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Bayesian Inference for Phase Unwrapping using Conjugate Gradient Method in One and Two Dimensions

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***Abstract*—**We investigated statistical performance of Bayesian inference using maximum entropy and MAP estimation for several models which approximated wave-fronts in remote sensing using SAR interferometry. Using Monte Carlo simulation for a set of wave-fronts generated by assumed true prior, we found that the method of maximum entropy realized the optimal performance around the Bayes-optimal conditions by using model of the true prior and the likelihood representing optical measurement due to the interferometer. Also, we found that the MAP estimation regarded as a deterministic limit of maximum entropy almost achieved the same performance as the Bayes-optimal solution for the set of wave-fronts. Then, we clarified that the MAP estimation perfectly carried out phase unwrapping without using prior information, and also that the MAP estimation realized accurate phase unwrapping using conjugate gradient (CG) method, if we assumed the model of the true prior appropriately.

***Keywords*—**Bayesian inference using maximum entropy, MAP estimation using conjugate gradient method, SAR interferometry.

I. Introduction

WAVE-FRONTS often carry information through noisy channel. Therefore, researchers [1]-[3] have developed many techniques to utilize wave-fronts for information communication both from theoretical and experimental points of view. Especially, many engineers have constructed optical instruments via interferometers to observe a set of phase differences in principal interval $[-\pi,+\pi]$. Also, researchers have constructed techniques to reconstruct original wave-fronts by making use of the set of principal phase differences from an interferogram observed by interferometer. Various techniques [1]-[3] have been proposed, such as least square estimation [4]-[7], Bayesian approaches [8], [9] using simulated annealing and a method of maximum entropy. In recent years, Saika and Nishimori [10], [11], Saika and Uezu [12] have studied phase unwrapping based on an analogy between statistical mechanics and Bayesian inference using the maximizer of the posterior marginal (MPM) estimate. Then, Marroquin and Rivera [13] have investigated the MAP estimation using conjugate gradient method (CG method) phase unwrapping in remote sensing using synthetic aperture radar (SAR) interferometry. They found that maximum of a posteriori (MAP) estimation via the CG method succeeded in phase unwrapping under several conditions. Sakaematsu and Saika [14] have improved performance of the MAP estimation using the CG method for two dimensional phase unwrapping. However, as they have not tried a systematic approach for this problem, it was not clarified criterion that the CG method based on the MAP estimation was effective for phase unwrapping in remote sensing using SAR interferometry.

Therefore, in this study, we tried a Bayesian inference using maximum entropy and maximum of a posteriori (MAP) estimation for one and two dimensional phase unwrapping in remote sensing using SAR interferometry both on the basis of the method of maximum entropy and the MAP estimation regarded as a deterministic limit of a method of maximum entropy. From the theoretical point of view, we first constructed the method of maximum entropy based on the Bayesian inference using a continuous spin system on the square lattice. Then, we estimated statistical performance of the present method based on performance measure using mean square error (MSE) for a set of wave-fronts generated by an assumed true prior expressed as a Boltzmann factor of the continuous spin model in one dimension. Especially, we tried the Bayesian inference using maximum entropy including the Bayes-optimal solution which was realized by using the model of the assumed true prior and the likelihood representing optical measurement using interferometer. Using numerical simulation for the set of the wave-fronts, we found that the optimal performance was achieved around the Bayes-optimal condition within statistical uncertainty, and that by making use of both an assumed model of the true prior and a likelihood representing optical measurement using the model prior. We found that the optimal performance was realized around the Bayes-optimal condition and also that the optimal performance achieved by the Bayes-optimal solution was almost similar to the performance of the MAP estimation regarded as the deterministic limit of the method of maximum entropy. Therefore, in order to construct a practical method, we tried the conjugate gradient method based on the MAP estimation for several artificial models approximating wave-fronts in remote sensing using the SAR interferometry. Using numerical simulation, we found that the MAP estimation was successful in phase unwrapping, if we set parameters appropriately.

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The content of this paper is organized as follows. First, we outline Bayesian inference using the method of maximum

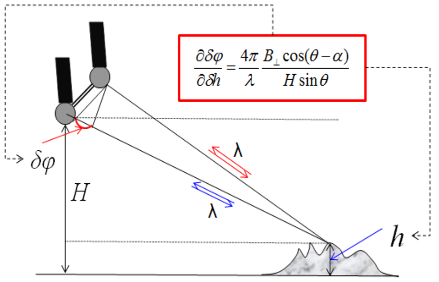

Fig. 1 Remote sensing using SAR interferometry

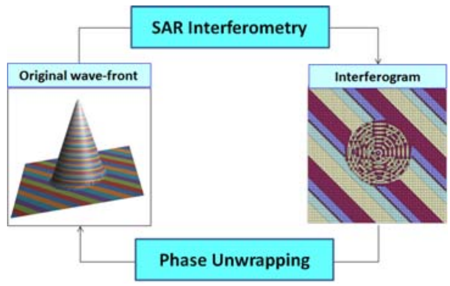

Fig. 2 Optical measurement via SAR interferometry and phase unwrapping

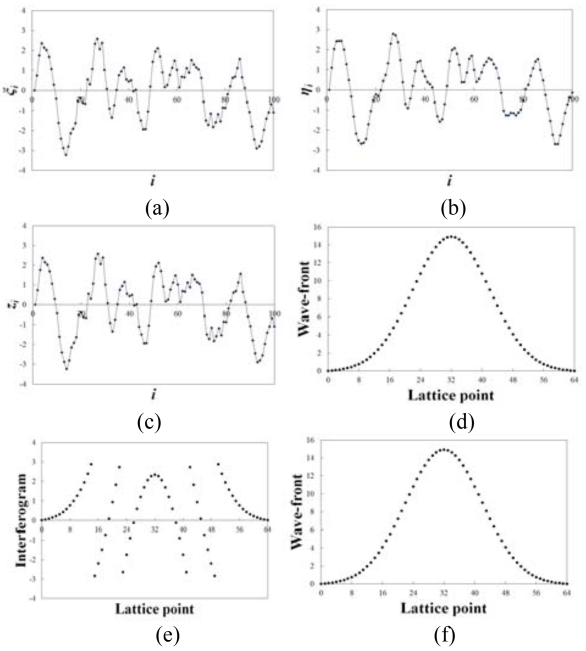

Fig. 3 (a) an original wave-front generated by the assumed true prior in eq. (1), (b) an interferogram of the wave-front in (a), (c) a reconstructed wave-front obtained by the method of maximum entropy under the Bayes-optimal condition, (d) an artificial model approximating one dimensional wave-front in remote sensing in SAR interferometry, (e) an interferogram of the original wave-front in (d), (f) a reconstructed wave-front obtained by the method of maximum entropy without using prior information

entropy for phase unwrapping for several artificial models representing wave-fronts in remote sensing using the SAR interferometry. Then, we show the statistical performance both of the method of maximum entropy and the MAP estimation for the set of wave-fronts in remote sensing using the SAR interferometry. Last part is devoted to summary and discussion.

II. General Formulation

In this chapter, we outlined general formulation in Fig. 1 for phase unwrapping in remote sensing using SAR interferometry (Fig. 2) both in one and two dimensions.

In this formulation, we first consider an original wave-front $\{\xi_i\}/\{\xi_{x,y}\}$ both in one and two dimensions, where $0<\xi_i(\xi_{x,y})<R_0$, $i=1,\ldots,L(x, y=1,\ldots,L)$. As shown in Fig. 3(a), we consider a set of one dimensional wave-fronts generated by an assumed true prior expressed by a Boltzmann factor of continuous spin system as

$$\Pr(\{\xi\}) = \frac{1}{Z_s}\exp\left[-\frac{J_s}{T_s}\sum_{i=1}^{L}(\xi_{i-1}-2\xi_i+\xi_{i+1})^2\right]. \quad (1)$$

Here, Z_s is a normalization factor and then J_s and T_s are hyper-parameters. Also, as shown in Fig. 3(d) and Fig. 4(a), we consider typical models approximating wave-fronts in remote sensing using SAR interferometry in one and two dimensions. Next, when these original wave-front $\{\xi_i\}/\{\xi_{x,y}\}$ are carried through noisy channel to optical measurement systems, they are corrupted by some noises, such as the Gaussian noise:

$$\eta_i = \xi_i + n_i(0,\sigma^2) \quad (2)$$

in one dimension and

$$\eta_{x,y} = \xi_{x,y} + n_{x,y}(0,\sigma^2) \quad (3)$$

in two dimensions. Next, by making use of interferometer, we observe corrupted interferograms (Figs. 3(b), (e) and Fig. 4(b)) as

$$\zeta_i = Wr(\eta_i) \quad (4)$$

in one dimension and

$$\zeta_{x,y} = Wr(\eta_{x,y}) \quad (5)$$

in two dimensions. Here,

$$\mathrm{Wr}(x) = \mathrm{mod}(x+\pi, 2\pi) - \pi \quad (6)$$

Then, as shown in Figs. 3(c) and 4(c), we derive sets of phase differences $\{g^x_i\}/\{g^x_{x,y}\}(\{g^y_{x,y}\})$ from the interferograms which are in eqs. (4) and (5) as

$$g_i^x = Wr(\zeta_{i+1} - \zeta_i) \tag{7}$$

in one dimension and

$$g_{x,y}^x = Wr(\zeta_{x+1,y} - \zeta_{x,y}) \tag{8}$$

$$g_{x,y}^y = Wr(\zeta_{x,y+1} - \zeta_{x,y}) \tag{9}$$

in two dimensions. Here, we note a relation between sampling points due to interferometer and the lattice points of the original wave-front in Fig. 5(a). Then, we assume that these phase differences are not corrupted by any noises, when we observe them due to the optical instruments via the interferometer.

Next, in order to carry out phase unwrapping by utilizing the set of the observed phase differences $\{g^x_i\}/\{g^x_{x,y}\}(\{g^y_{x,y}\})$, we

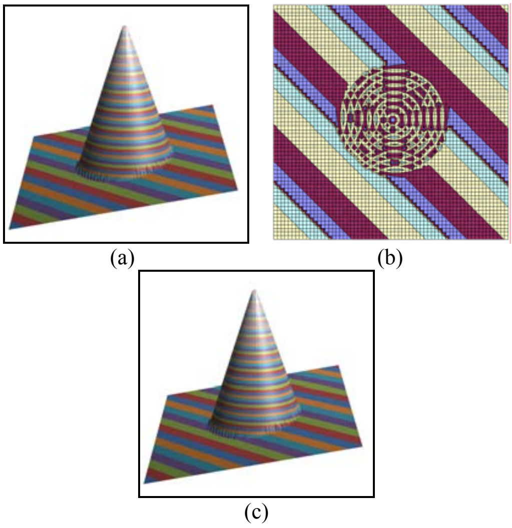

Fig. 4 (a) an original wave-front typical in remote sensing via SAR interferometry, (b) an interferogram obtained from the original wave-front in (a), (c) a reconstructed wave-front by the MAP estimation using the CG method under an optimal condition

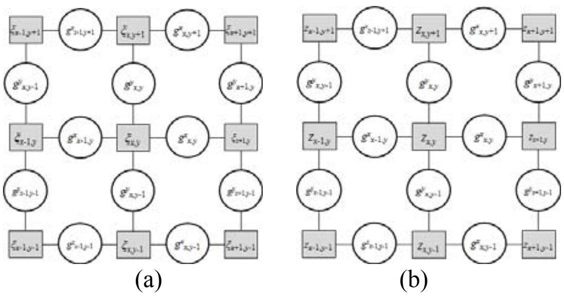

Fig. 5 (a) Lattice point (x,y) of the original wave-front $\{\xi_{x,y}\}$ and sampling points of phase differences $\{g^x_{x,y}\}$ and $\{g^y_{x,y}\}$ in two dimensions, (b) lattice point (x,y) of the original wave-front $\{z_{x,y}\}$ and sampling points of the phase differences $\{g^x_{x,y}\}$ and $\{g^y_{x,y}\}$ in two dimensions

try Bayesian inference via the method of maximum entropy. Here, we consider a model system $\{z_i\}/\{z_{x,y}\}$, where $0<z_i/z_{x,y}<R$, $i=1,\ldots,L$ / $x=1,\ldots,L$ and $y=1,\ldots,L$. In the two dimensions, the model system $\{z_{x,y}\}$ is arranged on the square lattice in Fig. 5(b). In this method, we reconstruct the original wave-front in one/two dimension(s) as an expectation value $z_i/z_{x,y}$ averaged over the posterior probability as

$$z_i = \arg\max \sum_{\{z\}\neq z_i} \Pr(\{z\} \mid \{g^x\}) \tag{10}$$

Here the posterior probability is estimated based on the Bayes-formula using the model prior and the likelihood as

$$\Pr(\{z\} \mid \{\tau\}) \propto \Pr(\{z\}) \Pr(\{g^x\} \mid \{z\}) \tag{11}$$

In this study, we assume the model prior which enhances smooth structures, as seen from natural wave-fronts appearing in remote sensing using the SAR interferometry, as

$$\Pr(\{z\}) = \frac{1}{Z} \exp\left[-\frac{J}{T_m} \sum_{i=1}^{L} (z_{i-1} - 2z_i + z_{i+1})^2 \right]. \tag{12}$$

Here Z is the normalization factor. Then, J and T_m are hyper-parameters. Then, we assume the likelihood enhancing wave-fronts from which principal phase differences $\{g^x_i\}$ are observed using the interferometer as

$$\Pr(\{g^x\} \mid \{z\}) \propto \exp\left[-\sum_{i=1}^{L} (f_i(\{z\}) - g_i^x)^2 \right], \tag{13}$$

$$f_i(\{z\}) = \mathrm{Wr}\big(\mathrm{Wr}(\{z_{i+1}\}) - \mathrm{Wr}(\{z_i\})\big). \tag{14}$$

On the other hand, in the two dimensional case, we also carry out phase unwrapping based on the Bayesian inference using the method of maximum entropy as

$$z_{x,y} = \arg\max \sum_{\{z\}\neq z_{x,y}} \Pr(\{z\} \mid \{g^x\}, \{g^y\}) \tag{15}$$

Here, the posterior probability is also estimated based on the Bayes formula:

$$\Pr(\{z\} \mid \{\tau\}) \propto \Pr(\{z\}) \Pr(\{g^x\}, \{g^y\} \mid \{z\}) \tag{16}$$

using the assumed model prior and the likelihood representing the optical measurement using the interferometer. Here, we use the model prior which enhances smooth structures as

$$\Pr(\{z\}) = \frac{1}{Z} \exp\left[-\frac{J}{T_m} E_{\mathrm{prior}}(\{z\}) \right], \tag{17}$$

$$E_{prior}(\{z\}) = J\sum_{(x,y)}(z_{x-1,y} - 2z_{x,y} + z_{x+1,y})^2 + J\sum_{(x,y)}(z_{x,y-1} - 2z_{x,y} + z_{x,y+1})^2 + \alpha J\sum_{(x,y)}(z_{x+1,y+1} - z_{x,y+1} - z_{x+1,y} + z_{x,y})^2 \quad (18)$$

Also, we assume the likelihood enhancing two dimensional wave-fronts from which principal phase differences $\{g^x\}$ and $\{g^y\}$ are observed by using the interferometer as

$$\Pr(\{g^x\},\{g^y\} \mid \{z\}) \propto \exp\left[-\frac{h}{T_m}\sum_{(x,y)}\left\{(f^x_{x,y}(\{z\}) - g^x_{x,y})^2 + (f^y_{x,y}(\{z\}) - g^y_{x,y})^2\right\}\right] \quad (19)$$

where

$$f^x_{x,y}(\{z\}) = \mathrm{Wr}(\mathrm{Wr}(z_{x+1,y}) - \mathrm{Wr}(z_{x,y})), \quad (20)$$

$$f^y_{x,y}(\{z\}) = \mathrm{Wr}(\mathrm{Wr}(z_{x,y+1}) - \mathrm{Wr}(z_{x,y})). \quad (21)$$

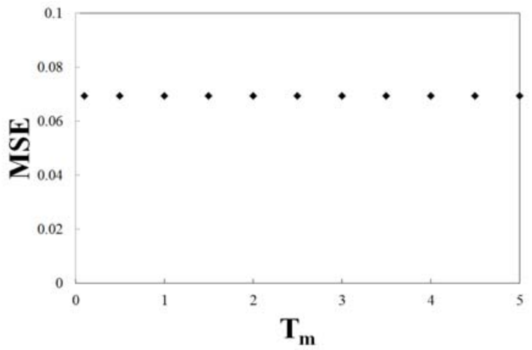

Fig. 6 Mean square error as a function of T_m obtained by the method of maximum entropy, if aliasing does not occur in optical measurements if σ=0.2 in one dimension

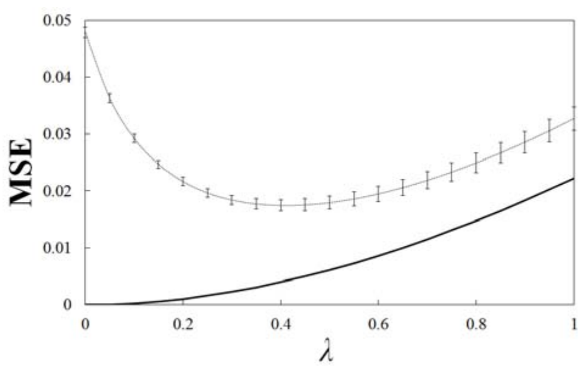

Fig. 7 Mean square error as a function of λ obtained by the two dimensional MAP estimation using the CG method. Bold (Dotted) line denoted the mean square error for σ=0 (0.2) at α= 1.2

Next, we also try the MAP estimation using the CG method for phase unwrapping in remote sensing via SAR interferometry. Althoug this strategy was already proposed by Saika, et al. [15], we here introduce this method using a generalized model of the true prior in the following. Here utilize the CG method based on the MAP estimation which is regarded as $T_m \to 0$ limit of the method of maximum entropy, i. e.,

$$z_i = \lim_{T_m \to 0}\langle z_i \rangle_{T_m} \quad (22)$$

$$\langle z_i \rangle_{T_m} = \frac{1}{Z}\prod_{i=1}^{L}\left(\int dz_i\right)\exp\left[-\frac{1}{T_m}E(\{z_i\})\right] z_i \quad (23)$$

where

$$E_{\mathrm{CG}}(\{z\}) = \sum_i (z_{i+1} - z_i - g^x_i)^2 + J\sum_i (z_{i-1} - 2z_i + z_{i+1})^2 \quad (24)$$

Here, we use the likelihood enhancing observed phase differences $\{g^x\}$. In this case, we carry out the CG method to obtain the MAP solution from the minimum condition of the one dimensional cost function in eq. (22) as

$$\frac{\partial E_{\mathrm{CG}}(\{z_i\})}{\partial z_i} = 0 \quad (25)$$

at every lattice site i. On the other hand, in the two dimensional case, we utilize the CG method to reconstruct the two dimensional wave-front as

$$z_{x,y} = \lim_{T_m \to 0}\langle z_{x,y} \rangle_{T_m} \quad (26)$$

$$\langle z_{x,y} \rangle_{T_m} = \frac{1}{Z}\prod_{(x,y)}^{L^2}\left(\int dz_i\right)\exp\left[-\frac{1}{T_m}E(\{z_{x,y}\})\right] z_{x,y} \quad (27)$$

using the two dimensional model $\{z_{x,y}\}(0<z_{x,y},<\infty,\ x,y=1,\ldots,L)$ on the square lattice, where

$$E_{CG}(\{z\}) = J\sum_{(x,y)}(z_{x-1,y} - 2z_{x,y} + z_{x+1,y})^2 + J\sum_{(x,y)}(z_{x,y-1} - 2z_{x,y} + z_{x,y+1})^2 + \alpha J\sum_{(x,y)}(z_{x+1,y+1} - z_{x,y+1} - z_{x+1,y} + z_{x,y})^2 + \sum_{(x,y)}(z_{x+1,y} - z_{x,y} - g^x_{x,y})^2 + \sum_{(x,y)}(z_{x,y+1} - z_{x,y} - g^y_{x,y})^2 \quad (28)$$

We note here that we utilize a generalized version of the model.

In this case, we carry out the CG method to obtain the MAP solution on the basis of the minimum condition of the cost functions in eq. (11) as

$$\frac{\partial E_{\mathrm{CG}}(\{z_{x,y}\})}{\partial z_{x,y}} = 0 \qquad (29)$$

at every lattice site (x, y).

In the following parts of this chapter, we note that a set of linear equations on $\{z_i\}/\{z_{x,y}\}$ as

$$Az = b, \qquad (30)$$

which represents both the minimum conditions of the cost function in eqs. (24), (28). Here, z is a L-dimensional vector which expresses a wave-front and b is a L-dimensional constant vector. Then, A is a $L\times L/L^2\times L^2$ square matrix. Next, we indicated how to obtain the MAP solution due to the CG method as below.

<u>Algorithm of the CG methodethodethod</u>
(i) First, we set to $\boldsymbol{z}_0 = 0$, $\boldsymbol{r}_0 = \boldsymbol{b}_0 - A\boldsymbol{z}_0$, $\boldsymbol{p}_0 = \boldsymbol{r}_0$.
(ii) Then, we set to k=0.
(iii) Next, we calculate

$$\alpha_k = \frac{\{r_k, p_k\}}{\{p_k, Ap_k\}},$$

where, $\{a, b\}=\sum_i a_i b_i$.
(iv) We set to $z_{k+1} = z_k + \alpha_k p_k$ and $r_{k+1} = r_k - \alpha_k A p_k$.
(v) If $\|r_{k+1}\|<\varepsilon$, stop. Otherwise, we calculate

$$\beta_k = -\frac{\{r_{k+1}, r_{k+1}\}}{\{r_k, r_k\}},$$

$$p_{k+1} = r_{k+1} + \beta_k p_k.$$

Next, in order to clarify efficiency of the present method, we evaluate a performance measure based on the mean square error as

$$\mathrm{MSE} = \frac{1}{L^2}\sum_{x=1}^{L}\sum_{y=1}^{L}(z_{x,y} - \xi_{x,y})^2 . \qquad (31)$$

On the other hand, we evaluate a performance measure based on the mean square error averaged over the assumed true prior as

$$\mathrm{MSE} = \sum_{\{\xi\}}\Pr(\{\xi\})\frac{1}{L^2}\sum_{x=1}^{L}\sum_{y=1}^{L}(z_{x,y} - \xi_{x,y})^2 \qquad (32)$$

If these variables become zero, if the phase unwrapping is carried out completely.

III. Performance

In this study, we investigated statistical performance of the MAP estimation using the CG method for phase unwrapping in remote sensing using the SAR interferometry. For this purpose, we first investigated statistical performance of Bayesian inference via maximum entropy including the Bayes-optimal solution. Then, we estimated the performance of the MAP estimation using the CG method for several models approximating wave-fronts in remote sensing using SAR interferometry.

As shown in Fig. 3 (a), we first considered the set of the one dimensional wave-fronts generated by the assumed true prior in eq. (1). Here, we observed the interferograms in Fig. 3(b). Next, as shown in Fig. 3(c), we found that the method of maximum entropy was effective for phase unwrapping for the set of wave-fronts generated by the assumed true prior around the Bayes-optimal condition. Further, as shown in Fig. 6, we evaluated how the mean square error depends on the parameter T_m around the Bayes-optimal condition. This result indicated that the method of maximum entropy realized the optimal performance around the Bayes-optimal condition, and also that the MAP estimation almost realized same statistical performance as the Bayesian inference under the Bayes-optimal condition. Therefore, we examined the performance of MAP estimation for phase unwrapping both for one and two dimensional models which approximated the wave-fronts in SAR interferometry in Fig. 3(d) and Fig. 4(a) whose interferograms were shown in Fig. 3(e) and Fig. 4(b). Then, as shown from the reconstructed wave-fronts in Fig. 3(d) and Fig. 4(c), we found that the MAP estimation using the CG method with high degree of accuracy. Also, as shown in Fig. 7, we evaluated statistical performance of the MAP estimation using the CG method for the wave-front (Fig. 4(a)) whose interferogram had no residue. Here, we evaluated λ dependence of the MSE averaged over 5 sets of the phase differences which were corrupted from the original wave-front (Fig. 4 (a)) by the Gaussian noise $n(0, \sigma^2)$, where σ=0, 0.2. We found that the CG method perfectly carried out phase unwrapping without using prior information, if observed information was not corrupted by any noises. On the other hand, if observed information was not corrupted by some noises, we found that the CG method was effective for phase unwrapping, if we tuned the parameter λ appropriately. Also, we noted that the performance was improved by introducing the parameter α(=1.2) into our model [15].

These results indicated that Bayesian inference using the maximum entropy realized the optimal performance around the Bayes-optimal condition, and also that the MAP estimation almost realized same performance as the Bayes-optimal solution.

IV. Summary and Discussion

In previous chapters, in order to construct a practical and useful method for phase unwrapping in remote sensing using SAR interferometry, we have constructed the MAP estimation using the CG method. For our purpose, we have first constructed the Bayesian inference using maximum entropy which included the Bayes-optimal solution for the sets of wave-fronts generated by the assumed true prior. Using Monte Carlo simulations for these wave-fronts, we have clarified that the method of maximum entropy realized the optimal performance around the Bayes-optimal condition within statistical uncertainty, and also that the MAP estimation almost realized the same performance as the method of maximum entropy under the Bayes-optimal condition. So, we have investigated the MAP estimation using the CG method for phase unwrapping. Using numerical simulations, we found that the CG method perfectly carried out phase unwrapping without using prior information, if observed information was not corrupted by any noises. On the other hand, if observed information was not corrupted by some noises, we found that the CG method was effective for phase unwrapping, if we tuned the parameter λ appropriately. In addition, we noted that the performance was improved by introducing the parameter $\alpha(=1.2)$ into our model [15].

These results have suggested that the MAP estimation using the CG method may be a practical and useful method for phase unwrapping by introducing a technique of parameter estimation, such as the EM algorithm. As a future problem, we are going to construct a method for phase unwrapping which can be applicable of realistic case, such as remote sensing using the SAR interferometry.

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