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BIBD's for $(13,5,5)$, $(16,6,5)$ and $(21,6,4)$ Possessing Possibly an Automorphism of Order 3

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Abstract—When trying to enumerate all BIBD's for given parameters, their natural solution space appears to be huge and grows extremely with the number of points of the design. Therefore, constructive enumerations are often carried out by assuming additional constraints on design's structure, automorphisms being mostly used ones. It remains a hard task to construct designs with trivial automorphism group – those with no additional symmetry – although it is believed that most of the BIBD's belong to that case. In this paper, very many new designs with parameters 2-$(13,5,5)$, 2-$(16,6,5)$ and 2-$(21,6,4)$ are constructed, assuming an action of an automorphism of order 3. Even more, it was possible to construct millions of such designs with no non-trivial automorphisms.

Keywords—BIBD, incidence matrix, automorphism group, tactical decomposition, deterministic algorithm.

I. Method of construction

In this paper our new enumeration results for 2-designs with parameters $(13,5,5)$, $(16,6,5)$ and $(21,6,4)$ are presented. In case of parameters 2-$(21,6,4)$ there was only one design known so far [6], whereas in case of the other two parameter triples there were dozens already known examples (a heuristic search for 2-$(13,5,5)$ has been carried out by V. Krčadinac in his thesis [4]). Although the numbers of new designs, constructed in this work, are counted in millions, for all three parameter triples, one should be aware of the fact that still many more remain to be found (this statement shall be supported in the following sections). Namely, all our distinct computer runs have been bounded to several days, on the other hand it is apparent that longer runs would bring many more BIBD's. Still, our feeling remains that the constructive enumeration results give a much better insight in the behavior of non-symmetric block designs, for which very few results of this kind are known so far.

These designs have been constructed assuming firstly in addition an action of an automorphism of order 3, performing an exhaustive computer search with the added constraint. The fact, that a group action always induces a tactical decomposition of the incidence matrix of the design, has been used. Modifying this approach by forgetting the assumption on the automorphism group action in a certain step of our construction procedure, designs with trivial automorphism group could be achieved, which are particularly interesting examples among all these obtained results.

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A balanced incomplete block design (BIBD) is a finite combinatorial structure with belonging parameters v, b, r, k and λ, consisting of a non-empty set $\mathcal{P}$ of v points and a set $\mathcal{B}$ of b k-element subsets of $\mathcal{P}$ which are called blocks. Every point is included in r blocks, and each 2-subset of $\mathcal{P}$ appears in exactly λ blocks. Since it can be shown that $r = \frac{\lambda(v-1)}{k-1}$ and $b = \frac{\lambda v(v-1)}{k(k-1)}$, a parameter triple (v,k,λ) is joined to every BIBD. BIBD's are often represented by their incidence matrices, which are 0-1 matrices of dimension $v \times b$, rows and columns labelled by points and blocks respectively, and entries equal to 1 indicating when a point lies on a block. More generally, a point p is said to be incident with a block B and denoted by $(p,B) \in I$, I being an incidence relation $I \subseteq \mathcal{P} \times \mathcal{B}$.

If a group G acts on such a design, then its orbits on the point set $\mathcal{P}$ and block set $\mathcal{B}$ form a tactical decomposition of that design and of its incidence matrix. A tactical decomposition is a partition of rows and columns of the given matrix, having the property that the row and column sum of each submatrix of the partition is constant. The orbits are denoted by $\mathcal{P} = \mathcal{P}_1 \sqcup \mathcal{P}_2 \sqcup \cdots \sqcup \mathcal{P}_m$ and $\mathcal{B} = \mathcal{B}_1 \sqcup \mathcal{B}_2 \sqcup \cdots \sqcup \mathcal{B}_n$; their cardinalities as $|\mathcal{P}_1|, |\mathcal{P}_2|, \ldots, |\mathcal{P}_m|$ and $|\mathcal{B}_1|, |\mathcal{B}_2|, \ldots, |\mathcal{B}_n|$. Further the blocks incident with a point p are denoted by $\langle p \rangle = \{B \in \mathcal{B} \mid (p,B) \in I\}$, for any $p \in \mathcal{P}$ and $\langle B \rangle = \{p \in \mathcal{P} \mid (p,B) \in I\}$, for any $B \in \mathcal{B}$. Now, because of the tactical decomposition property, the following coefficients are well defined:

$$\rho_{ij} = |\langle p \rangle \cap \mathcal{B}_j|, \;\; p \in \mathcal{P}_i \,; \quad \kappa_{ij} = |\langle B \rangle \cap \mathcal{P}_i|, \;\; B \in \mathcal{B}_j \,.$$

The coefficients ρ_{ij} play an important role in our construction procedure and fulfil (for details, see e.g. [1] and [3]) the following equations:

$$\sum_{j=1}^{n} \rho_{ij} = k, \;\; \forall i \tag{1}$$

$$\sum_{j=1}^{n} \frac{|\mathcal{P}_l|}{|\mathcal{B}_j|} \rho_{ij} \rho_{lj} = \lambda \cdot |\mathcal{P}_l| + \delta_{il}(k-\lambda) \,. \tag{2}$$

Dual equations are valid for the coefficients κ_{ij}.

Our two-step construction procedure finds firstly all matrices $[\rho_{ij}]$, the entries ρ_{ij} of which explain how many blocks from the block orbit $\mathcal{B}_j$ need to be incident with every point from the point orbit $\mathcal{P}_i$. These matrices $[\rho_{ij}]$ are called tactical decomposition matrices (TDM's), being aware of the fact that the design to which that matrix refers may even not exist.

The second step refines the TDM's by determining exactly the incidences between points and blocks (in fact, trying to extend every TDM to a block design incidence matrix, fulfilling its defining properties).

Own program based on a backtracking strategy was developed. Since the first non-fixed row of the tactical decomposition matrix can be expanded in a unique way, without loss of generality, a filtering procedure of every further row candidate with that beginning (non-fixed but unique) row could be introduced. This idea reduced the algorithm complexity quite largely.

After finding all designs with an automorphism of order 3 (that could be constructed), it was intriguing to construct designs with these parameters having the trivial automorphism group. For this purpose, it shows to be sufficient to "forget" the group action in the second step of our construction procedure. Namely, the achieved TDM's were taken with the goal to expand them to incidence matrices not taking into account that the block submatrices of order 3, which are substitutes for the coefficients ρ_{ij}, have to be circulant. Herewith, for every entry $\rho_{ij} = 1$ or 2, the number of possibilities for substituting it with a 0-1 matrix was doubled. For example, if $\rho_{ij} = 1$, there were 3 circulant matrices of order 3, but 6 permutation matrices of that order. So it was not surprising that an exhaustive search could not be carried out for all cases where it was possible when dealing only with circulant matrices.

II. Results for $(13, 5, 5)$

If it is assumed that an automorphism of order 3 acts on a $(13, 5, 5)$ block design, then its number of fixed points $F_p \in \{1, 4, ...\}$, because all point orbits are of length 1 or 3. Similarly, as $b = 39$, for the number of fixed blocks it holds $F_b \in \{0, 3, ...\}$. We claim that there is no fixed block. Namely, it would consist of at least 2 fixed points, hence $F_p \geq 4$. But it is not possible to construct more than 2 fixed points, each being incident with 15 blocks and each pair being incident with 5 blocks. After we are convinced of the fact that $F_b = 0$, it is clear that only one fixed point can be constructed of orbits of length 3, otherwise the target intersection of $\lambda = 5$ cannot be reached. Hence, the following proposition has been proven.

Proposition 1: If an automorphism of order 3 acts on a block design with parameters $(13, 5, 5)$, then it fixes 1 point and acts fixed-block-free.

Clearly, our fixed point is incident with all blocks from 5 block orbits of length 3. Based on this fixed structure, 342 tactical decomposition matrices were obtained, fulfilling the system of equations (1)-(2), but only 43 out of them led to a design. More precisely, when the second step of our construction procedure was attended, the 342 TDM's being their input, altogether 4086378 incidence matrices were obtained, leading to 1084129 non-isomorphic copies of designs. Among these non-isomorphic structures, there are 1021266 simple designs. Namely, some of the TDM's have equal columns which can lead to equal blocks after the second construction step. Our computational result gives a complete classification of BIBD's for $(13, 5, 5)$ possessing an automorphism of order 3.

TABLE I
Group orders of all $(13, 5, 5)$ designs, C case

$\lvert Aut(\mathcal{D})\rvert$	$frequency$
3	1021120
6	144
24	61857
39	1
48	9
192	990
384	3
1536	4
3072	1
$\sum$	1084129

TABLE II
Group orders of obtained $(13, 5, 5)$ designs, CAC case

$\lvert Aut(\mathcal{D})\rvert$	$frequency$
1	191668
2	22914
3	9069
4	82
8	1799
16	6
24	722
192	2
$\sum$	226262

Theorem 1: There are 1084129 block designs for $(13, 5, 5)$ admitting an action of an automorphism of order 3. 1021266 of them are simple.

Table 1 shows a complete list of the automorphism group orders appearing for designs we have already constructed, as well as their frequencies.

After this complete classification of $(13, 5, 5)$ designs admitting an action of an automorphism group of order 3, it was still interesting for us to find some such designs with no automorphisms. Therefore, as explained in the previous section, the group action in the second step of our construction procedure was forgotten, when extending the TDM's to incidence matrices, which means that our intention was to construct incidence matrices from tactical decomposition matrices, allowing not only circulant matrices of order 3 to replace the coefficients ρ_{ij}, but also anticirculant matrices with ρ_{ij} ones in each row. In the previous case the solution space had a size of 3^{13^3}, whereas it increases now to 6^{13^3}, since every element of the variable part (first two rows are kept fixed) of the 5×13 tactical decomposition matrix is now trying to be replaced with one of the 6 circulant or anticirculant matrices (CAC) instead of only 3 circulant matrices (C). Under these circumstances, an exhaustive search could not have been done any more, not even for one tactical decomposition matrix in a realistic time frame. One chosen TDM led us, adding some additional constraints by choosing only some of the possibilities for the first non-fixed row, to 226262 non-isomorphic designs (200737 out of them being simple). It could be seen, to our satisfaction, that in most of the cases these designs have only a trivial automorphism group. A complete list of automorphism group orders of these designs are presented in Table 2.

By adding all the frequencies not divisible by 3 and having in mind our former classification given in Theorem 1, our enumeration results can be summarized as follows.

Proposition 2: There are at least 1300598 2-designs with parameters $(13, 5, 5)$.

At the end, incidence matrices of two examples of found 2-$(13, 5, 5)$ designs are listed, the one with the largest full automorphism group and one with the trivial automorphism group, respectively. The TDM's which led to these designs are listed firstly (note that the first row of our TDM's belongs to the fixed point).

TDM_1	TDM_2
3333300000000	3333300000000
2210022211110	3110022111111
1111120022202	1003111221111
1111101121032	0221010112221
0012212201211	0110312111112

2-$(13, 5, 5)$ design with $|Aut(\mathcal{D}_1)| = 3072$

111111111111111000000000000000000000000
110110100000000110110110100100100100000
011011010000000011011011010010010010000
101101001000000101101101001001001001000
100100010100100011000000110011101000110
010010001010010101000000011101110000011
001001100001001110000000101110011000101
1001000001001001000010010011010000111011
0100101001001000000001001101001000111101
001001010010010000100100110100000111110
000000100110110001110110000010011001001
000000010011011100011011000001101100100
000000001101101010101101000100110010010

2-$(13, 5, 5)$ design with $|Aut(\mathcal{D}_2)| = 1$

111111111111111000000000000000000000000
111100100000000110110100100100100100100
111010010000000011011010010010010010010
111001001000000101101001001001001001001
100000000111100100100110110010010001001
010000000111010010010011011001001100100
001000000111001001001101101100100010010
0001101101000000100000001001110011101010
0000110110100000010000100100011101110001
0001011010010000001000010010101110011100
0001000100000111100011001100001010010101
000010001000111010110010001100100001011
0000011000000111001101100010010001100110

III. Results for $(16, 6, 5)$

At the beginning it is assumed for a cyclic group of order 3 to act on a block design with parameters $(16, 6, 5)$. Unfortunately, it wasn't possible, as in the previous case, to specify the number of fixed points and fixed blocks, since there are many possibilities here. Firstly, the case where $F_p = F_b = 4$ was under focus. In this case, 891 tactical decomposition matrices have been achieved.

TABLE III
Automorphism group orders of a chosen design sample

$\lvert Aut(\mathcal{D})\rvert$	$frequency$
3	21254100
6	37
12	2
24	3490
192	52

TABLE IV
Result comparison for one TDM

	$numincmat$	$niso$	$\lvert Aut(\mathcal{D})\rvert$
C	450808	225404	3(225404)
CAC (partly)	51986104	$\geq$ 472971	1(430134),3(42837)

Again, as in case of previous design parameters, our first aspiration was to develop these TDM's into incidence matrices, i.e. to classify block designs for $(16, 6, 5)$ admitting an action of an automorphism of order 3 fixing four points and blocks. The first and second part of our algorithm finished in an appropriate time frame, but the main challenge here was extraction of non-isomorphic designs among the obtained structures. Summarizing our computational results gives the following theorem.

Theorem 2: There are 203398868 block designs for $(16, 6, 5)$ admitting an action of an automorphism of order 3 fixing 4 points and 4 blocks.

Table 3 shows the orders of automorphism groups for a chosen design sample among the huge number of non-isomorphic designs achieved above when assuming a cyclic group action.

For the purpose of having an impression of the relation between the number of designs admitting an automorphism of order 3 (circulant case) and those with trivial automorphism group (circulant and anticirculant case), the second experiment that takes into account also anticirculant matrices was conducted, for one chosen tactical decomposition matrix. This test finished successfully finding 51986104 incidence matrices. The first one million to test them on isomorphism were chosen, and got 430134 non-isomorphic designs with trivial automorphism group and some further ones with a nontrivial automorphism group. A comparison of the obtained results for the same TDM is given in Table 4, for both C and CAC case. It can be denoted by $numincmat$ the number of all constructed incidence matrices, whereas $niso$ stays for the number of non-isomorphic ones.

Taking into account that different automorphism group orders lead to non-isomorphic designs, the following result was obtained.

Proposition 3: There are at least 203829002 2-designs with parameters $(16, 6, 5)$.

We want to stress that this statement is a result of only one possible kind of action of an automorphism of order 3. Knowing further that there are 9467 TDM's with 1 fixed point and 1 fixed block - which give plenty of non-isomorphic designs as well, one can imagine how many structures probably remain to be revealed.

IV. Tests for $(21, 6, 4)$

An automorphism of order 3 can act on a block design for $(21, 6, 4)$ in many different ways. Since we were aware of the size of the problem, we had a closer look at a case when the number of fixed points and blocks are considerably large, namely $F_p = 6$ and $F_b = 8$. Even with these additional assumptions on the group action, we didn't find all TDM's, but only some particularly interesting ones. Adding some constraints of symmetry to the fixed structure, altogether 318 TDM's have been achieved. It takes too long to continue with an exhaustive search for indicence matrices for all these TDM's (although they all give rise to incidence matrices of BIBD's). Therefore, to get a better insight in the thickness of the designs, it was decided to make 3 kinds of experiments with these TDM's.

Firstly, we wanted to analyze one chosen tactical decomposition matrix (since it was obvious that the solution space is too huge for an exhaustive search of all TDM's we have found before). Our algorithm for the second step of the construction procedure was able to manage that: for the first TDM, 241314960 incidence matrices were obtained (causing problems with static computer memory, because of the fact that 287GB were needed to store these matrices). We have chosen the first million of these matrices, to isolate those among them which are non-isomorphic, in a suitable time period, and established that all 1000000 matrices are non-isomorphic, all having an automorphism group of order 3 and no order larger.

Our second experiment's aim was to get to know how many designs, up to isomorphism, could be revealed constructing the first 1000 incidence matrices from every particular TDM. In this second test, 318000 matrices were obtained, all non-isomorphic and again all with an automorphism group of order 3.

For the third search, permitting anticirculant matrices on a place of a TDM's element as well, we have chosen the first TDM - which in the first (C) test brought 241314960 incidence matrices. Since it was out of range to make an exhaustive search for the whole matrix, one more row in the variable part of TDM was taken fixed. With this constraint, 5895429 incidence matrices were constructed, but 978653 non-isomorphic copies were isolated among them (Table 5).

TABLE V
Result comparison for one TDM

	$numincmat$	$niso$	$\|Aut(\mathcal{D})\|$
C	241314960	$\geq$ 1000000	3(1000000)
CAC (partly)	5895429	978653	1(700745),3(277908)

Having now in mind the results of all three experiments, the next statement can be pronounced.

Proposition 4: There are at least 1700745 2-designs with parameters $(21, 6, 4)$.

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