Bifurcation analysis of a delayed predator-prey fishery model with prey reserve in frequency domain

Changjin Xu

Abstract—In this paper, applying frequency domain approach, a delayed predator-prey fishery model with prey reserve is investigated. By choosing the delay τ as a bifurcation parameter, It is found that Hopf bifurcation occurs as the bifurcation parameter τ passes a sequence of critical values. That is, a family of periodic solutions bifurcate from the equilibrium when the bifurcation parameter exceeds a critical value. The length of delay which preserves the stability of the positive equilibrium is calculated. Some numerical simulations are included to justify the theoretical analysis results. Finally, main conclusions are given.

Keywords—predator-prey model; stability; Hopf bifurcation; frequency domain; Nyquist criterion

I. Introduction

In recent years, the dynamics properties of the predator-prey models which have significant biological background have received much attention from many applied mathematicians and ecologists. Many interesting results have been reported[9-11,13,15-16,18]. In 2007, Zhang et al.[17] have investigated the existence of biological and bionomic equilibrium and the local and global stability of the following non-autonomous predator-prey system with prey dispersal in a two-patch environment

$$
\begin{cases}
\dot{x}_1(t) = r_1 x_1(1 - \frac{x_1}{K_1}) - \ x_1 x_3 - \ _1 x_1 + \ _2 x_2 \\
\qquad - q_1 E_1 x_1, \\
\dot{x}_2(t) = r_2 x_2(1 - \frac{x_2}{K_2}) + \ _1 x_1 - \ _2 x_2, \\
\dot{x}_3(t) = -dx_3 + k \ x_1 x_3 - q_2 E_2 x_3,
\end{cases} \tag{1}
$$

where $x_1(t)$ and $x_3(t)$ are biomass densities of prey species and predator species inside the unreserved area which is an open-access fishing zone, respectively, at time t. $x_2(t)$ is the biomass density of prey species inside the reserved area where no fishing is permitted at time t. All the parameters are assumed to be positive. r_1 and r_2 are the intrinsic growth rates of prey species inside the unreserved and reserved areas, respectively. d, and k are the death rate, capturing rate and conversion rate of predators, respectively. K_1 and K_2 are the carrying capacities of prey species in the unreserved and reserved areas, respectively. $_1$ and $_2$ are migration rates from the unreserved area to the reserved area and the reserved area to the unreserved area, respectively. E_1 and E_2 are the effects applied to harvest the prey species and predator species in the unreserved area. q_1 and q_2 are the catch-ability coefficients.

Changjin Xu is with the Guizhou Key Laboratory of Economics System Simulation, Guizhou College of Finance and Economics, Guiyang 550004, PR China e-mail: xcj403@126.com.

We must point out that system (1) only assumes that the mature of the prey is instantaneous, but in the natural word, it it more realistic to require time lag for mature of prey, Based on this viewpoint, then system (1) becomes the following delayed autonomous predator-prey system

$$
\begin{cases}
\dot{x}_1(t) = r_1 x_1 \left[1 - \frac{x_1(t-\tau)}{K_1} \right] - \ x_1 x_3 - \ _1 x_1 \\
\qquad + \ _2 x_2 - q_1 E_1 x_1, \\
\dot{x}_2(t) = r_2 x_2 \left[1 - \frac{x_2(t-\tau)}{K_2} \right] + \ _1 x_1 - \ _2 x_2, \\
\dot{x}_3(t) = -dx_3 + k \ x_1 x_3 - q_2 E_2 x_3.
\end{cases} \tag{2}
$$

It is well known that the research on the existence of Hopf bifurcation is very critical. To obtain a deep and clear understanding of dynamics of predator-prey system with time delay, in this paper, we shall investigate the existence of Hopf bifurcation for system (2). It is worth pointing out that many early work on Hopf bifurcation of the delayed differential equations is used the state-space formulation for delayed differential equations, known as the "time domain" approach[2,9-11,13-15]. But there exists another approach that comes from the theory of feedback systems known as frequency domain method which was initiated and developed by Allwright[1], Mees and Chua[6] and Moiola and Chen [6,7] and is familiar to control engineers. This alternative representation applies the engineering feedback systems theory and methodology: an approach described in the "frequency domain"–the complex domain after the standard Laplace transforms having been taken on the state-space system in the time domain. This new methodology has some advantages over the classical time-domain methods[4,5,8,12]. A typical one is its pictorial characteristic that utilizes advanced computer graphical capabilities thereby bypassing quite a lot of profound and difficult mathematical analysis.

In this paper, we will devote our attention to finding the Hopf bifurcation point for models (2) by means of the frequency-domain approach. We found that if the coefficient is used as a bifurcation parameter, then Hopf bifurcation occurs for the model (2). That is, a family of periodic solutions bifurcates from the equilibrium when the bifurcation parameter exceeds a critical value. Some numerical simulations are carried out to illustrate the theoretical analysis. We believe that it is the first time to investigate Hopf bifurcation of the model (2) using the frequency-domain approach. Throughout the paper, we assume that

(H1) $r_1 - \ _1 - q_1 E_1 > 0, r_2 - \ _2 > 0$;

(H2) $(r_1 - {}_1 - q_1E_1)x_1^* + {}_2x_2^* > \dfrac{r_1}{K_1}x_1^{*2}$,

where

$$x_1^* = \dfrac{d + q_2E_2}{k}, x_2^* = \dfrac{K_2\left[r_2 - {}_2 + \sqrt{(r_2 - {}_2)^2 + \frac{4r_2\sigma_1x_1^*}{K_2}}\right]}{2r_2}.$$

The remainder of the paper is organized as follows: in Section 2, applying the frequency-domain approach formulated by Moiola and Chen [7], the existence of Hopf bifurcation parameter is determined and shown that Hopf bifurcation occurs when the bifurcation parameter exceeds a critical value. In Section 3, some numerical simulation are carried out to verify the correctness of theoretical analysis result. Finally, some conclusions are included in Section 4.

II. THE EXISTENCE OF HOPF BIFURCATION

It is obvious that under the conditions (H1) and (H2), system (2) has a unique positive equilibrium $E_*(x_1^*, x_2^*, x_3^*)$, where

$$x_3^* = \dfrac{(r_1 - {}_1 - q_1E_1)x_1^* + {}_2x_2^* - \frac{r_1}{K_1}x_1^{*2}}{x_1^*}.$$

We can rewrite the nonlinear system (1.2) as a matrix form

$$\dfrac{dx(t)}{dt} = Ax(t) + H(x), \qquad (3)$$

where $x = (x_1(t), x_2(t), x_3(t))^T$,

$$A = \begin{pmatrix} r_1 - {}_1 - q_1E_1 & {}_2 & 0 \\ {}_1 & r_2 - {}_2 & 0 \\ 0 & 0 & -d - q_2E_2 \end{pmatrix},$$

$$H(x) = \begin{pmatrix} -\frac{r_1x_1(t-\tau)}{K_1} - x_1x_3 \\ -\frac{r_2x_2x_2(t-\tau)}{K_2} \\ k\ x_1x_3 \end{pmatrix}.$$

Choosing the coefficient as a bifurcation and introducing a "state-feedback control" $u = g[y(t - \ ); \ ]$, where $y(t) = (y_1(t), y_2(t), y_3(t))^T$, we obtain a linear system with a nonlinear feedback as follows

$$\begin{cases} \frac{dx}{dt} = Ax + Bu, \\ y = -Cx, \\ u = g[y(t - \ ); \ ], \end{cases} \qquad (4)$$

where

$$B = C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$u = g[y(t - \ ), \ ] = \begin{pmatrix} -\frac{r_1y_1y_1(t-\tau)}{K_1} - y_1y_3 \\ -\frac{r_2y_2y_2(t-\tau)}{K_2} \\ k\ y_1y_3 \end{pmatrix}.$$

Next, taking Laplace transform on (4), we obtain the standard transfer matrix of the linear part of the system

$$G(s; \ ) = C[sI - A]^{-1}B.$$

Then

$$G(s; \ ) = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}, \qquad (5)$$

where

$$g_{13} = g_{23} = g_{31} = g_{32} = 0,$$

$$g_{11} = \dfrac{s - (r_2 - {}_2)}{[s - (r_1 - {}_1 - q_1E_1)][s - (r_2 - {}_2)] - {}_1{}_2},$$

$$g_{12} = \dfrac{{}_2}{[s - (r_1 - {}_1 - q_1E_1)][s - (r_2 - {}_2)] - {}_1{}_2},$$

$$g_{21} = \dfrac{{}_1}{[s - (r_1 - {}_1 - q_1E_1)][s - (r_2 - {}_2)] - {}_1{}_2},$$

$$g_{22} = \dfrac{s - (r_1 - {}_1 - q_1E_1)}{[s - (r_1 - {}_1 - q_1E_1)][s - (r_2 - {}_2)] - {}_1{}_2}.$$

If this feedback system is linearized about the equilibrium $y = -C(x_1^*, x_2^*, x_3^*)^T$, then the Jacobian of $u = g[y(t - \ ); \ ]$ is given by $J(\ ) = \frac{\partial g}{\partial y}\Big|_{y=\tilde{y}=-C(x_1^*,x_2^*,x_3^*)^T} =$

$$\begin{pmatrix} \frac{r_1}{K_1}(1 + e^{-s\tau})x_1^* + x_3^* & 0 & x_1^* \\ 0 & \frac{r_2}{K_2}(1 + e^{-s\tau})x_2^* & 0 \\ -k\ x_3^* & 0 & -k\ x_1^* \end{pmatrix}.$$

Let $h(\ , s; \ ) = \det|\ I - G(s; \ )J(\ )| =$

$$\begin{cases} [(s - r_1 + {}_1 + q_1E_1)(s - r_2 + {}_2) - {}_1{}_2] \\ -(s - r_2 + {}_2)\left[\dfrac{r_1}{K_1}(1 + e^{-s\tau})x_1^* + x_3^*\right] \end{cases}$$

$$\times \begin{cases} [(s - r_1 + {}_1 + q_1E_1)(s - r_2 + {}_2) - {}_1{}_2] \\ -(s - r_1 + {}_1 + q_1E_1)\left[\dfrac{r_2}{K_2}(1 + e^{-s\tau})x_2^*\right] \end{cases}$$

$$\times [(s + d + q_2E_2) + k\ x_1^*]$$

$$- {}_1{}_2k\ {}^2x_1^*x_2^*x_3^*\dfrac{r_2}{K_2}(1 + e^{-s\tau})x_2^*$$

$$+ \begin{cases} [(s - r_1 + {}_1 + q_1E_1)(s - r_2 + {}_2) - {}_1{}_2] \\ -(s - r_1 + {}_1 + q_1E_1)\left[\dfrac{r_2}{K_2}(1 + e^{-s\tau})x_2^*\right] \end{cases}$$

$$\times (s - r_2 + {}_2)\ {}^2kx_1^*x_3^* + (s + d + q_2E_2)$$

$$+ k\ {}_1{}_2\ x_1^*\dfrac{r_2}{K_2}(1 + e^{-s\tau})x_2^*$$

$$\times \left[\dfrac{r_1}{K_1}(1 + e^{-s\tau})x_1^* + x_3^*\right] = 0.$$

Applying the generalized Nyquist stability criterion with $s = i\ $, we obtain the following results.

Lemma 2.1. [7] *If an eigenvalue of the corresponding Jacobian of the nonlinear system, in the time domain, assumes a purely imaginary value $i\ _0$ at a particular $\ = \ _0$, then the corresponding eigenvalue of the constant matrix $G(i\ _0; \ _0)J(\ _0)$ in the frequency domain must assume the value $-1 + i0$ at $\ = \ _0$.*

To apply Lemma 2.1, let $\hat{\ } = \hat{\ }(i\ ; \ )$ be the eigenvalue of $G(i\ ; \ )J(\ )$ that satisfies $\hat{\ }(i\ _0; \ _0) = -1 + 0i$. Then

$$h(-1, i\ _0; \ _0) = 0.$$

Separating the real and imaginary parts and rearranging, we obtain

$$
\begin{aligned}
&[(k\ x_1^* - d - q_2E_2)(m_1n_1 - n_1n_2) + h_2]\cos 2\ _0 _0 \\
&+ _0[(m_1n_1 - n_1n_2)]\sin 2\ _0 _0 \\
&[(k\ x_1^* - d - q_2E_2)(l_2m_1 + l_1m_2 - p_2n_1 - p_1n_2) \\
&+ _0(n_1l_2 + n_2l_1 - m_1p_2 - p_1m_2) \\
&+ v_1 + s_1 + h_1]\cos\ _0 _0 + [(k\ x_1^* - d - q_2E_2) \\
&\times(n_1l_2 + n_2l_1 - m_1p_2 - p_1m_2) + _0(l_2m_1 \\
&+ l_1m_2 - p_2n_1 - p_1n_2) + t_1]\sin\ _0 _0 \\
&+(k\ x_1^* - d - q_2E_2)(l_1l_2 - p_1p_2) + _0(l_1p_2 - l_1p_1) \\
&+ v_1 + w_1 + h_0 = 0, \quad (6)
\end{aligned}
$$

$$
\begin{aligned}
&_0(m_1n_1 - n_1n_2)\cos 2\ _0 _0 + [(k\ x_1^* - d - q_2E_2) \\
&\times(m_1n_1 - n_1n_2) + f_2]\sin 2\ _0 _0 + [(l_2m_1 + l_1m_2 \\
&-p_2n_1 - p_1n_2 + (k\ x_1^* - d - q_2E_2)(n_1l_2 + n_2l_1 \\
&-m_1p_2 - p_1m_2) + t_1]\cos\ _0 _0 + [\ _0(n_1l_2 + n_2l_1 \\
&-m_1p_2 - p_1m_2) + (k\ x_1^* - d - q_2E_2)(l_2m_1 + l_1m_2 \\
&-p_2n_1 - p_1n_2) + v_1 + s_1 + f_1]\sin\ _0 _0 \\
&+ _0(l_1l_2 - p_1p_2) + (k\ x_1^* - d - q_2E_2) \\
&\times(l_1p_2 - l_1p_1) + w_2 - _0 = 0, \quad (7)
\end{aligned}
$$

where

$$
\begin{aligned}
m_1 &= (r_2 - _2)\frac{r_2}{K_2}x_1^*, m_2 = (_1 + q_1E_1 - r_1)\frac{r_2}{K_2}x_1^*, \\
n_1 &= \frac{r_2}{K_2}x_1^*\ _0, n_2 = \frac{r_2}{K_2}x_2^*\ _0, \\
l_1 &= _1 _2 + (r_1 - _1 - q_1E_1)(_2 - r_2) - _0^2 \\
&\quad -(_2 - r_2)(\frac{r_2}{K_2}x_1^* + x_3^*), \\
l_2 &= _1 _2 + (r_1 - _1 - q_1E_1)(_2 - r_2) + _0 \\
&\quad -(_1 + q_1E_1 - r_1)\frac{r_2}{K_2}x_2^*, \\
p_1 &= (r_1 - _1 - q_1E_1)\ _0 + (_2 - r_2)\ _0, \\
p_2 &= (r_1 - _1 - q_1E_1)\ _0 - (_2 - r_2)\ _0, \\
v_1 &= -k\ _1 _2x_1^*x_2^{*2}x_3^*\frac{r_2}{K_2}, \\
s_1 &= (_2 - r_2)\ ^2kx_1^*x_2^*x_3^*(_1 + q_1E_1 - r_1)\frac{r_2}{K_2}, \\
t_1 &= (_2 - r_2)\ ^2kx_1^*x_2^*x_3^*\frac{r_2}{K_2}, \\
w_1 &= (_2 - r_2)\ ^2kx_1^*x_3^*[(r_1 - _1 - q_1E_1) \\
&\quad \times(_2 - r_2 - _1 _2) + _0^2 - (_1 + q_1E_1 - r_1)x_2^*\frac{r_2}{K_2}] \\
&\quad + _0\ ^2kx_1^*x_3^*[(r_1 - _1 - q_1E_1)\ _0 \\
&\quad - _0(_2 - r_2 - _1 _2) + \frac{r_2}{K_2}\ _0x_2^*], \\
w_2 &= k\ _0\ ^2x_1^*x_3^*[(r_1 - _1 - q_1E_1)(_2 - r_2 - _1 _2) \\
&\quad + _0^2 - (_1 + q_1E_1 - r_1)x_2^*\frac{r_2}{K_2}] \\
&\quad +(_2 - r_2)\ ^2kx_1^*x_3^*[(r_1 - _1 - q_1E_1)\ _0 \\
&\quad - _0(_2 - r_2 - _1 _2) + \frac{r_2}{K_2}\ _0x_2^*], \\
h_0 &= k\ _1 _2\ ^2x_1^*x_2^*x_3^*\frac{r_2}{k_2} - d - q_2E_2,
\end{aligned}
$$

$$
\begin{aligned}
h_1 &= k\ _1 _2\ x_1^*x_2^*\left(\frac{r_1r_2}{K_1K_2}x_1^* + x_3^*\right), \\
h_2 &= k\ _1 _2\ x_1^*x_3^*\frac{r_1r_2}{K_1K_2}, f_1 = k\ _1 _2\ ^2x_1^*x_3^*\frac{r_1r_2}{K_1K_2}, \\
f_2 &= 2k\ _1 _2\ ^2x_1^{*2}x_3^*\frac{r_1r_2}{K_1K_2} - x_3^*.
\end{aligned}
$$

Similar to paper[14], by (6) and (7), we can obtain the expression of $\cos\ _0 _0$, say

$$
\cos\ _0 _0 = f_1(_0), \quad (8)
$$

where $f_1(_0)$ is a function with respect to $_0$. Substitute (8) into (6), then we can easily get the expression of $\sin\ _0 _0$, say

$$
\sin\ _0 _0 = f_2(_0), \quad (9)
$$

where $f_2(_0)$ is a function with respect to $_0$. Thus we obtain

$$
f_1^2(_0) + f_2^2(_0) = 1. \quad (10)
$$

If the coefficients of the system (2) are given, it is easy to use computer to calculate the roots of (10) (say $_0$). Then from (8), we derive

$$
_k = \frac{1}{_0}[\arccos f_1(_0) + 2k\ ]\ (k = 0, 1, 2, \cdots). \quad (11)
$$

Theorem 2.1. (Existence of Hopf bifurcation parameter) *For system (2), if conditions (H1)-(H2) hold and $_0$ is positive real roots of (10), then Hopf bifurcation point of system (2) is*

$$
_k = \frac{1}{_0}[\arccos f_1(_0) + 2k\ ]\ (k = 0, 1, 2, \cdots),
$$

where $f_1(_0)$ is defined by (8).

III. Estimation of the length of delay to preserve stability

In the present section, we will obtain an estimation $_+$ for the length of the delay which preserves the stability of the positive equilibrium $E_*(x_1^*, x_2^*, x_3^*)$, i.e., $E_*(x_1^*, x_2^*, x_3^*)$ is asymptotically stable if $< _+$.

We consider system (2) in $C([-\ , \infty), R^3)$ with the initial values $x_i(\ ) = _i(\ )$, $_i(0) \geq 0, i = 1, 2, 3$, $\in [-\ , 0]$. Let $\bar{x}_i(t) = x_i(t) + x_1^*, (i = 1, 2, 3)$, for simplification, we still denote $\bar{x}_i(t)$ as $x_i(t)$, then the linear equation of (2) at $E_*(x_1^*, x_2^*, x_3^*)$ takes the form

$$
\begin{cases}
\dot{x}_1(t) = a_1x_1 + a_2x_2 + a_3x_3 + a_0x_1(t - \ ), \\
\dot{x}_2(t) = b_1x_1 + b_2x_2 + b_0x_2(t - \ ), \\
\dot{x}_3(t) = c_1x_1 + c_2x_3,
\end{cases} \quad (12)
$$

where

$$
\begin{aligned}
a_1 &= r_1 - \frac{r_1x_1^*}{K_1} - x_3^* - _1 - q_1E_1, a_2 = _2, \\
a_3 &= - x_1^*, a_0 = -\frac{r_1x_1^*}{K_1}, b_1 = - _1, b_2 = r_2 - \frac{r_2x_2^*}{K_2} - _2, \\
b_0 &= -\frac{r_2x_2^*}{K_2}, c_1 = k\ x_3^*, c_2 = k\ x_1^* - d - q_2E_2.
\end{aligned}
$$

Taking Laplace transform of system (12), we get

$$\begin{cases} (s-a_1)\tilde{x}_1 = a_2\tilde{x}_2 + a_3\tilde{x}_3 + a_0 e^{-s\tau}\tilde{x}_1 \\ \qquad\qquad + a_0 e^{-s\tau}K_1(s) + {}_1(0), \\ (s-b_2)\tilde{x}_2 = b_1\tilde{x}_1 + b_0 e^{-s\tau}\tilde{x}_2 + b_0 e^{-s\tau}K_2(s) \\ \qquad\qquad + {}_2(0), \\ (s-c_2)\tilde{x}_3 = c_1\tilde{x}_1 + {}_3(0), \end{cases} \qquad (13)$$

where $\tilde{x}_i (i=1,2,3)$ are the Laplace transform of $x_i(t)(i=1,2,3)$, respectively, $K_1(s) = \int_{-\tau}^0 e^{-s\tau}x_1(t)dt$, $K_2(s) = \int_{-\tau}^0 e^{-s\tau}x_2(t)dt$. Solving (13) for $\tilde{x}_1$ leads to $\tilde{x}_1 = \frac{K(s,\tau)}{J(s)}$, where

$$\begin{aligned} K(s,\tau) &= [a_0 k_1 e^{-s\tau} + {}_1(0)](s-b_2 - b_0 e^{-s\tau}M_1(s)) \\ &\quad \times (s-c_2) + a_3 {}_3(0)(s-b_2-b_0 e^{-s\tau}) \\ &\quad - a_2(s-c_2)[b_0 e^{-s\tau}K_2(s) + {}_2(0)], \\ J(s) &= (s-a_1-a_0 e^{-s\tau})(s-b_2-b_0 e^{-s\tau})(s-c_2) \\ &\quad -a_3 c_1(s-b_2-b_0 e^{-s\tau}) - a_2 b_1(s-c_2). \end{aligned}$$

Following along the lines of [3] and using the Nyquist criterion, we obtain that the conditions for local asymptotic stability of $E_*(x_1^*, x_2^*, x_3^*)$ are given by

$$\text{Im}\{J(i {}_0)\} > 0, \qquad (14)$$

$$\text{Re}\{J(i {}_0)\} = 0, \qquad (15)$$

where $\text{Im}\{J(i {}_0)\}$ and $\text{Re}\{J(i {}_0)\}$ are the imaginary part and real part of $J(i {}_0)$, respectively and ${}_0$ is the small positive root of (14). It follows from (15) that

$$\begin{aligned} \mu_1 \cos 2 {}_0 &+ \mu_2 \sin 2 {}_0 + \mu_3 \cos {}_0 \\ &+ \mu_4 \sin {}_0 = \mu_0, \end{aligned} \qquad (16)$$

where

$$\begin{aligned} \mu_0 &= 2[(a_1+b_2+c_2) {}_0^2 + a_3 b_2 c_1 {}_0 + a_1 b_2 c_2)] \\ &\quad + a_0 c_2(b_2-b_0), \\ \mu_1 &= a_0 c_2(b_2+b_0), \ \mu_2 = a_0(b_2 {}_0 - b_0), \\ \mu_3 &= 2 {}_0(a_1 b_2 - a_0 c_2 - b_2 c_2 - a_2 b_2), \\ \mu_4 &= c_2(a_0 b_2 - a_1 b_0) - (a_0+b_0) {}_0^2 - a_3 b_2 c_1. \end{aligned}$$

Hence

$$\begin{aligned} &|a_0 c_2(b_2+b_0)| + |a_0 b_2| {}_0 + |a_0 b_0| \\ &+ 2 {}_0|a_1 b_2 - a_0 c_2 - b_2 c_2 - a_2 b_2| \\ &+ |c_2(a_0 b_2 - a_1 b_0)| + |a_0+b_0| {}_0^2 + |a_3 b_2 c_1| \\ &\geq 2[(a_1+b_2+c_2) {}_0^2 + a_3 b_2 c_1 {}_0 + a_1 b_2 c_2)] \\ &+ a_0 c_2(b_2-b_0), \end{aligned}$$

which leads to

$$_2 {}_0^2 + {}_1 {}_0 + {}_0 \leq 0, \qquad (17)$$

where

$$\begin{aligned} {}_0 &= a_0 c_2(b_2-b_0) + 2a_1 b_2 c_2 - |a_0 c_2(b_2+b_0)| \\ &\quad - |a_0 b_0| - |c_2(a_0 b_2 - a_1 b_0)| - |a_3 b_2 c_1|, \\ {}_1 &= 2a_3 b_2 c_1 - |a_0 b_2| - 2|a_1 b_2 - a_0 c_2 - b_2 c_2 - a_2 b_2|, \\ {}_2 &= 2(a_1+b_2+c_2) - |a_0+b_0|. \end{aligned}$$

We assume that

(H3) $2(a_1+b_2+c_2) - |a_0+b_0| > 0.$

If we denote

$$_+ = \frac{| {}_1| + \sqrt{{}_1^2 + 4| {}_0 {}_2|}}{2 {}_2} \qquad (18)$$

Under the condition (H3), we have ${}_0 \leq {}_+$. From (14) we derive

$$\begin{aligned} {}_0^2 &< {}_1 \cos 2 {}_0 + {}_2 \sin 2 {}_0 + {}_3 \cos {}_0 \\ &\quad + {}_4 \sin {}_0 + {}_0, \end{aligned} \qquad (19)$$

where

$$\begin{aligned} {}_0 &= \left(b_2 c_2 - \frac{a_0 b_0}{2} - a_1 b_2 - a_1 c_2 - a_3 c_1\right) {}_0 + \frac{a_0 b_2}{2}, \\ {}_1 &= \frac{a_0 b_0 {}_0 + a_0 b_0}{2}, \ {}_2 = \frac{a_0 b_2 c_2 + a_0 b_0 c_2}{2}, \\ {}_3 &= (a_0 b_2 - a_1 b_0 + a_0 c_2 + b_0 c_2) {}_0, \\ {}_4 &= a_0 b_2 c_2 + a_3 b_0 c_1 - a_0 b_2 - a_1 b_2 c_2 - a_0 {}_0. \end{aligned}$$

It is easy to see from (16) that

$$_0^2 = \frac{}{2(a_1+b_2+c_2)}, \qquad (20)$$

where

$$\begin{aligned} &= \mu_1 \cos 2 {}_0 + \mu_2 \sin 2 {}_0 + \mu_3 \cos {}_0 + \mu_4 \sin {}_0 \\ &\quad - a_0 c_2(b_2-b_0) - 2a_3 b_2 c_1 {}_0 - 2a_1 b_2 c_2. \end{aligned}$$

Substituting (20) into (19) and rearranging, we obtain

$$\begin{aligned} {}_1(\cos 2 {}_0 &- 1) + {}_2 \sin 2 {}_0 \\ &+ {}_3(\cos {}_0 - 1) + {}_4 \sin {}_0 < {}_0, \end{aligned} \qquad (21)$$

where

$$\begin{aligned} {}_0 &= 2(a_1+b_2+c_2) {}_0 + a_0 c_2(b_2-b_0) \\ &\quad + 2a_3 b_2 c_1 {}_0 + 2a_1 b_2 c_2 \\ &\quad - [\mu_1 - 2(a_1+b_2+c_2) {}_1] \\ &\quad - [\mu_3 - 2(a_1+b_2+c_2) {}_3], \\ {}_1 &= \mu_1 - 2(a_1+b_2+c_2) {}_1, \\ {}_2 &= \mu_1 - 2(a_1+b_2+c_2) {}_2, \\ {}_3 &= \mu_3 - 2(a_1+b_2+c_2) {}_3, \\ {}_4 &= \mu_4 - 2(a_1+b_2+c_2) {}_4. \end{aligned}$$

Using the bounds

$$\begin{aligned} {}_1(\cos 2 {}_0 - 1) &\leq 2 {}_1 \sin^2\left(\frac{{}_0}{2}\right) \leq \frac{1}{2}| {}_1| {}_+^2 {}^2, \\ {}_3(\cos 2 {}_0 - 1) &\leq 2 {}_3 \sin^2\left(\frac{{}_0}{2}\right) \leq \frac{1}{2}| {}_3| {}_+^2 {}^2, \\ {}_2 \sin 2 {}_0 &\leq 2| {}_2| {}_+, \ {}_4 \sin 2 {}_0 \leq 2| {}_4| {}_+, \end{aligned}$$

we have

$$L_1 {}^2 + L_2 \leq L_3,$$

where

$$
\begin{aligned}
L_1 &= \frac{1}{2}[|a_0c_2(b_2+b_0)|+2|a_1+b_2+c_2| \\
&\quad \times \frac{|a_0b_0|_{\;+}+|a_0b_0|}{2} \\
&\quad +|2_{\;+}|a_1b_2-a_0c_2-b_2c_2-a_2b_2| \\
&\quad -2|a_1+b_2+c_2| \\
&\quad \times|a_0b_2-a_1b_0+a_0c_2+b_0c_2|_{\;+}]|_{\;+}^2, \\
L_2 &= 2[|a_0(|b_2|_{\;+}+|b_0|)|+2|a_1+b_2+c_2| \\
&\quad \times \frac{|a_0b_2c_2+a_0b_0c_2|}{2}+|c_2(a_0b_2-a_1b_0)| \\
&\quad +|(a_0+b_0)|_{\;+}^2+|a_3b_2c_1|+2|a_1+b_2+c_2| \\
&\quad \times(|a_0b_2c_2|+|a_3b_0c_1|+|a_0b_2|+|a_1b_2c_2| \\
&\quad +|a_0|_{\;+})]_{\;+}, \\
L_3 &= 2|a_1+b_2+c_2|\left(|b_2c_2-\frac{a_0b_0}{2}-a_1b_2-a_1c_2\right. \\
&\quad \left. -a_3c_1|_{\;+}+\frac{|a_0b_2|}{2}\right) \\
&\quad +|a_0c_2(b_2-b_0)|+2|a_3b_2c_1|_{\;+}+2|a_1b_2c_2|.
\end{aligned}
$$

It is easy to see that if $<_{\;+}=\frac{-L_2+\sqrt{L_2^2+4L_1L_3}}{2L_1}$, the stability of $E_*(x_1^*,x_2^*,x_3^*)$ of system (1.2) is preserved.

Theorem 3.1. *For system (2), under the conditions (H1)-(H3), if there exists a τ in $0\le\tau\le\tau_+$ such that $L_1\tau^2+L_2\tau\le L_3$, then τ_+ is the maximum value(length of delay) of τ for which $E_*(x_1^*,x_2^*,x_3^*)$ is asymptotically stable.*

IV. NUMERICAL EXAMPLES

In this section, we present some numerical results of system (2) to verify the analytical predictions obtained in the previous section. Let us consider the following system

$$
\begin{cases}
\dot{x}_1(t)=0.5x_1(1-0.2x_1(t-\tau))-0.8x_1y-0.4x_1 \\
\qquad +2x_2-0.06x_1, \\
\dot{x}_2(t)=0.5x_2(1-0.9x_2(t-\tau))+0.4x_1-2x_2, \\
\dot{x}_3(t)=-0.6y+1.9x_1y-0.6y,
\end{cases}
\tag{22}
$$

which has a positive equilibrium $E_*(x_1^*,x_2^*,x_3^*) \approx (0.2020,0.5711,1.0417)$ and satisfies the conditions indicated in Theorem 2.1. The positive equilibrium $E_* = (0.2020,0.5711,1.0417)$ is asymptotically stable for $\tau < \tau_0 \approx 2.6$. Figs.1-7 show that the positive equilibrium $E_0 \approx (0.6316,0.1945,0.2789)$ is asymptotically stable when $\tau = 2.5 < \tau_0 \approx 2.6$. When τ passes through the critical value τ_0, the positive equilibrium $E_* \approx (0.6316,0.1945,0.2789)$ loses its stability and a Hopf bifurcation occurs, i.e., a family of periodic solutions bifurcate from the positive equilibrium $E_* \approx (0.6316,0.1945,0.2789)$. Figs.8-14 show that a family of periodic solutions bifurcate from the positive equilibrium $E_* \approx (0.6316,0.1945,0.2789)$ when $\tau = 2.658 > \tau_0 \approx 2.6$.

Fig.1

Fig.2

Fig.3

Fig.4

Fig.5

Fig.6

Fig.7

Fig.10

Fig.11

Fig.12

Fig.13
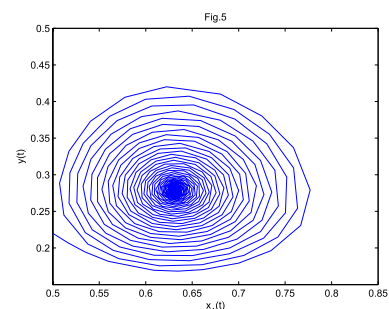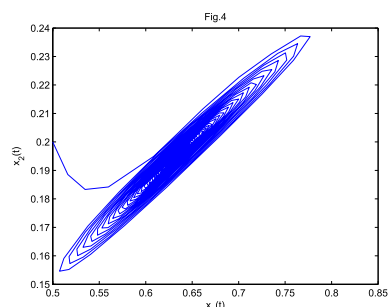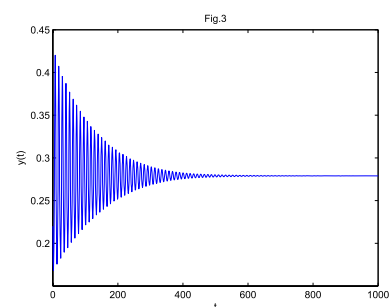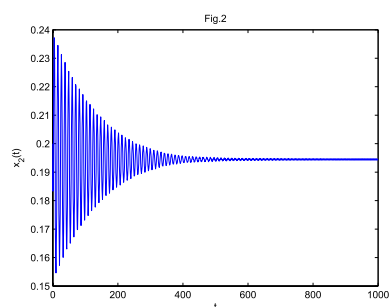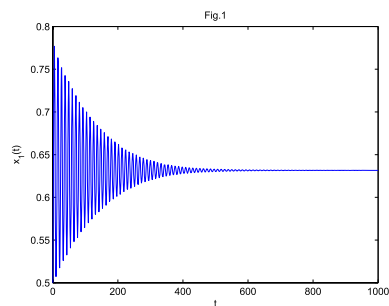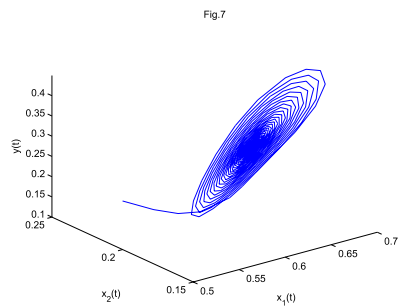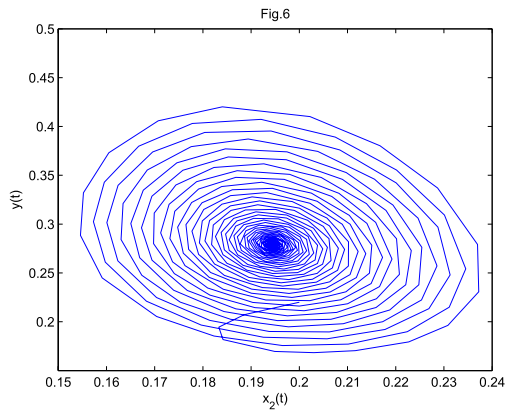
Figs.1-7 Behavior and phase portrait of system (22) with $\tau = 2.5 < \tau_0 \approx 2.6$. The positive equilibrium $E_* \approx (0.6316, 0.1945, 0.2789)$ is asymptotically stable. The initial value is (0.5,0.2,0.22).

Fig.8

Fig.9

Fig.14
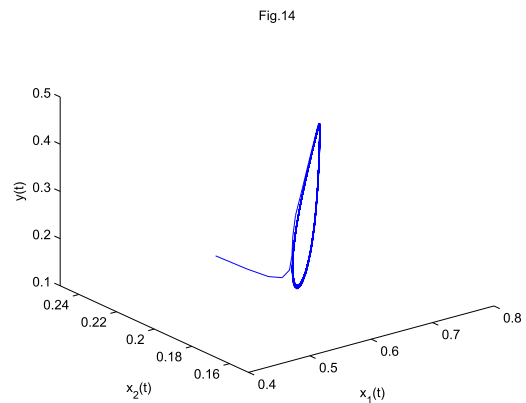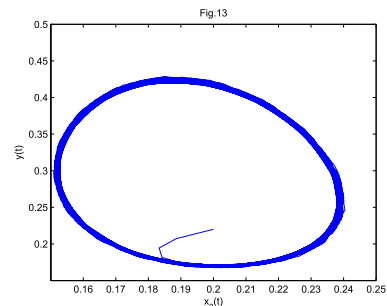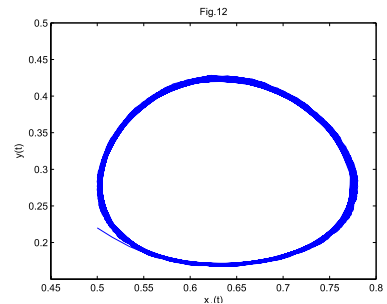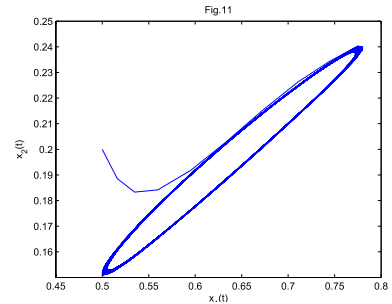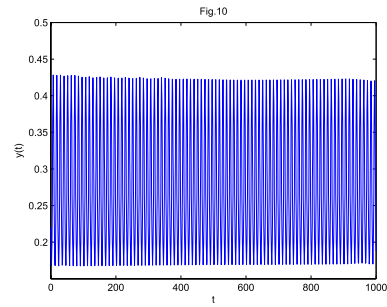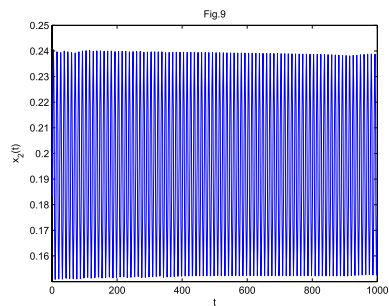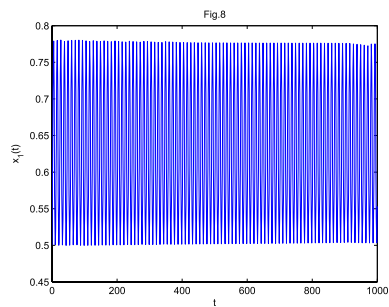
Figs.8-14 Behavior and phase portrait of system (22) with $\tau = 2.658 > \tau_0 \approx 2.6$. Hopf bifurcation occurs from the positive

equilibrium $E_* \approx (0.6316, 0.1945, 0.2789)$. The initial value is (0.5,0.2,0.22).

V. Conclusions

In this paper, by choosing the coefficient as a bifurcating parameter, we investigated a delayed predator-prey fishery model with prey reserve in frequency domain approach. It is found that a Hopf bifurcation occurs when the bifurcating parameter passes through a critical value. Meanwhile, the length of delay preserving the stability of the positive equilibrium $E_*(x_1^*, x_2^*, x_3^*)$, is estimated. Considering computational complexity, the direction and the stability of the bifurcating periodic orbits for system (2) have not been investigated. It is beyond the scope of the present paper and will be further investigated elsewhere in the near future.

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Changjin Xu received the M. S. degree from Kunming University of Science and Technology, Kunming, in 2004., the Ph.D.degree from Central South University, Changsha, China, all in the mathematics or applied mathematics. He is an associate professor at Guizhou College of Finance and Economics. His current research interests focus on the stability and bifurcation theory of delayed differential and equation. E-mail: xcj403@126.com