International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:8, No:1, 2014

Best Co-approximation and Best Simultaneous Co-approximation in Fuzzy Normed Spaces

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Abstract—The main purpose of this paper is to consider the t-best co-approximation and t-best simultaneous co-approximation in fuzzy normed spaces. We develop the theory of t-best co-approximation and t-best simultaneous co-approximation in quotient spaces. This new concept is employed us to improve various characterisations of t-co-proximinal and t-co-Chebyshev sets.

Keywords—Fuzzy best co-approximation, fuzzy quotient spaces, proximinality, Chebyshevity, best simultaneous co-approximation.

I. Introduction

The concept of best co-approximation was introduced by Franchetti and Furi [3], in order to study some characteristic properties of real Hilbert spaces among real reflexive Banach spaces, and such problems were considered further by Papini and Singer, [16] and Geetha S. Rao and Saravanan [8].Subsequently, Geetha S. Rao et. al have developed the theory of best co-approximation to a considerable extent[4], [5], [6]. Further there are some results on co-approximation in [13], [15]. In [7], Geetha S. Rao and Saravanan obtained some theorems on best co-approximation in quotient spaces. Modarres and Dehghani [14] introduced and to discussed the concept of the best simultaneous co-approximation in normed linear spaces, which is the generalization of best co-approximation in normed spaces and also studied best simultaneous co-approximation in quotient spaces.

One of the most important problems in fuzzy topology is to obtain an appropriate concept of fuzzy metric space and fuzzy normed spaces.This problem has been investigated by many authors [1], [2], [9], [11], [12] from different point of view. In particular, George and Veeramani [9] had introduced and studied the notion of fuzzy metric space with the help of continuous t- norms, which constitutes a slight but appealing modification of the one due to Kramosil and Michalak [12]. Veeramani [19] in 2001 introduced the concept of t-best approximation in fuzzy metric spaces and Also, Vaezpour and Karimi [18] have introduced the concept of t-best approximation in fuzzy normed spaces. Goudarzi and Vaezpour [10] considered the set of all t-best simultaneous approximation in fuzzy normed linear spaces.

In this paper we consider the set of all t-best co-approximation and t-best simultaneous co-approximation in fuzzy normed spaces and we develop the theory of t-best co-approximation and t-best simultaneous co-approximation in quotient spaces. This new concept is employed us to improve various characterizations of t-co-proximinal and t-co-Chebyshev sets.

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II. Preliminaries

Definition 1: A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is said to be continues t-norm if $([0,1], *)$ is a topological monoid with unit 1 such that $a*b \leq c*d$ whenever $a \leq c$ and $b \leq d$ $(a, b, c, c \in [0,1])$. We call $*_1 \leq *_2$ if $a *_1 b \geq a *_2 b$ for all $a, b \in [0,1]$.

Definition 2: The 3-tuple $(X, N, *)$ is said to be a fuzzy normed space if X is a vector space, $*$ is a continuous t-norm and N is a fuzzy set on $X \times (0, \infty)$ satisfying the following conditions for every $x, y \in X$ and $s, t > 0$

(i) $N(x,t) > 0$
(ii) $N(x,t) = 1 \Leftrightarrow x = 0$
(iii) $N(\alpha x, t) = N(x, t/ \mid \alpha \mid)$ for all $\alpha \neq 0$
(iv) $N(x,t) * N(y,s) \leq N(x+y, t+s)$
(v) $N(x, \cdot) : (0,\infty) \to [0,1]$ is continuous
(vi) $lim_{t\to\infty} N(x,t) = 1$.

Lemma 1: Let N be a fuzzy norm. Then:

(i) $N(x,t)$ is non decreasing with respect to t for each $x \in X$.
(ii) $N(y-x,t) = N(x-y,t)$.

Example 1: Let $(X, \|\cdot\|)$ be a normed space. We define $a*b = ab$ or $a*b = min\{a,b\}$ and

$$N(x,t) = \frac{kt^n}{kt^n + m\|x\|}, k, m, n \in \mathbb{R}^+.$$

Then $(X, N, *)$ is a fuzzy normed space. In particular if $k = m = n = 1$ we have

$$N(x,t) = \frac{t}{t + \|x\|},$$

which is called the standard fuzzy norm induced by the norm $\|\cdot\|$.

Remark 1: In [17], it was shown that every fuzzy norm induces a fuzzy metric and so every fuzzy normed space is a topological space.

Definition 3: Let $(X, N, *)$ be a fuzzy normed space. The open and closed ball $B(x,r,t)$ and $B[x,r,t]$ with the center $x \in X$, radius $0 < r < 1$ and $t > 0$ are defined as follows:

$$B(x,r,t) = \{y \in X : N(x-y,t) > 1-r\}.$$

$$B[x,r,t] = \{y \in X : N(y-x,t) \geq 1-r\}.$$

Proposition 1: [9] Let $(X, N, *)$ be a fuzzy metric space. Then M is a continuous function on $X \times X \times (0, \infty)$.

Corollary 1: [9] Let $(X, N, *)$ be a fuzzy normed space. Then $N : X \times (0, \infty) \to [0, 1]$ is continuous.

Remark 2: [9] For any $r_1 > r_2$, we can find r_3 such that $r_1 * r_3 \geq r_2$ and for any r_4 we can find r_5 such that $r_5 * r_5 \geq r_4$, $(r_1, r_2, r_3, r_4, r_5 \in (0, 1))$.

III. t-Best Co-approximation

Definition 4: Let $(X, N, *)$ be a fuzzy normed space. A subset $G \subseteq X$ is called F-bounded if there exists $t > 0$ and $0 < r < 1$ such that $N(x, t) > 1 - r$ for all $x \in G$.

Definition 5: Let $(X, N, *)$ be a fuzzy normed space, G be a nonempty subset of X. An element $g_0 \in G$ is called a t-best co-approximation to x from G if for $t > 0$, we define,

$$N(g_0 - g, t) \geq N(x - g, t).$$

for all $g \in G$

The set of all t-best simultaneous co-approximation to x from G will be denoted by $R^t_G(x)$ and we have

$$R^t_G(x) = \{g_0 \in G : N(g_0 - g, t) \geq N(x - g, t), \forall g \in G\}.$$

The set-valued function R^t_G which is associated with each $x \in X$, the set of all its t-best co-approximations, is called the t-cometric projection operator. For $t > 0$ putting $\check{G}^t_x = \{x \in X : N(g, t) \geq N(g - x, t), \forall g \in G\} = (R^t_G)^{-1}(\{0\})$. It is clear that $g_0 \in R^t_G(x)$ if and only if $x - g_0 \in \check{G}^t_x$.

Definition 6: Let G be a subset of $(X, N, *)$. If each $x \in X$ has at least (respectively exactly) one t-best co-approximation in G, the G is called a t-co-proximinal (respectively t-co-Chebyshev) set.

Definition 7: Let $(X, N, *)$ be a fuzzy normed space. A subset G of X is said to be convex if $(1 - \lambda)x + \lambda g_0 \in G$ whenever $g_0 \in G, x \in X$ and $0 < \lambda < 1$.

Theorem 1: Let $(X, N, *)$ be a fuzzy normed space and G is nonempty subset of X. Let K be a F-bounded in X. If $g_0 \in R^t_G(x)$ and $(1 - \lambda)x + \lambda g_0 \in G$ for $0 < \lambda < 1$, then $(1 - \lambda)x + \lambda g_0 \in R^t_G(x)$.

Proof: Since $g_0 \in R^t_G(x)$, then we have:

$$N(g_0 - g, t) \geq N(x - g, t) \text{ for all } g \in G \tag{1}$$

Therefore, for a given $t > 0$, take the natural number n such that $t > \frac{1}{n}$. By assumptions and Definition 2, we have,

$$\begin{aligned}
&N([(1 - \lambda)x + \lambda g_0] - g, t) \\
&= N([1 - \lambda)x - \lambda g + \lambda g + \lambda g_0] - g, t) \\
&= N((1 - \lambda)x - (1 - \lambda)g + \lambda(g_0 - g), t) \\
&= N((1 - \lambda)(x - g) + \lambda(g_0 - g), t) \\
&\geq N(x - g, \frac{1}{(1 - \lambda)n}) * N(g_0 - g, \frac{t}{\lambda n}) \\
&\geq N(x - g, \frac{1}{(1 - \lambda)n}) * N(x - g, \frac{t}{\lambda n}), \text{ Since (1)} \\
&\geq \lim_{n \to \infty} N(x - g, \frac{t}{\lambda n}) \\
&= N(x - g, t)
\end{aligned}$$

Thus, $(1 - \lambda)x + \lambda g_0 \in R^t_G(x)$ ■

Theorem 2: Let $(X, N, *)$ be a fuzzy normed space. If G is convex subset of X, then $R^t_G(x)$ is a convex subset G of X.

Proof: It is obvious by Theorem 1. Hence $R^t_G(x)$ is convex. ■

Example 2: Let $X = \mathbb{R}^2$. For $a * b = ab$. Define $N : \mathbb{R}^2 \times (0, \infty) \to [0, 1]$ by

$$N((x, y), t) = \frac{| x | + | y |}{t}$$

Then $(X, N, *)$ is a fuzzy normed space. Let $G = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$ is a subset of $X$and $x = (-1, 1)$. Then for any $(g_1, g_2) \in G$ and every $t > 0$,

$$\begin{aligned}
N((0, 1) - (g_1, g_2), t) &= N((-g_1, 1 - g_2), t) \\
&= \frac{| g_1 | + | 1 - g_2 |}{t} \\
&= \frac{g_1 + | 1 - g_2 |}{t} \\
&\geq \frac{1 + g_1 + | 1 - g_2 |}{t} \\
&= N((-1 - g_1, 1 - g_2), t) \\
&= N((-1, 1) - (g_1, g_2), t)
\end{aligned}$$

So for every $t > 0$, $g_0 = (0, 1)$ is t-best co-approximation of $(-1, 1)$ from G. Therefore $g_0 = (0, 1) \in R^t_G(-1, 1)$ and for $\lambda = \frac{1}{2}$, by Definition 7, we have, $(1 - \lambda)(-1, 1) + \lambda(0, 1) = (1 - \frac{1}{2})(-1, 1) + \frac{1}{2}(0, 1) = (\frac{1}{2}, 0) \in R^t_G(-1, 1)$. Therefore $(\frac{1}{2}, 0) \notin \partial(G)$.

Theorem 3: (Invariance by translation and scalar multiplication) Let G be a nonempty subset of a fuzzy normed space $(X, N, *)$. Then:

1) $R^{|\alpha| t}_{\alpha G}(\alpha x) = \alpha R^t_G(x)$ for every $x \in X$, $t > 0$ and $\alpha \in \mathbb{R} \backslash \{0\}$.
2) $R^t_{G+y}(x + y) = R^t_G(x) + y$ for every $x, y \in X$ and $t > 0$.

Proof: (1) $g_0 \in R^{|\alpha| t}_{\alpha G}(\alpha x)$ if and only if, $g \in \alpha G$ and $N(g_0 - g, | \alpha | t) \geq N(\alpha x - g, | \alpha | t)$, if and only if $\frac{1}{\alpha} g \in G$ and $N(\frac{1}{\alpha} g_0 - \frac{1}{\alpha} g, t) \geq N(x - \frac{1}{\alpha} g, t)$, if and only if $g_1 = \frac{1}{\alpha} g \in G$ and $N(\frac{1}{\alpha} g_0 - g_1, t) \geq N(x - g_1, t)$, if and only if $\frac{1}{\alpha} g_0 \in R^t_G(x)$. However, this is equivalent to $g_0 \in \alpha R^t_G(x)$. Therefore $R^{|\alpha| t}_{\alpha G}(\alpha x) = \alpha R^t_G(x)$.
(2) $g_0 \in R^t_{G+y}(x + y)$ if and only if $g + y \in G + y$ and $N(g_0 - (g + y), t) \geq N(x - g, t)$ if and only if $N((g_0 - y) - g), t) \geq N(x - g, t)$, for all $g \in G$, if and only if $g_0 - y \in R^t_G(x)$, and this is equivalent to $g_0 \in R^t_G(x) + y$. Therefore $R^t_{G+y}(x + y) = R^t_G(x) + y$. ■

Corollary 2: Let G be a nonempty subset of a fuzzy normed space $(X, N, *)$. The following statements are hold:

1) G is t-co-proximinal (resp. t-co-Chebyshev) if and only if $| \alpha | G$ is $| \alpha |$t-co-proximinal (resp. $| \alpha |$t-co-Chebyshev) for any scalar $\alpha \in \mathbb{R} \backslash \{0\}$.
2) G is t-co-proximinal (resp. t-co-Chebyshev) if and only if $G + y$ is t-co-proximinal (resp. t-co-Chebyshev) for every $y \in X$.

Proof: (1) G is t-co-proximinal if and only if $R^t_G(x) \neq \emptyset$, if and only if $\alpha R^t_G(x) \neq \emptyset$, if and only if $R^{|\alpha| t}_{\alpha G}(\alpha x) \neq \emptyset$ (by theorem 3 (1)). Hence $| \alpha | G$ is t-co-proximinal.

(2) G is t-co-proximinal if and only if $R^t_G(x) \neq \emptyset$, if and only if $\alpha R^t_G(x) + y \neq \emptyset$, if and only if $R^t_G(x+y) \neq \emptyset$. Hence $G + y$ is t-co-proximinal. ■

Corollary 3: Let G be a nonempty subspace of X. Then for $t > 0$,

1) $R^{|\alpha|t}_G(\alpha x) = \alpha R^t_G(x)$, for $\alpha \neq 0 \in \mathbb{C}$
2) $R^t_{G+y}(x+y) = R^t_G(x) + y$ for every $x, y \in X$.

IV. t-co-proximinality and t-co-Chebyshevity in Quotient Spaces

In this section we give characterizations of simultaneous t-co-proximinality and simultaneous t-co-Chebyshevity in quotient spaces. First we remind that if $(X, N, *)$ is a fuzzy normed space and M is a linear manifold in X, for $t > 0$ and $x \in X$, let $d(x, M, t)$ denote the distance between x and M, i.e.,

$$d(x, M, t) = \sup_{y \in M} N(x - y, t)$$

Then the quotient space X/M is equipped with the fuzzy norm

$$N(x + M, t) = \sup_{y \in M} N(x + y, t).$$

It has been proved in [17] that N is a fuzzy norm on X/M. Also $Q : X \to X/M$ is the natural map, $Qx = x + M$ and the followings hold,

1) $N(Qx, t) \geq N(x, t)$.
2) If $(X, N, *)$ is a fuzzy Banach space then $(X/M, N, *)$ is a fuzzy Banach space.

Theorem 4: Let M be a closed subspace of $(X, N, *)$ and $G \supseteq M$ a subspace of X. If G is t-co-proximinal of X, then G/M is a t-co-proximinal subspace of X/M.

Corollary 4: Let M be a closed subspace of $(X, N, *)$ and $G \supseteq M$ a subspace of X. If G/X is t-co-proximinal with X/M, then G is t-co-proximinal with X.

Theorem 5: Let M be a t-co-Chebyshev subspace of $(X, N, *)$ and $G \supseteq M$ a subspace of X. If G/X is t-co-Chebyshev with X/M, then G is t-co-Chebyshev with X.

Proof: By hypothesis G/M is t-co-Chebyshev. Then some F-bounded subset K of X has distinct t-best co-approximations such as y_1 and y_2 in G/M. Thus we have,

$$y_1, y_2 \in R^t_G(K).$$

It is clear that,

$$y_1 + M, y_2 + M \in R^t_{G/M}(K/M).$$

Since G/M is t-co-Chebyshev, $y_1 + M = y_2 + M$, then $y_1 - y_2 \in M$. Now since $y_1, y_2 \in R^t_G(M)$, then there exists $x - y_1$ and $x - y_2$ in $\check{G}$ and $G \supseteq M$; therefore $x - y_1$ and $x - y_2$ are in $\check{M}$. So there exists $0 \in R^t_G(\check{M})$ and also $y_1 - y_2 \in R^t_G(x - y_2)$. Since M is t-co-Chebyshev, $y_1 = y_2$. ■

Theorem 6: Let M be a closed subspace of $(X, N, *)$ and $G \supseteq M$ a subspace of X. If G is t-co-Chebyshev of X, then G/M is a t-co-Chebyshev with X/M.

Proof: Assume that the claim is does not hold. Then for some F-bounded subset K of $X, K/M$ has two distinct t-best co-approximations such as $g_1 + M$ and $g_2 + M$ in G/M. Thus $g_1 - g_2$ is not in M. By Corollary 4, there exist t-best co-approximations m_1 and m_2 form M, such that $m_1 + M$ and $m_2 + M$ are in $R^t_{G/M}(K/M)$. But G is t-co-Chebyshev. Thus $g_1 + m_1 = g_2 + m_2$ and so $g_1 - g_2$ belongs to M, which is a contradiction. ■

Theorem 7: Let M be a t-co-proximinal subspace of $(X, N, *)$, $G \supseteq M$ a subspace of X. Then for each F-bounded set K in X,

$$Q(R^t_G(M)) = R^t_{G/M}(K/M).$$

Proof: It does not need the t-coproximinality of M for the inclusion

$$Q(R^t_G(M)) \subseteq R^t_{G/M}(K/M).$$

Suppose $g \in R^t_G(M), a \in G$; then for every $b \in M$,

$$\begin{aligned} N((g + M) - (a + M), t) &= d(g - a, M) \\ &\geq N(g - (a + b), t) \\ &\geq N(x - (a + b), t) \\ &= N((x - a) + b, t) \\ &\geq N((x + M) - (a + M), t) \end{aligned}$$

It follows that

$$Q(R^t_G(M)) = R^t_{G/M}(K/M).$$

■

Theorem 8: Let M and G be subspaces of a fuzzy normed space $(X, N, *)$ such that $M \subset G$ and let $x \in X/G$ and $g_1 \in G$. If g_1 is a t-best co-approximation to x from G, then $g_1 + M$ is a t-best co-approximation to $x + M$ from the quotient space G/M.

Proof: Assume that $g_1 + M$ is not a t-best co-approximation to $x + G$ from G/M. Then there exists $g'_1 + M \in G/M$ such that

$$N(g'_1 + M - (g_1 + M), t) < N(x + M - (g'_1 + M), t)$$

That is,

$$N(g'_1 - g_1 + M, t) < N(x - g'_1 + M, t)$$

That is,

$$d(x - g'_1, M, t) > d(g'_1 - g_1, M, t).$$

This implies that there exists $g \in M$ such that

$$N(x - g'_1 - g, t) > d(g'_1 - g_1, M, t) > N(g'_1 - g_1 + g, t).$$

That is,

$$N((g + g'_1) - g_1, t) < N(x - (g + g'_1), t).$$

Thus g_1 is not a t-best approximation to x from G, a contradiction. ■

Theorem 9: Let $(X, N, *)$ is a fuzzy normed space and M is a linear manifold in X and let G be a t-coproximinal subspace of X containing M. Then

$$Q(\check{G}^t_x) \subseteq (R^t_{(G/M)})^{-1}(M).$$

Proof: If $x \in \check{G}^t_x$ and $g \in G$, then for every $m \in M$,

$$\begin{aligned} N(g + M, t) &\geq N(g + m, t) \\ &\geq N(g - x + m, t) \\ &\geq N((g - x) + M, t) \end{aligned}$$

This implies that $Q(x) \subseteq (R^t_{(G/M)})^{-1}(M)$. ■

Theorem 10: Let M be a t-proximinal subspace of a fuzzy normed space $(X, N, *)$ and let G be a t-coproximinal subspace of X containing M such that $R^t_G(x)$ is t-compact, for every $x \in X$. Then, $R^t_{G/M}(x+M)$ is t-compact for every $x \in X$.

V. t-Best Simultaneous Co-approximation

Definition 8: Let $(X, N, *)$ be a fuzzy normed space, G be a subset of X and M be a F-bounded subset in X. An element $g_0 \in G$ is called a t-best simultaneous co-approximation to M from G if for $t > 0$, we define,

$$N(g_0 - g, t) \geq \inf_{m \in M} N(m - g, t).$$

The set of all t-best simultaneous co-approximation to M from G will be denoted by $\mathcal{R}^t_G(M)$ and we have

$$\mathcal{R}^t_G(M) = \left\{ g_0 \in G : N(g_0 - g, t) \geq \inf_{m \in M} N(m - g, t) \right\}.$$

Definition 9: Let G be a subset of $(X, N, *)$. It is called a simultaneous t-co-proximial subset of X if for each F-bounded set M in X, there exists at least one t-best simultaneous co-approximation from G to M. Also it is called a simultaneous t-co-Chebyshev subset of X if for each F-bounded set M in X there exists a unique simultaneous t-best co-approximation from G to M.

Theorem 11: Suppose that G is a subset of $(X, N, *)$ and M is F-bounded in X. Then $\mathcal{R}^t_G(M)$ is a F-bounded subset of X and if G is convex and is a closed subset of X and $*$ has the condition $a * b \geq a$ for all $a, b \in [0, 1]$, then $\mathcal{R}^t_G(M)$ is closed and is convex for each F-bounded subspace M of X.

Proof: Since M is F-bounded, there exist $t > 0$ and $< r < 1$ such that $N(x, t) > 1 - r$, for all $x \in M$. If $g_0 \in \mathcal{R}^t_G(M)$, then

$$N(g_0 - g, t) \geq \inf_{m \in M} N(m - g, t).$$

Now, for all $m \in M$ and $g_0 \in \mathcal{R}^t_G(M)$. Then it follows that for every $g \in G$,

$$\begin{aligned} N(g_0, 3t) &\geq N(g_0 - m, 2t) * N(m, t) \\ &\geq N(g_0 - g + g - m, 2t) * (1 - r) \\ &\geq N(g_0 - g, t) * N(g - m, t) * (1 - r) \end{aligned}$$

$$\begin{aligned} N(g_0, 3t) &\geq \inf_{m \in M} N(m - g, t) * N(m - g, t) * (1 - r) \\ &\geq \inf_{m \in M} N(m - g, t) * (1 - r) \\ &\geq (1 - r_0). \end{aligned}$$

for some $0 < r_0 < 1$. Then $\mathcal{R}^t_G(M)$ is F-bounded.

Suppose that G is convex and is closed subset of X. We show that $\mathcal{R}^t_G(M)$ is convex and closed. Let $x, y \in \mathcal{R}^t_G(M)$ and $0 < \lambda < 1$. Since G is convex, there exists $(1-\lambda)x + \lambda g_0 \in G$ such that for each $0 < \lambda < 1$ and for a given $t > 0$, take the natural number n such that $t > \frac{1}{n}$. By assumption and definition of 2, we have

$$\begin{aligned} & N([(1 - \lambda)m + \lambda g_0] - g, t) \\ &= N([1 - \lambda)m - \lambda g + \lambda g + \lambda g_0] - g, t) \\ &= N((1 - \lambda)m - (1 - \lambda)g + \lambda(g_0 - g), t) \\ &= N((1 - \lambda)(m - g) + \lambda(g_0 - g), t) \\ &\geq N(m - g, \frac{1}{(1 - \lambda)n}) * N(g_0 - g, \frac{t}{\lambda n}) \end{aligned}$$

$$\begin{aligned} & N([(1 - \lambda)m + \lambda g_0] - g, t) \\ &\geq N(m - g, \frac{1}{(1 - \lambda)n}) * \inf_{m \in M} N(m - g, \frac{t}{\lambda n}) \\ &= \lim_{n \to \infty} \inf_{m \in M} N(m - g, \frac{t}{\lambda n}) \\ &= \inf_{m \in M} N(m - g, t) \end{aligned}$$

So $\mathcal{R}^t_G(M)$ is convex.

Finally, let $\{g_{0_n}\} \subset \mathcal{R}^t_G(M)$ and suppose $\{g_{0_n}\}$ converges to some g_0 in X. Since $\{g_{0_n}\} \subset G$ and G is closed to $g_0 \in G$. Therefore by Corollary 1 for $t > 0$ we have

$$\begin{aligned} N(g_0 - g, t) &= N(\lim_{n \to \infty} g_{0_n} - g, t) \\ &= \lim_{n \to \infty} N(g_{0_n} - g, t) \\ &\geq \inf_{m \in M} N(m - g, t) \end{aligned}$$

■

Theorem 12: The following assertions are hold for $t > 0$,

1) If $0 \in G$ or if G is a subset of $(X, N, *)$, then
$$N(g_0, t) \geq \inf_{m \in M} N(m, t),$$
2) $\mathcal{R}^t_{G+x}(M + x) = \mathcal{R}^t_G(M) + x$, $\forall x \in X$,
3) $\mathcal{R}^{|\alpha|t}_{\alpha G}(\alpha M) = \alpha \mathcal{R}^t_G(M)$, $\forall \alpha \in \mathbb{R}$.

Proof: The proof of (1) is obvious.

(2) The proof is obvious. Indeed, let $g_0 \in \mathcal{R}^t_G(M)$. Then for every $g_1 \in G$,

$$\begin{aligned} N(g_1 - (g_0 + x), t) &= N(g_1 - x - g_0, t) \\ &\geq \inf_{m \in M} N(m - (g_1 - x), t) \\ &= \inf_{m \in M} N(m + x - g_1, t) \end{aligned}$$

Thus $g_0 + x \in \mathcal{R}^t_G(M + x)$.

Conversely, let $g_0 + x \in \mathcal{R}^t_G(M + x)$. Then it is sufficient to prove that $g_0 \in \mathcal{R}^t_G(M)$. For every $g_1 \in G$, it follows that

$$\begin{aligned} N(g_1 - g_0, t) &= N(g_1 + x - (g_0 + x), t) \\ &\geq \inf_{m \in M} N(m + x - (g_1 + x), t) \\ &= \inf_{m \in M} N(m - g_1, t) \end{aligned}$$

Therefore $\mathcal{R}^t_G(M + x) = \mathcal{R}^t_G(M) + x$, $\forall x \in X$.

(3) Clearly equality holds for $\alpha = 0$, so suppose that $\alpha \neq 0$. Then, $g_0 \in \mathcal{R}^{|\alpha|t}_{\alpha G}(\alpha M)$ if and only if $g_0 \in \alpha G$ and $N(g_0 - \alpha g, t) \geq \inf_{m \in M} N(\alpha m - \alpha g, t)$ for all $g \in G$ if and only if $\frac{g_0}{\alpha} \in G$ and $N(\frac{g_0}{\alpha} - g, \frac{t}{|\alpha|}) \geq \inf_{m \in M} N(m - g, \frac{t}{|\alpha|})$ for

all $g \in G$ if and only if $\frac{g_0}{\alpha} \in \mathcal{R}^t_G(M)$ if and only if $g_0 \in \alpha\mathcal{R}^t_G(M)$. Therefore, $\mathcal{R}^{|\alpha|t}_{\alpha G}(\alpha M) = \alpha\mathcal{R}^t_G(M)$, ■

Remark 3: Theorem 7 (1) and (2) can be restated as:

$$\mathcal{R}^t_G(\alpha M + g) = \alpha\mathcal{R}^t_G(M) + g, \forall g \in G.$$

Corollary 5: Let A be a non empty subset of a fuzzy normed space $(X, N, *)$. The following statements are hold.

1) A is simultaneous t-co-proximinal (resp. simultaneous t-co-Chebyshev) if and only if $A + y$ is simultaneous t-co-proximinal (resp. simultaneous t-co-Chebyshev), for each $y \in X$.
2) A is simultaneous t-co-proximinal (resp. simultaneous t-co-Chebyshev) if and only if αA is simultaneous $| \alpha |$t-co-proximinal (resp. simultaneous $| \alpha |$t-co-Chebyshev), for each $\alpha \in \mathbb{R}$.

Proof: (1) A is simultaneous t-co-proximinal (resp. simultaneous t-co-Chebyshev) if and only if $\mathcal{R}^t_A(x) \neq \emptyset$ if and only if $\mathcal{R}^t_A(x) + y \neq \emptyset$, for any $y \in X$ if and only if $\mathcal{R}^t_{A+y}(x + y) \neq \emptyset$ if and only if $A + y$ is simultaneous t-co-proximinal (resp. simultaneous t-co-Chebyshev)
(2) A is simultaneous t-co-proximinal (resp. simultaneous t-co-Chebyshev) if and only if $\mathcal{R}^t_A(x) \neq \emptyset$ if and only if $\alpha\mathcal{R}^t_A(x) \neq \emptyset$, for any $y \in X$ if and only if $\mathcal{R}^t_{\alpha A}(\alpha x) \neq \emptyset$ if and only if αA is simultaneous $| \alpha |$t-co-proximinal (resp. simultaneous $| \alpha |$t-co-Chebyshev) ■

Corollary 6: Let G be a nonempty subspace of X and M be a $F-$bounded subset of X. Then for $t > 0$.

1) $\mathcal{R}^t_G(M + x) = \mathcal{R}^t_G(M) + x, \forall x \in G$,
2) $\mathcal{R}^{|\alpha|t}_G(\alpha M) = \alpha\mathcal{R}^t_G(M)$, for $\alpha \in \mathbb{R}\backslash\{0\}$.

Proposition 2: Let G be a nonempty subspace of $(X, N, *)$ and M be a F-bounded subset of X. If $g_0 \in \mathcal{R}^t_G(M)$, then $g_0 \in \mathcal{R}^t_G(\alpha^n m + (1 - \alpha^n)g_0)$, $\alpha > 1$, $n = 0, 1, ...$

Proof: Let $g_0 \in \mathcal{R}^t_G(M)$. Claim: $g_0 \in \mathcal{R}^t_G(\alpha m + (1 - \alpha)g_0)$ Now

$$\begin{aligned}
&N(g_0 - g, 2t)\\
&\geq N(m - g, 2t)\\
&\geq \inf_{m\in M} N(\alpha m + (1 - \alpha)g_0 - g, 2t)\\
&= \inf_{m\in M} N(\alpha(m - g) + (1 - \alpha)(g_0 - g), 2t)\\
&\geq \inf_{m\in M} N(\alpha(m - g), t) * N((1 - \alpha)(g_0 - g), t)\\
&\geq \inf_{m\in M} N(m - g, \frac{t}{\alpha}) * N(g_0 - g, \frac{t}{(1 - \alpha)})\\
&\geq N(g_0 - g, \frac{t}{\alpha}) * N(g_0 - g, \frac{t}{(1 - \alpha)})\\
&= N(g_0 - g, t)
\end{aligned}$$

By repeated application of the claim it follows that

$$g_0 \in \mathcal{R}^t_G(\alpha^n m + (1 - \alpha^n)g_0).$$

■

Theorem 13: Let G is a subset of $(X, N, *)$ and M is F-bounded in X. If $g_0 \in G$ is a t-best co-approximation to $\alpha m_1 + (1 - \alpha)m_2$ for some $\alpha \in [0, 1]$ and $m_1, m_2 \in M$ and $*$ has the condition $m = \min\{m_1, m_2\}$ for all $m \in M$, then g_0 is a t-best simultaneous co-approximation to M from G.

Proof: Assume that g_0 is a t-best co-approximation to $\alpha m_1 + (1 - \alpha)m_2$ for some $\alpha \in [0, 1]$. Then for every $g \in G$, it follows that

$$\begin{aligned}
&N(g - g_0, 2t)\\
&\geq N(\alpha m_1 + (1 - \alpha)m_2 - g, 2t)\\
&= N(\alpha(m_1 - g) + (1 - \alpha)(m_2 - g), 2t)\\
&\geq N(\alpha(m_1 - g), t) * N((1 - \alpha)(m_2 - g), t)\\
&= N(m_1 - g, \frac{t}{| \alpha |}) * N(m_2 - g, \frac{t}{| 1 - \alpha |})\\
&\geq \inf_{m_1\in M} N(m_1 - g, \frac{t}{| \alpha |}) * \inf_{m_2\in M} N(m_2 - g, \frac{t}{| 1 - \alpha |})\\
&\geq \inf_{m\in M} N(m - g, t)
\end{aligned}$$

Thus g_0 is a t-best simultaneous co-approximation to M. ■

VI. Simultaneous t-co-proximinality and Simultaneous t-co-Chebyshevity in Quotient Spaces

Lemma 2: Let $(X, N, *)$ be a fuzzy normed space and M be a t-co-proximinal subspace of X. For each nonempty F-bounded set S in X and $t > 0$,

$$N(g_0 - g, t) = \sup_{g\in M} \inf_{s\in S} N(s - g, t).$$

Proof: Since M is t-co-proximinal it follows that for each $m \in M$ there exists $m_g \in \mathcal{R}^t_M(S)$ such that for $t > 0$,

$$N(s - g_s, t) = \sup_{g\in M} N(s - g, t).$$

So,

$$\begin{aligned}
N(g_0 - g, t) &\geq \inf_{s\in S} N(s - g, t)\\
&\geq \inf_{s\in S} N(s - g_s, t)\\
&= \inf_{s\in S} \sup_{g\in M} N(s - g, t)\\
&\geq \sup_{g\in M} \inf_{s\in S} N(s - g, t)\\
&\geq \sup_{g\in M} N(g_0 - g, t)\\
&\geq N(g_0 - g, t)
\end{aligned}$$

This implies that,

$$N(g_0 - g, t) = \sup_{g\in M} \inf_{s\in S} N(s - g, t)$$

■

Lemma 3: Let $(X, N, *)$ be a fuzzy normed space, M a t-co-proximinal subspace of X and S be an arbitrary subset of X. The following asserations are equivalent:

(i) S is a F-bounded subset of X.
(ii) S/M is a F-bounded subset of X/M.

Proof: Suppose that S be a F-bounded subset of X. Then there exist $t > 0$, $0 < r < 1$ such that,

$$N(x, t) > 1 - r, \quad (\forall x \in S)$$

But,

$$\begin{aligned} N(x+M,t) &= \sup_{y\in M} N(x+y,t) \\ &\geq N(x,t) \\ &\geq (1-r) \end{aligned}$$

So (i)→(ii) is proved. (ii)→(i). Let S/M be a F-bounded subset of X/M. Since M is t-co-proximinal, then for each $s \in S$ there exists $g_s \in M$ such that, $g_s \in \mathcal{R}^t_M(S)$. So for each $s \in S$,

$$N(s-g_s,t) = \sup_{g\in M} N(s-g,t) \tag{2}$$

Now from Lemma 2 we conclude that for $t > 0$,

$$\begin{aligned} \inf_{s\in S} N(s-g_s,t) &= \inf_{s\in S}\sup_{g\in M} N(s-g,t) \\ &= \sup_{g\in M}\inf_{s\in S} N(s-g,t) \end{aligned}$$

Then for $0 < r < 1$ such that $\inf_{s\in S} N(s-g_s,t) \geq r$ and $t > 0$ there exists $g_r \in M$ such that

$$\begin{aligned} \inf_{s\in S} N(s-g_r,t) &\geq \inf_{s\in S} N(s-g_s,t) - r \\ &\geq 0 \end{aligned}$$

So by (2), for all $s \in S$ we have,

$$\begin{aligned} N(s,t) &\geq N(s-g_r,\frac{t}{2}) * N(g_r,\frac{t}{2}) \\ &\geq \inf_{s\in S} N(s-g_r,\frac{t}{2}) * N(g_r,\frac{t}{2}) \\ &\geq (\inf_{s\in S} N(s-g_s,\frac{t}{2}) - r) * N(g_r,\frac{t}{2}) \\ &\geq (\inf_{s\in S}\sup_{g\in M} N(s-g,\frac{t}{2}) - r) * N(g_r,\frac{t}{2}) \\ &= (\inf_{s\in S} N(s+M,\frac{t}{2}) - r) * N(g_r,\frac{t}{2}) \end{aligned} \tag{3}$$

Since S/M is F-bounded, by its definition and remark 2, we can find $0 < r_0 < 1$ such that the last equation in the right hand side of (3) be greater than or equal to $1 - r_0$ and this completes the proof. ■

Theorem 14: Let M be a t-co-proximinal subspace of $(X, N, *)$ and $G \supseteq M$ a subspace of X. Let K be a F-bounded in X. If $g_0 \in \mathcal{R}^t_G(K)$, then $g_0 + M \in \mathcal{R}^t_{G/M}(K/M)$.

Proof: Since K is a F-bounded by Lemma 3, K/M is F-bounded in X/M. Assume that $g_0 \in \mathcal{R}^t_G(K)$, then g_0+M not in $\mathcal{R}^t_{G/M}(K/M)$. Thus there exists $g_1 \in G$ such that for $t > 0$,

$$\begin{aligned} \inf_{k\in K} N(k-(g_1+M,t) &> \inf_{k\in K} N(k-(g_0+M),t) \\ &\geq \inf_{k\in K} N(k-g_0,t) \end{aligned} \tag{4}$$

such that for each $k \in K$ and for $t > 0$,

$$N(k-(g_1+M),t) = \sup_{m\in M} N(k-(g_1+m),t).$$

Then for each $0 < \varepsilon < 1$ and $k \in K$ there exists $m_k \in M$ such that for $t > 0$,

$$N(k-(g_1+m_k),t) \geq N(k-(g_1+M),t) - \varepsilon.$$

Since $g_1 \in m_k \in G$ we conclude that

$$\begin{aligned} N(g-g_0,t) &\geq \inf_{k\in K} N(k-(g_1+m_k),t) \\ &\geq \inf_{k\in K} N(k-(g_1+M),t) - \varepsilon. \end{aligned} \tag{5}$$

Thus,

$$N(g-g_0,t) \geq \inf_{k\in K} N(k-(g_1+M),t)$$

■

Corollary 7: Let M be a t-proximinal subspace of $(X, N, *)$ and $G \subseteq M$ subspace of X. If G is simultaneous t-proximinal then for each F-bounded set K in X,

$$Q(\mathcal{R}^t_G(K)) \subseteq \mathcal{R}^t_{G/M}(K/M).$$

Theorem 15: Let M be a t-co-proximinal subspace of $(X, N, *)$. If $G \supseteq M$ subspace of X. Then for each F-bounded set K in X,

$$Q(\mathcal{R}^t_G(K)) = \mathcal{R}^t_{G/M}(K/M).$$

Proof: By Corollary 7 we obtain,

$$Q(\mathcal{R}^t_G(K)) \subseteq \mathcal{R}^t_{G/M}(K/M).$$

Also by Lemma 3, G/M is simultaneous t-co-proximinal in X/M. Now let,

$$g_0 + M \in \mathcal{R}^t_{G/M}(K/M),$$

where $g_0 \in G$. By simultaneous t-proximinality of M there exists $m_0 \in M$ such that $m_0 \in \mathcal{R}^t_M(K-g_0)$. Then in view of Theorem 14 we conclude that $g_0 + m_0 \in \mathcal{R}^t_G(K)$. Therefore $g_0 + M \in Q(\mathcal{R}^t_G(K))$ and the proof is complete. ■

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