3) represents the value of default leg protection received by the buyer at time t.

$$\sum_{t<T_m} B(t,T_m)(1-R)P(T_{m-1} < \tau_i < T_m \mid F_t) \tag{3}$$

Where:

- $B(t,T_m)$ represents the discount factor from time T_m to t
- P is a risk neutral probability
- Each T_m represents the different deadline payments
- R is the recovery rate
- F_t represents the flow of information

And (4) represents the value of premium leg received by the investor.

$$s_o^i \sum_{t<T_m} B(t,T_m)(T_m - T_{m-1})P(T_i > T_m \mid F_t) \tag{4}$$

Consider the tranche $Tr \equiv [K_d — K_u]$ of a CDO [16], whose underlying portfolio is supposed to be made of N CDSs, each with an identical nominal value $\mathcal{P}$. Thus the underlying portfolio is supposed homogeneous and the recovery rate associated to each entity is R.

It is assumed that the purchase and sale operations associated to hedging on tranche Tr are made for maturity t_m at rate π [14].

Let t_k be the payment moments with $k = 1,..,m$, and $\delta = t_k - t_{k-1}$ the time lapse between two payments. Let n_k be the number of CDSs, which default at time t_k [14].

For each moment t_k, the seller and the buyer must pay respectively the amounts $\mathcal{DL}_k$ et $\mathcal{PL}_k$ [14] such that:

$$\mathcal{DL}_k = \mathcal{L}_k - \mathcal{L}_{k-1} \tag{5}$$

$$\mathcal{PL}_k = \tfrac{1}{2} \times \pi \times (\mathcal{N}_k + \mathcal{N}_{k-1}) \times \delta \tag{6}$$

Where:

- $\mathcal{L}_k$ the associated loss to tranche Tr at time t_k
- $\mathcal{N}_k$ the reference nominal for the premium payment at time t_k

The legs in (5) and (6) depend on n_k. Thus it is essential to determine bounds for n_k with respect to those of Tr: K_d and K_u.

Introduce two threshold values:

$$x_d = N \times K_d \tag{7}$$

$$x_u = N \times K_u \tag{8}$$

and

$$\mathcal{K} = (1-R) \times \mathcal{P} \tag{9}$$

$\mathcal{L}_k$ is linked to $\mathcal{N}_k$ by the following relation:

$$\mathcal{N}_k = \mathcal{K} \times (x_u - x_d) - \mathcal{L}_k \tag{10}$$

Besides, note that $k \in [1, m]$, $n_k \in \mathbb{N}$, which is not necessarily the case of x_d and x_u. Therefore two integers n_d and n_u are built such that:

$$n_d = \lfloor x_d \rfloor \quad \lfloor x_d \rfloor \text{ is the integer part of } x_d \tag{11}$$

$$n_u = \begin{cases} \lfloor x_u \rfloor - 1 & \text{if } x_u \in \mathbb{N} \\ \lfloor x_u \rfloor & \text{otherwise} \end{cases} \tag{12}$$

Express $\mathcal{L}_k$ with respect to the number of defaults n_k:

$$\mathcal{L}_k = \begin{cases} 0 & \text{if } 0 \leq n_k \leq n_d \\ \mathcal{K} \times (n_k - x_d) & \text{if } 1 + n_d \leq n_k \leq n_u \\ \mathcal{K} \times (x_u - x_d) & \text{otherwise} \end{cases} \tag{13}$$

(14)

Finally, thanks to (10) the following expression of $\mathcal{N}_k$ with respect to n_k is deduced:

$$\mathcal{N}_k = \begin{cases} \mathcal{K} \times (x_u - x_d) & \text{if } 0 \leq n_k \leq n_d \\ \mathcal{K} \times (x_u - n_k) & \text{if } 1 + n_d \leq n_k \leq n_u \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

In this way the cash flow increase associated to the sale and purchase of hedging at each time t_k of payment is described with respect to the number of defaults n_k [16].

II. Methodology

In order to determine the default probability curve for later calculations, existing synthetic CDO market quotes are used to bootstrap the curve (see Fig. 3). The market quotes are expressed as a list of maturity dates with their corresponding CDS market spreads. The estimation uses the standard model of the survival probability [2], [10], [13].

Inputs for bootstrapping the curve are:

- Market dates and their associated market spreads
- Settlement dates that are earlier than or equal to the aforementioned market dates
- Zero dates and Zero Rates corresponding to zero coupon bonds

To calculate the probability of having N defaults at time t, (16) is used [15], [16]:

$$\mathbb{P}[N_t(\cdot) = j\,] = C_N^j \int_{-\infty}^{\infty} [f(\bar{p}_t, \rho_t; z)]^j \, [1 - f(\bar{p}_t, \rho_t; z)]^{N-j} \varphi(z)\, dz \tag{16}$$

Where:

- $N_t(\cdot)$ is a random variable, which represents the number of defaults at time t
- ρ_t is a real number such that $-1 \leq \rho < 1$ and corresponds to a correlation of default
- $\bar{p}_t$ is is a real number such that $-1 < \bar{p}_t < 1$ and represents the unconditional probability that a CDS defaults
- $\varphi(z) = \frac{1}{\sqrt{2\pi}} exp[-\frac{1}{2}z^2]$

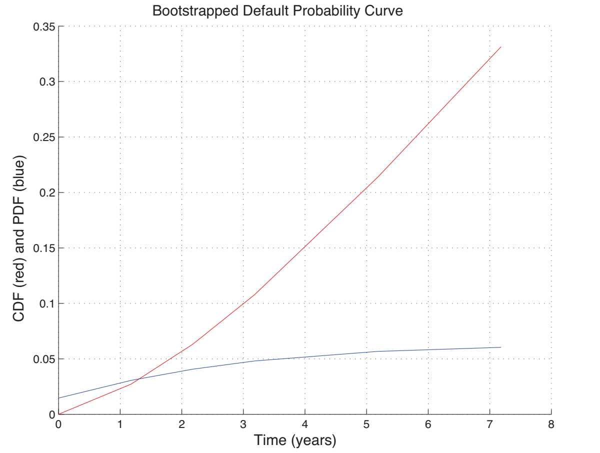

Fig. 3: Cumulative density function of the bootstrapped default probability curve.

- $f(\bar{p}_t, \rho_t; z) = \Phi(\frac{\Phi^{-1}(\bar{p}_t) - \sqrt{\rho_t} z}{\sqrt{1-\rho_t}})$
- $\Phi(u) = \int_{-\infty}^{u} \varphi(z) dz$
- $C_N^j = \frac{N!}{j!(N-j)!}$

Parameters ρ_t and $\bar{p}_t$ are defined as follows:

1) ρ_t is the same for each firm and each time horizon. Therefore $\rho_t = \rho$. Besides the fair assumption is made that its value varies between 0 and 30% based on historical data [16]. This parameter is set to different reasonable values due to lack of data.
2) $\bar{p}_t$ varies at each period of time and is determined with market data. Concretely its value is determined from the bootstraping operation described above.

Letting $\pi_j = \mathbb{P}[N_t(\cdot) = j\,]$, it becomes possible to draw the Probability and Cumulative Density Functions for a given set of ρ_t and $\bar{p}_t$ values as can be seen in Fig. 4.

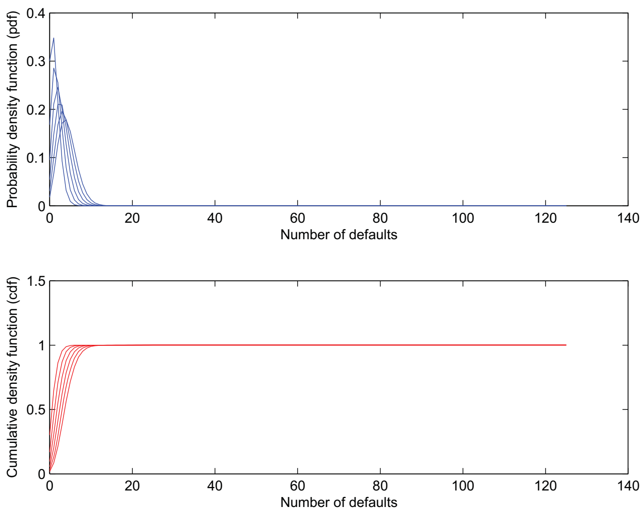

Fig. 4: PDF and CDF of Number of Defaults for Different Parameters.

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The CDF of $N_t(\cdot)$ drawn in red in Fig. 4 is derived from Equation 17:

$$q_j = \sum_{j=0}^{d} \pi_j \tag{17}$$

$$0 < q_0 < q_1 < \cdots < q_{d-1} < q_d = 1 \tag{18}$$

Thanks to a random variable u, which follows a uniform probability such that $u \sim uniform(0,1)$, the values of $N_t(\cdot)$ are determined by following the following rule:

$$\text{When} \quad q_i \leq u < q_{i+1} \Rightarrow N_t(\cdot) = i + 1$$

This simulation is performed as long as scenarios are needed, and for each period of time. Notice that $\bar{p_t}$ is empirically changed at each period of time t.

One then ends up with a graph where X-axis represents the time and Y-axis the number of defaults. However one does not obtain a single curve but a scatter of points as (16) will be simulated many times (see Fig. 6). Thus, there is a consequent number of ways to link each point and this will give multiple paths. The following hypothesis is assumed from this point: there cannot be a strictly inferior number of defaults from one time to another.

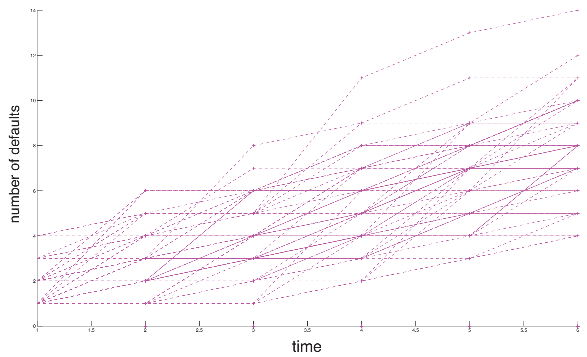

Fig. 5: Evolution of the number of defaults at each period of time for different simulations.

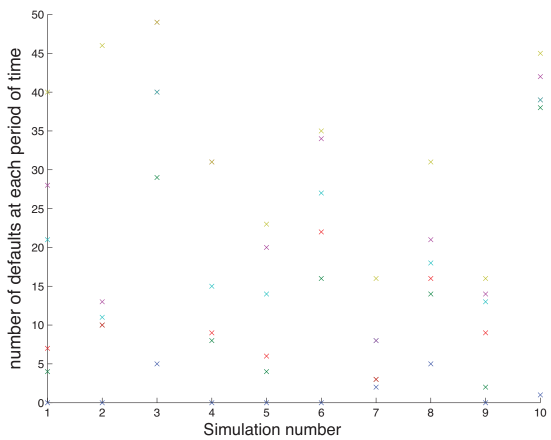

Fig. 6: Number of defaults for 10 simulations.

For example, if Equation 16 simulated only once, at time t_1, one will choose the first number of the matrix. At time t_2, if the second number of the matrix is not strictly inferior, it will be selected. Otherwise, the third number will be used, or the fourth etc. At the end there will be one path and (16) will be simulated again (see. Fig. 5).

To proceed two additional simplifying assumptions will be made. First $\bar{p_t}$ and ρ_t are not recomputed at each time t and they stay the same for all maturities T. Then it is assumed that cash flows at each payment date are independent from previous cash flows. They are only dependent on the occurring number of defaults when they are considered. The simplex enables determine the best strategy and the associated cash flow with regard to ρ and T. Once ρ is fixed one is able to compute default scenarios and the cash flows, whether sold or bought, cause in each slice.

Denote:

- P_k^i is the protection i bought at time k, $(0 < i \leq 6)$
- P_k^{i+6} is the protection $i + d$ sold at time k
- $\phi_k^i = 0, 1$ is the quantity of each P_k^i at time k

It is important to notice that each P_k^i can be computed at time k if the number of defaults n_k is known at time k.
Now consider the cash flows of period k:

$$V_k = \sum_{i=0}^{2d} \phi_k^i P_k^i$$

$max(V_k)$ can now be found by Simplex method:

$$max(V_k) = \sum_{i=0}^{2d} \phi_k^i P_k^i \tag{19}$$

$$\begin{cases} \phi_k^1 + \phi_k^7 \leq 1 \\ \ldots \\ \phi_k^i + \phi_k^{i+6} \leq 1 \\ \ldots \\ \phi_k^6 + \phi_k^{12} \leq 1 \\ \forall i \quad \phi_k^i \geq 0 \end{cases} \tag{20}$$

Conditions from (20) express the fact that it is not possible to buy and sell the same protection at the same time. Simplex method is used at each time k and outputs each ϕ_k^i and $max(V_k)$.

The final output is a vector containing the best cash flow at each time. It is repeated for each simulation generated by a fixed value of ρ. Then the vector values are averaged and one ends up with a matrix of maximum cash flows for each pair of (T,ρ).

Now it is interesting to determine the best sets of (T,ρ) leading to best cash flows. They are located on the steepest slope of the curve corresponding to $Max(\frac{\partial V}{\partial T})$ and $Max(\frac{\partial V}{\partial \rho})$.

The surface obtained with the simplex is plotted on Fig. 7.

As understandable the expected gain decreases when the maturity time and correlation of default increase. The equi-gain lines on the surface are giving the investor the possibility to trade between maturity and correlation of default for fixed return.

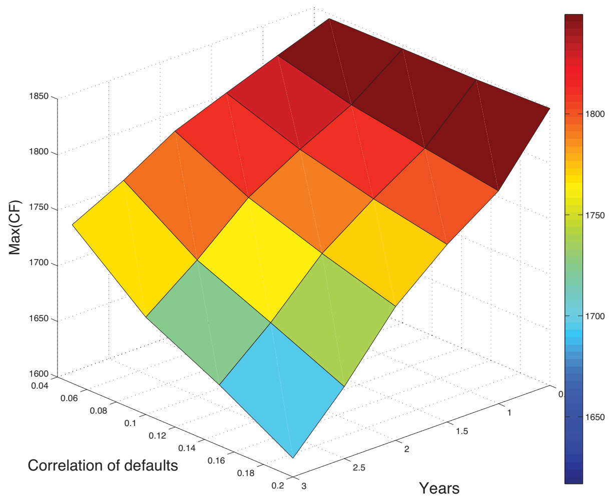

Fig. 7: Cash Flow Distribution

III. Conclusion and further Perspectives

To achieve proposed goal – to represent the behavior of a CDO in order to help the investor take decisions – several simplifications have been made. First the CDO is considered as synthetic and each CDS has the same nominal. Then a Gaussian model is used to determine the default probabilities instead of an intensity model. Moreover it is assumed that an investor can only buy or sell protection on the whole tranche of the CDO whereas it is possible to operate it on a fraction of a tranche in real market. Finally the correlation coefficient which distinguisheing the different scenarios is randomly determined. From technical point of view, a common linear optimization method – the Simplex algorithm – is used to evaluate the best cash flow projections with regard to several parameters. Despite the assumptions made throughout present work, this paper points out a way to represent the P&L of a synthetic CDO regarding several market configurations.

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