Axisymmetric Nonlinear Analysis of Point Supported Shallow Spherical Shells

M. Altekin, R. F. Yükseler

***Abstract*—**Geometrically nonlinear axisymmetric bending of a shallow spherical shell with a point support at the apex under linearly varying axisymmetric load was investigated numerically. The edge of the shell was assumed to be simply supported or clamped. The solution was obtained by the finite difference and the Newton-Raphson methods. The thickness of the shell was considered to be uniform and the material was assumed to be homogeneous and isotropic. Sensitivity analysis was made for two geometrical parameters. The accuracy of the algorithm was checked by comparing the deflection with the solution of point supported circular plates and good agreement was obtained.

***Keywords*—**Bending, nonlinear, plate, point support, shell.

I. Introduction

SINCE plate and shell structures have been extensively used in various disciplines like aerospace, marine, civil, nuclear, and automotive engineering, a large number of studies on plates and shells have been published (e.g. [1]-[17]). The studies on the large deflection bending of shallow spherical shells have basically focused on conventional boundary conditions (i.e., simply supported or clamped along the edge). However, for design purposes or for increasing the load carrying capacity of the structure, point supports have frequently been used. In the current study a shallow spherical shell with a circular plan form supported by a point support at the apex under linearly varying axisymmetric load undergoing large deflection was investigated. The influence of the geometrical parameters on the deflection and on the bending moment was examined by sensitivity analysis. The effect of the boundary conditions was studied. The accuracy of the algorithm was checked by comparing the deflection of a circular plate with those available in the literature.

II. Formulation

The smallest radius of curvature of a shallow shell at every point is larger than the greatest lengths measured along the midsurface of the shell [17]. The depth of a shallow spherical shell is limited by the relation [7] given by $\eta < 1/8$. The geometrically nonlinear shallow spherical shell equations were used in the current study. The equilibrium equations, the stress-strain relations, and the strain-displacement relations presented by Huang [7] were reorganized and rearranged in terms of three displacement components (w, u, β) where $\beta = w'$, and three stress resultants (n_r, q_r, m_r) given by

$$L_1 = m_r + r\,m_r' + D\left(\frac{w'}{r} + \nu\,\beta'\right) - r\,q_r = 0 \tag{1}$$

$$L_2 = (1-\nu)\,n_r + r\,n_r' - E\,t\,\frac{u}{r} = 0 \tag{2}$$

$$\begin{aligned} L_3 &= q_r + r q_r' + r\beta' n_r + 2h\frac{r}{a^2}(1+\nu)\,n_r \\ &+ \frac{Et}{r} w' u + 2Et\frac{h}{a^2} u + \nu w' n_r + r q_0 = 0 \end{aligned} \tag{3}$$

$$L_4 = m_r + D\,\beta' + D\frac{\nu}{r}\beta = 0 \tag{4}$$

$$\begin{aligned} L_5 &= u' + 2h\frac{r}{a^2}\beta + \frac{1}{2}\beta^2 \\ &- \frac{(1-\nu^2)}{E\,t} n_r + \nu\frac{u}{r} = 0 \end{aligned} \tag{5}$$

$$L_6 = w' - \beta = 0 \tag{6}$$

were obtained where

$$D = \frac{Et^3}{12(1-\nu^2)}, \qquad q_0 = q_0(r) = \frac{r}{a} q. \tag{7}$$

Substituting the nondimensional parameters given by

$$W = \frac{w}{t}, \qquad U = \frac{u}{t}, \qquad c = \frac{a}{t}, \tag{8}$$

$$Q = \frac{q}{E}, \qquad \eta = \frac{h}{2a}, \qquad \xi = \frac{r}{a}, \tag{9}$$

$$N_r = \frac{n_r}{Et}, \qquad Q_r = \frac{q_r}{Et}, \qquad M_r = \frac{m_r}{Et^2} \tag{10}$$

into (1)-(6), the differential operators $(L_1, L_2, L_3, L_4, L_5, L_6)$ are obtained as follows:

$$\begin{aligned} L_1 &= M_r + \xi\frac{dM_r}{d\xi} + \frac{1}{12(1-\nu^2)c^2\,\xi}\frac{dW}{d\xi} \\ &+ \frac{\nu}{12(1-\nu^2)c}\frac{d\beta}{d\xi} - c\,\xi\,Q_r = 0 \end{aligned} \tag{11}$$

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$$L_2 = (1-\nu) N_r + \xi \frac{dN_r}{d\xi} - \frac{1}{c\xi} U = 0 \tag{12}$$

$$L_3 = Q_r + \xi \frac{dQ_r}{d\xi} + \xi \frac{d\beta}{d\xi} N_r + 4\eta (1+\nu) \xi N_r + \frac{1}{c^2 \xi} \frac{dW}{d\xi} U + 4\frac{\eta}{c} U + \frac{\nu}{c} \frac{dW}{d\xi} N_r + c\xi^2 Q = 0 \tag{13}$$

$$L_4 = 12\left(1-\nu^2\right) c M_r + \frac{d\beta}{d\xi} + \frac{\nu}{\xi}\beta = 0 \tag{14}$$

$$L_5 = \frac{1}{c}\frac{dU}{d\xi} + 4\eta \xi \beta + \frac{1}{2}\beta^2 - \left(1-\nu^2\right) N_r + \frac{\nu}{c\xi} U = 0 \tag{15}$$

$$L_6 = \frac{1}{c}\frac{dW}{d\xi} - \beta = 0 \tag{16}$$

The boundary conditions are satisfied exactly at the

i. clamped edge: at $\xi = 1;\ \ W = U = \beta = 0$
ii. simply supported edge: at $\xi = 1;\ \ W = U = M_r = 0$
iii. apex: at $\xi = 0;\ \ W = U = \beta = 0$

III. Numerical Results

Equations (11)-(16) were transformed to algebraic equations by the forward, and backward finite difference formula. Six unknowns $(W, U, \beta, N_r, Q_r, M_r)$ were defined at each grid point located at ξ_i. Numerical solutions were evaluated for clamped (C) and simply supported (S) shallow spherical shells of $a = 1$ m, and $\nu = 0.3$.

$$\xi_i = \frac{n-i}{n-1}, \qquad i = 1, 2, ..., n \tag{17}$$

The convergence studies were performed (Table I), and it was revealed that $n = 101$ was sufficient for admissable accuracy. Circular plates were examined by setting $\eta = 0$. The central deflections of point supported circular plates computed in the current study agree well with those presented in [12]. Figs. 1-2 were plotted by using the formulae of Szilard [12].

TABLE I
Convergence Study ($Q = 10\text{x}10^{-7}$, $\nu = 0.3$, $c = 50$)

$\eta = 0.05$	W at $\xi = 0.5$ (S)	W at $\xi = 0.5$ (C)
$n = 41$	0.0146	0.0147
$n = 51$	0.0144	0.0147
$n = 81$	0.0138	0.0145
$n = 101$	0.0143	0.0147
$n = 121$	0.0143	0.0147

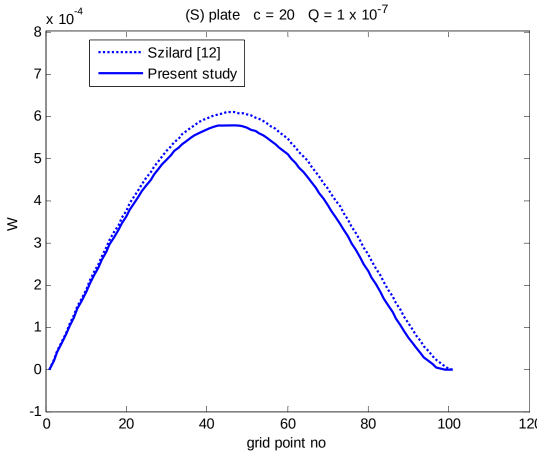

Fig. 1 Comparison of the results ((S) plate deflection)

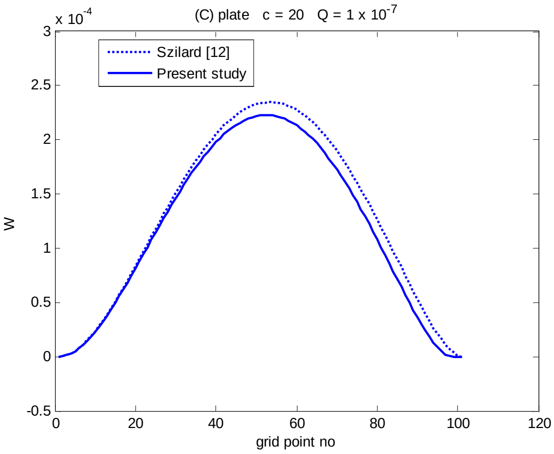

Fig. 2 Comparison of the results ((C) plate deflection)

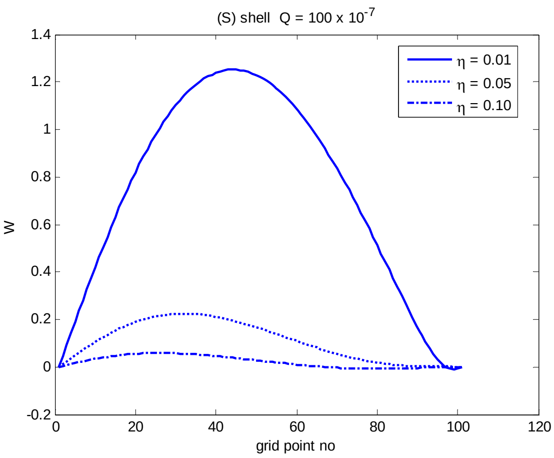

Fig. 3 Influence of depth on deflection (S) shell (c = 50)

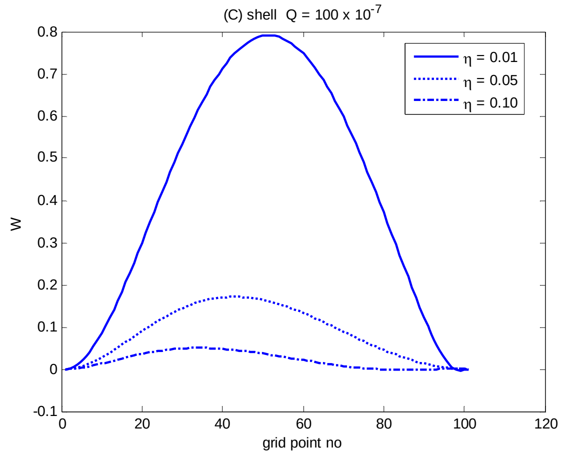

Fig. 4 Influence of depth on deflection (C) shell (c = 50)

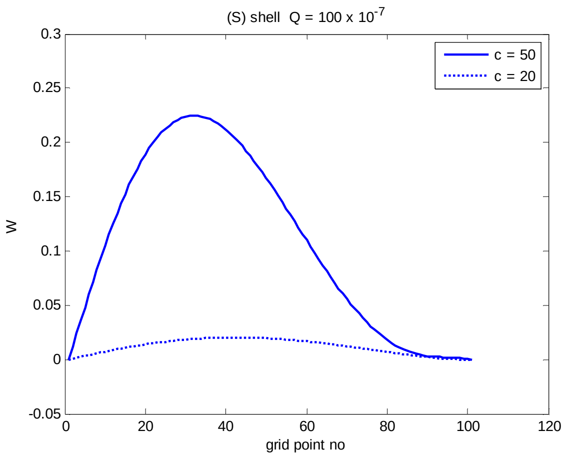

Fig. 5 Influence of thickness on deflection (S) shell (η = 0.05)

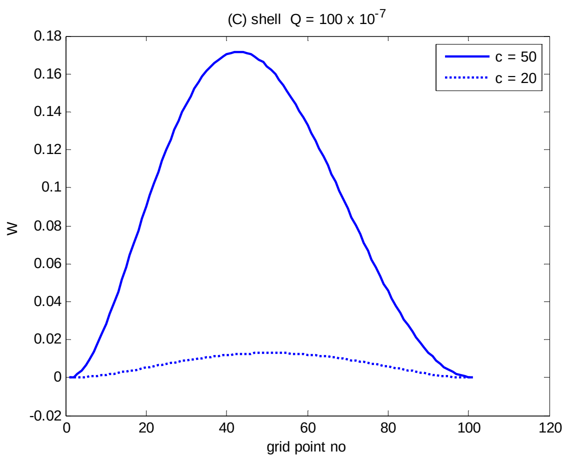

Fig. 6 Influence of thickness on deflection (C) shell (η = 0.05)

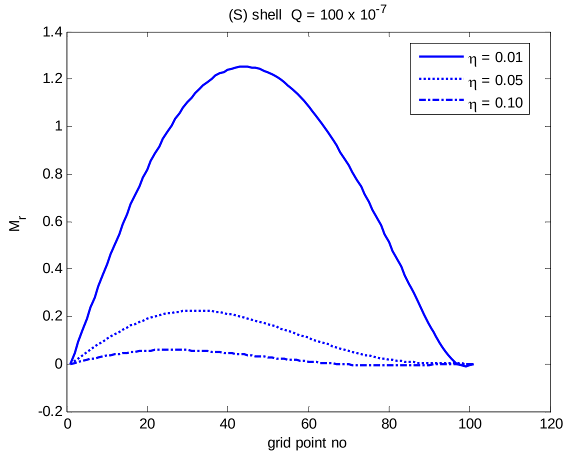

Fig. 7 Influence of depth on M_r (S) shell (c = 50)

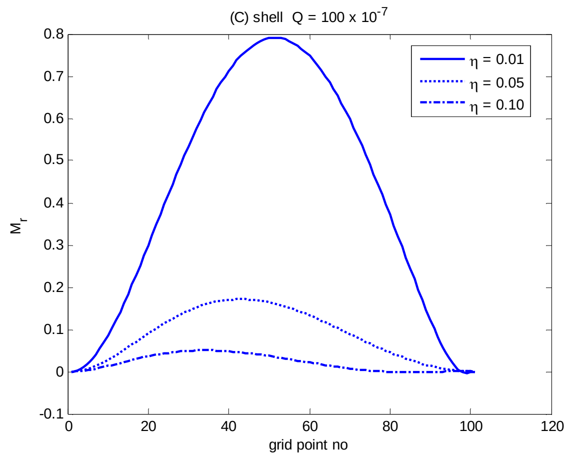

Fig. 8 Influence of depth on M_r (C) shell (c = 50)

IV. Conclusion

The convergence studies reveal that the contribution of the point support requires more grid points than the case in which there is no point support at the apex, and the external load is uniform (e.g. [3]). As the depth of the shell decreases, the location of maximum deflection moves to the center of the shell (Figs. 3, 4). The relation between the depth and deflection is nonlinear (Figs. 3, 4). The influence of the boundary conditions at the edge of the shell on deflection is almost identical in the vicinity of the point support (Figs. 3-6). The thickness has a nonlinear influence on deflection (Figs. 5, 6). As the depth of the shell decreases, the bending moment M_r increases (Figs. 7, 8). As the shell becomes deeper, the influence of the boundary conditions at the edge of the shell on M_r decreases (Figs. 7, 8).

Nomenclature

D, E	flexural rigidity, Young's modulus
Q	nondimensional counterpart of q
R, W	radius of the shell, nondimensional deflection

U nondimensional horizontal radial displacement
L_i differential operator (i=1,2,...,6)
M_r nondimensional meridional moment per unit length of the shell
N_r nondimensional membrane force per unit length of the shell
Q_r nondimensional transverse shear force per unit length of the shell
a, c base radius of the shell, parameter of thickness,
t thickness of the shell
h, n rise of the apex, total number of grid points located along the meridian on the middle surface of the shell
q, w maximum intensity of external load, deflection
r, u radial coordinate, horizontal radial displacement
m_r meridional moment per unit length of the shell
n_r membrane force per unit length of the shell
q_r transverse shear force per unit length of the shell
q_0 linearly varying axisymmetric load
β, η rotation, parameter of depth
ν, ξ Poisson's ratio, nondimensional radial coordinate
ξ_i nondimensional radial coordinate of the i^{th} grid point
( )' differentiation with respect to r

References

[1] N. Akkas, and R. Toroslu, "Snap-through buckling analyses of composite shallow spherical shells", *Mechanics of Composite Materials and Structures*, vol. 6 (4), 1999, pp. 319-330.
[2] M. Altekin, and R. F. Yükseler, "Axisymmetric large deflection analysis of an annular circular plate subject to rotational symmetric loading", *Proceedings of the Eleventh Int. Conference on Computational Structures Technology*, Dubrovnik, September 2012.
[3] M. Altekin, and R. F. Yükseler, "Axisymmetric large deflection analysis of fully and partially loaded shallow spherical shells", *Structural Engineering and Mechanics*, vol. 47 (4), 2013, pp. 559-573.
[4] R. A. Arciniega, and J. N. Reddy, "Large deformation analysis of functionally graded shells", *Int. J. of Solids and Structures*, vol. 44 (6), 2007, pp. 2036-2052.
[5] D. H. Bich, D. V. Dung, and L. K. Hoa, "Nonlinear static and dynamic buckling analysis of functionally graded shallow spherical shells including temperature effects", *Composite Structures*, vol. 94 (9), 2012, pp. 2952-2960.
[6] N. D. Duc, and T. Q. Quan, "Nonlinear dynamic analysis of imperfect functionally graded material double curved thin shallow shells with temperature-dependent properties on elastic foundation", *J. of Vibration and Control*, DOI: 10.1177/1077546313494114.
[7] N. C. Huang, "Unsymmetrical buckling of thin shallow spherical shells", *J. of Applied Mechanics*, vol. 31 (3), 1964, pp. 447-457.
[8] V. A. Maksimyuk, E. A. Storozhuk, and I. S. Chernyshenko, "Using mesh-based methods to solve nonlinear problems of statics for thin shells", *Int. Applied Mechanics*, vol. 45 (1), 2009, pp. 32-56.
[9] G. H. Nie, and J. C. Yao, "An asymptotic solution for non-linear behavior of ımperfect shallow spherical shells", *J. of Mechanics*, vol. 26 (2), 2010, pp. 113-122.
[10] L. S. Ramachandra, and D. Roy, "A novel technique in the solution of axisymmetric large deflection analysis of a circular plate", *J. of Applied Mechanics*, vol. 68 (5), 2001, pp. 814-816.
[11] K. Y. Sze, X. H. Liu, and S. H. Lo, "Popular benchmark problems for geometric nonlinear analysis of shells", *Finite Elements in Analysis and Design*, vol. 40 (11), 2004, pp. 1551-1569.
[12] R. Szilard, *Theory and Analysis of Plates*. Englewood Cliffs, New Jersey: Prentice-Hall, 1974, pp 616-708.
[13] E. Ramm, and W. A. Wall, "Shell structures- a sensitive interrelation between physics and numerics", *Int. J. for Numerical Methods in Engineering*, vol. 60 (1), 2004, pp. 381-427.
[14] D. Chakravortya, J.N. Bandyopadhyaya, and P.K. Sinha, "Finite element free vibration analysis of point supported laminated composite cylindrical shells", *J. of Sound and Vibration*, vol. 181 (1), 1995, pp. 43–52.
[15] T. Irie, G. Yamada, and Y. Kudoh, "Free vibration of a point-supported circular cylindrical shell", *J. of the Acoustical Society of America*, vol. 75 (4), 1984, pp. 1118-1123.
[16] Y. Narita, and A. W. Leissa, "Vibrations of corner point supported shallow shells of rectangular planform", *Earthquake Engineering & Structural Dynamics*, vol. 12 (5), 1984, pp. 651-661.
[17] Y. Kobayashi, and A. W. Leissa, "Large amplitude free vibration of thick shallow shells supported by shear diaphragms", *Int. J. of Non-Linear Mechanics*, vol. 30 (1), 1995, pp. 57-66.