

Asymptotic Properties of a Stochastic Predator-Prey Model with Bedding-DeAngelis Functional Response

Xianqing Liu, Shouming Zhong, Lijiang Xiang

Abstract—In this paper, a stochastic predator-prey system with Bedding-DeAngelis functional response is studied. By constructing a suitable Lyapunov function, sufficient conditions for species to be stochastically permanent is established. Meanwhile, we show that the species will become extinct with probability one if the noise is sufficiently large.

Keywords—Stochastically permanent, extinct, white noise, Bedding-DeAngelis functional response.

I. INTRODUCTION

IN mathematical biology, the predator's functional response which is the rate of prey consumption by an average predator is one of the significant elements of the predator-prey relationship. Generally, the functional response can be classified into two types: prey-dependent and predator-dependent. And Bedding-DeAngelis functional response belongs to predator-dependent functional response. As a matter of fact, the phenomenon that predators have to share or compete for food is common. Therefore, studying Bedding-DeAngelis functional response is meaningful.

A. The Model

However, we have no choice but to admit that all population systems are often subject to environmental noises. So, considering the corresponding stochastic population is necessary and important[1]-[13]. In [1], Liu and Wang introduced global stability of a nonlinear stochastic predator-prey system with Bedding-DeAngelis functional response. As we all know, stochastically permanent and extinct are also very important. There are two noise sources in [1], but their coupled mode is very simple. We know one noise source not only has influence on the growth rate of predator but also on the prey's. Therefore, from the argument above, we study the following form in this paper:

$$\begin{cases} dx = x \left[r_1 - b_1 x - \frac{a_1 y}{1 + \beta x + \gamma y} \right] dt \\ \quad + x [\sigma_1 dB_1(t) + \mu_2 dB_2(t)], \\ dy = y \left[r_2 - \frac{a_2 x}{1 + \beta x + \gamma y} - b_2 y \right] dt \\ \quad + y(t) [\sigma_2 dB_1(t) + \mu_1 dB_2(t)], \end{cases} \quad (1)$$

X. Liu is with School of Mathematical Science, University of Electronic Science and Technology of China, Chengdu 611731, PR China. (e-mail: xianqing_liu87@163.com).

S. Zhong and L. Xiang are with Electronic Science and Technology of China.

where $x(t)$ and $y(t)$ stand for the population densities of prey and predator at time t , respectively; $r_i, b_i, a_i, \beta, \gamma$ are positive parameters, $i = 1, 2$. μ_i^2 and σ_i^2 represent the intensities of the white noises, $i = 1, 2$. Let $(\Omega, F, \{F_t\}_{t \geq 0}, P)$ be a complete probability space with a filtration $\{F_t\}_{t \geq 0}$ satisfying the usual conditions, i.e. it is right continuous and increasing while F_0 contains all P -null sets. We denote by R_+^2 the positive cone in R^2 , and also denote by $X(t) = (x(t), y(t))$ and $|X(t)| = (x^2(t) + y^2(t))^{\frac{1}{2}}$.

1) *The Preliminaries:* In this section, we give some definitions, lemmas, assumptions and notations. The proof of Lemma 1, Lemma 2 and Lemma 3 are similar to [14]. Here, we omit them.

Definition 1 (see [14]) The solution $X(t) = (x(t), y(t))$ of (1) are said to be stochastically permanent, if for any $\epsilon \in (0, 1)$, there exists a pair of positive constants $\delta = \delta(\epsilon)$ and $\chi = \chi(\epsilon)$ such that for any initial value $X(0) = (x(0), y(0)) \in R_+^2$, the solution $X(t)$ to (1) has the properties that

$$\liminf_{t \rightarrow \infty} P\{|X(t)| \geq \delta\} \geq 1 - \epsilon, \quad \liminf_{t \rightarrow \infty} P\{|X(t)| \leq \chi\} \geq 1 - \epsilon.$$

Lemma 1 For any initial value $x_0 > 0, y_0 > 0$, there is a unique positive local solution $(x(t), y(t))$ for $t \in [0, \tau_e)$ of model (1) almost surely (a.s.).

Lemma 2 For any given initial value $X_0 = (x_0, y_0) \in R_+^2$, there is a unique solution $X(t) = (x(t), y(t))$ to model (1) on $t \geq 0$ and the solution will remain in R_+^2 with probability 1.

Lemma 3 The solutions of model (1) are stochastically ultimately bounded for any initial value $X_0 = (x_0, y_0) \in R_+^2$.

Assumption (A₁):

$$\frac{1}{2} \max \left\{ (\sigma_1^2 + \mu_2^2 + \sigma_1 \sigma_2 + \mu_1 \mu_2), (\sigma_2^2 + \mu_1^2 + \sigma_1 \sigma_2 + \mu_1 \mu_2) \right\} < \min \left\{ r_1 + \frac{a_1}{\gamma}, r_2 \right\},$$

Assumption (A₂): $r_1 - \frac{\sigma_1^2 + \mu_2^2}{2} < 0$,

Assumption (A₃): $r_2 + \frac{a_2}{\beta} - \frac{\sigma_2^2 + \mu_1^2}{2} < 0$.

For convenience of statement, we introduce some notations: let

$$M(x, y) = \frac{a_1 y}{1 + \beta x + \gamma y}, \quad N(x, y) = \frac{a_2 x}{1 + \beta x + \gamma y}.$$

II. CONCLUSION

Theorem 1 Under Assumption (A_1) , for any initial value $X(0) = (x(0), y(0)) \in R_+^2$, the solution $X(t) = (x(t), y(t))$ satisfies that

$$\limsup_{t \rightarrow \infty} E\left(\frac{1}{|X(t)|^\alpha}\right) \leq K, \quad (2)$$

where α is an arbitrary positive constant satisfying

$$\frac{\alpha+1}{2} \max \left\{ (\sigma_1^2 + \mu_2^2 + \sigma_1\sigma_2 + \mu_1\mu_2), (\sigma_2^2 + \mu_1^2 + \sigma_1\sigma_2 + \mu_1\mu_2) \right\} < \min \left\{ r_1 + \frac{a_1}{r}, r_2 \right\}. \quad (3)$$

there exist an arbitrary positive constant $s > 0$ satisfying

$$\alpha \min \left\{ r_1 + \frac{a_1}{r}, r_2 \right\} - \frac{\alpha(\alpha+1)}{2} \max \left\{ (\sigma_1^2 + \mu_2^2 + \sigma_1\sigma_2 + \mu_1\mu_2), (\sigma_2^2 + \mu_1^2 + \sigma_1\sigma_2 + \mu_1\mu_2) \right\} - s > 0. \quad (4)$$

Theorem 2 Assume (A_1) hold, equation(1) is Stochastically permanent.

The proof is application of the well-known Chebyshev inequality, Lemma 3 and Theorem 1. Here, we omit it.

Theorem 3 Assume (A_2) and (A_3) hold. For any given initial value $(x_0, y_0) \in R_+^2$, the solution $(x(t), y(t))$ to (1) will be extinct exponentially with probability one.

APPENDIX A

PROOF OF THEOREM 1

Proof: The proof is motivated by the method of [12]. Define $V_1(x, y) = x + y$, for $(x, y) \in R_+^2$, by the Itô's formula, we compute

$$dV_1(x, y) = \left\{ x[r_1 - b_1x - M(x, y)] + y[r_2 + N(x, y) - b_2y] \right\} dt + x[\sigma_1 dB_1(t) + \mu_2 dB_2(t)] + y[\sigma_2 dB_1(t) + \mu_1 dB_2(t)].$$

Then define $W(x, y) = \frac{1}{V_1(x, y)}$, dropping $x(t)$ from $U(x(t), y(t))$, $V_3(x(t), y(t))$ and t from $x(t), y(t)$, we have

$$dW = LWdt - W^2 \left\{ x[\sigma_1 dB_1(t) + \mu_2 dB_2(t)] + y[\sigma_2 dB_1(t) + \mu_1 dB_2(t)] \right\},$$

where

$$LW = -W^2 \left[x(r_1 - b_1x - M(x, y)) + y(r_2 + N(x, y) - b_2y) \right] + W^3 \left[(\sigma_1^2 + \mu_2^2)x^2 + (\sigma_2^2 + \mu_1^2)y^2 + 2xy(\sigma_1\sigma_2 + \mu_1\mu_2) \right].$$

Under Assumption (A_1) , let us choose a positive constant α such that it obeys (3). By the Itô formula, we get

$$L(1+W)^\alpha = \alpha(1+W)^{\alpha-1}LW + \frac{\alpha(\alpha-1)}{2}W^4 \times (1+W)^{\alpha-2} \left[(\sigma_1^2 + \mu_2^2)x^2 + (\sigma_2^2 + \mu_1^2)y^2 + 2xy(\sigma_1\sigma_2 + \mu_1\mu_2) \right].$$

Then we choose $s > 0$ sufficiently small such that it satisfies (4). Consequently,

$$Le^{st}(1+W)^\alpha = se^{st}(1+W)^\gamma + e^{st}L(1+W)^\gamma = e^{st}(1+W)^{\gamma-2} \left[s(1+W)^2 + H \right],$$

where

$$H = -\alpha W^2 \left[x(r_1 - b_1x - M(x, y)) + y(r_2 + N(x, y) - b_2y) \right] - \alpha W^3 \left[x(r_1 - b_1x - M(x, y)) + y(r_2 + N(x, y) - b_2y) \right] + \alpha W^3 \left[(\sigma_1^2 + \mu_2^2)x^2 + (\sigma_2^2 + \mu_1^2)y^2 + 2xy(\sigma_1\sigma_2 + \mu_1\mu_2) \right] + \frac{\alpha(\alpha+1)}{2}W^4 \times \left[(\sigma_1^2 + \mu_2^2)x^2 + (\sigma_2^2 + \mu_1^2)y^2 + 2xy(\sigma_1\sigma_2 + \mu_1\mu_2) \right].$$

In the following analysis, we will discuss the upper boundedness of the function $(1+W)^{\alpha-2}[s(1+W)^2 + H]$. It is easy to imply that

$$W^3 \left[(\sigma_1^2 + \mu_2^2)x^2 + (\sigma_2^2 + \mu_1^2)y^2 + 2xy(\sigma_1\sigma_2 + \mu_1\mu_2) \right] \leq \left(\max \left\{ (\sigma_1^2 + \mu_2^2\sigma_1\sigma_2 + \mu_1\mu_2), (\sigma_2^2 + \mu_1^2\sigma_1\sigma_2 + \mu_1\mu_2) \right\} \right) U$$

and

$$\frac{\alpha(\alpha+1)}{2}W^4 \left[(\sigma_1^2 + \mu_2^2)x^2 + (\sigma_2^2 + \mu_1^2)y^2 + 2xy(\sigma_1\sigma_2 + \mu_1\mu_2) \right] \leq \frac{\alpha(\alpha+1)}{2} \left(\max \left\{ (\sigma_1^2 + \mu_2^2 + \sigma_1\sigma_2 + \mu_1\mu_2), (\sigma_2^2 + \mu_1^2 + \sigma_1\sigma_2 + \mu_1\mu_2) \right\} \right) W^2.$$

Hence,

$$Le^{st}(1+W)^\alpha = e^{st}(1+W)^{\alpha-2} \left[s(1+W)^2 + H \right] \leq e^{st}(1+W)^{\alpha-2} \left\{ \left[s + \alpha \max \left\{ b_1, b_2 \right\} \right] + \left[2s - \alpha \min \left\{ r_1 + \frac{a_1}{r}, r_2 \right\} + \alpha \max \left\{ b_1, b_2 \right\} \right] + \alpha \max \left\{ (\sigma_1^2 + \mu_2^2 + \sigma_1\sigma_2 + \mu_1\mu_2), (\sigma_2^2 + \mu_1^2 + \sigma_1\sigma_2 + \mu_1\mu_2) \right\} \right] W - \left[-\alpha \min \left\{ r_1 + \frac{a_1}{r}, r_2 \right\} - \frac{\alpha(\alpha+1)}{2} \max \left\{ (\sigma_1^2 + \mu_2^2 + \sigma_1\sigma_2 + \mu_1\mu_2), (\sigma_2^2 + \mu_1^2 + \sigma_1\sigma_2 + \mu_1\mu_2) \right\} - s \right] U^2 \right\}.$$

From (4), we know that there exists a positive constant S such that $Le^{st}(1+W)^\alpha \leq Se^{st}$.

Therefore,

$$E \left[e^{st} (1+W(t))^\alpha \right] \leq (1+W(0))^\alpha + \frac{S}{s} e^{st} = (1+W(0))^\alpha + K_1 e^{st},$$

where $K_1 = \frac{S}{s}$.

So, we have

$$\limsup_{t \rightarrow \infty} EW^\alpha(t) \leq \limsup_{t \rightarrow \infty} E(1+W(t))^\alpha \leq K_1.$$

Note that $(x+y)^\alpha \leq 2^\alpha(x^2+y^2)^{\frac{\alpha}{2}} = 2^\alpha|X|^\alpha$, where $X = (x, y) \in R^2$. Now, we can obtain that

$$\limsup_{t \rightarrow \infty} E\left(\frac{1}{|X(t)|^\alpha}\right) \leq 2^\alpha \limsup_{t \rightarrow \infty} EW^\alpha(t) \leq 2^\alpha K_1 =: K.$$

Theorem 1 is proved.

APPENDIX B PROOF OF THEOREM 3

Proof: Define Lyapunov function $V_2 = \ln x$. Applying Itô formula leads to

$$\begin{aligned} dV_2 &= d(\ln x) \\ &= \left[\left(r_1 - \frac{\sigma_1^2 + \mu_2^2}{2} \right) - b_1 x - M(x, y) \right] dt + \sigma_1 dB_1(t) \\ &\quad + \mu_2 dB_2(t). \end{aligned}$$

Integrating it from 0 to t , yields

$$\begin{aligned} \ln x(t) &= \ln x_0 + \left(r_1 - \frac{\sigma_1^2 + \mu_2^2}{2} \right) t - b_1 \int_0^t x(s) ds \\ &\quad - c_1 \int_0^t M(x(s), y(s)) ds \\ &\quad + \sigma_1 \int_0^t dB_1(s) + \mu_2 \int_0^t dB_2(s). \end{aligned}$$

Consequently,

$$\ln x(t) \leq \ln x_0 + \left(r_1 - \frac{\sigma_1^2 + \mu_2^2}{2} \right) t + \sigma_1 B_1(t) + \mu_2 B_2(t).$$

Dividing t on the both sides and letting $t \rightarrow \infty$, we can obtain

$$\limsup_{t \rightarrow \infty} \frac{\ln x(t)}{t} \leq r_1 - \frac{\sigma_1^2 + \mu_2^2}{2} < 0 \text{ a.s.}$$

Similarly, define Lyapunov function $V_3 = \ln y$, by the Itô formula, we have

$$\begin{aligned} \ln y(t) &= \ln y_0 + \left(r_2 - \frac{\sigma_2^2 + \mu_1^2}{2} \right) t - b_2 \int_0^t y(s) ds \\ &\quad - c_2 \int_0^t N(x(s), y(s)) ds \\ &\quad + \sigma_2 \int_0^t dB_1(s) + \mu_1 \int_0^t dB_2(s). \end{aligned}$$

Therefore,

$$\frac{\ln y(t)}{t} \leq \frac{\ln y_0}{t} + r_2 - \frac{\sigma_2^2 + \mu_1^2}{2} + \frac{a_2}{\beta} + \frac{\sigma_2 B_1(t)}{t} + \frac{\mu_1 B_2(t)}{t}.$$

Let $t \rightarrow \infty$, we have

$$\limsup_{t \rightarrow \infty} \frac{\ln y(t)}{t} \leq r_2 + \frac{a_2}{\beta} - \frac{\sigma_2^2 + \mu_1^2}{2} < 0 \text{ a.s.}$$

The desired assertion is derived.

ACKNOWLEDGMENT

The authors would like to thank the associate editor and the anonymous reviewers for their detailed comments and suggestions.

REFERENCES

- [1] M. Liu and K. Wang, Global stability of a nonlinear stochastic predator-prey system with Beddington-DeAngelis functional response, *Commun. Nonlinear Sci. Numer. Simulat.*, 2011, 16, 1114-1121.
- [2] C. Ji, D. Q. Jiang and X. Li, Qualitative analysis of a stochastic ratio-dependent predator-prey system, *Journal of Computational and Applied Mathematics*, 2011, 235, 1326-1341.
- [3] J. Lv and K. Wang, Asymptotic properties of a stochastic predator-prey system with Holling II functional response, *Commun Nonlinear Sci Numer Simulat.*, 2011, 16, 4037-4048.
- [4] D. Q. Jiang, N. Shi and X. Li, Global stability and stochastic permanence of a non-autonomous logistic equation with random perturbation, *J. Math. Anal. Appl.*, 2008, 340, 588-597.
- [5] C. Ji, D. Q. Jiang and N. Jiang, Analysis of a predator-prey model with modified Leslie-Gower and Holling-type II schemes with stochastic perturbation, *J Math Anal Appl.*, 2009, 359, 482-498.
- [6] Z. Xing, J. Peng, Boundedness, persistence and extinction of a stochastic non-autonomous logistic system with time delays, *Applied Mathematical Modelling*, 2012, 36, 3379C3386.
- [7] M. Liu and K. Wang, Extinction and permanence in a stochastic non-autonomous population system, *Applied Mathematics Letters*, 2010, 23, 1464C1467.
- [8] M. Liu and K. Wang, Global asymptotic stability of a stochastic Lotka-CVolterra model with infinite delays, *Commun Nonlinear Sci Numer Simulat.*, 2012, 17, 3115C3123.
- [9] M. Liu and K. Wang, On a stochastic logistic equation with impulsive perturbations, *Computers and Mathematics with Applications*, 2012, 63, 871C886.
- [10] C. Zhu and G. Yin, On competitive Lotka-CVolterra model in random environments, *J. Math. Anal. Appl.*, 2009, 357, 154C170.
- [11] M. Liu and K. Wang, Persistence and extinction in stochastic non-autonomous logistic systems, *J. Math. Anal. Appl.*, 2011, 375, 443C457.
- [12] X. Liu, S. Zhong, B. Tian, F. Zheng, Asymptotic properties of a stochastic predator-prey model with Crowley-Martin functional response, *Journal of Applied Mathematics and Computing* (2013) doi:10.1007/s12190-013-0674-0.
- [13] X. Liu, S. Zhong, F. Zhong, Z. Liu, Properties of a stochastic predator-prey system with Holling II functional response, *Inter. J. Math. Sci.*, 2013, 7, 73-79.
- [14] X. Li, X. Mao, Population dynamical behavior of non-autonomous Lotka-Volterra competitive system with random perturbation. *Discrete Contin Dyn Syst*, 2009;24:523-545.

Xianqing Liu was born in Hubei Province, China, in 1987. She received the B.S. degree from Hubei University for Nationalities, Enshi, in 2011, in applied mathematics. She is currently pursuing the M.S. degree with School of Mathematical Science, University of Electronic Science and Technology of China. Her research interests include stochastically and delay dynamic systems.

Shouming Zhong was born in 1955 in Sichuan, China. He received B.S. degree in applied mathematics from UESTC, Chengdu, China, in 1982. From 1984 to 1986, he studied at the Department of Mathematics in Sun Yatsen University, Guangzhou, China. From 2005 to 2006, he was a visiting research associate with the Department of Mathematics in University of Waterloo, Waterloo, Canada. He is currently as a full professor with School of Applied Mathematics, UESTC. His current research interests include differential equations, neural networks, biomathematics and robust control. He has authored more than 80 papers in reputed journals such as the International Journal of Systems Science, Applied Mathematics and Computation, Chaos, Solitons and Fractals, Dynamics of Continuous, Discrete and Impulsive Systems, Acta Automatica Sinica, Journal of Control Theory and Applications, Acta Electronica Sinica, Control and Decision, and Journal of Engineering Mathematics.

Lijiang Xiang was born in Zhejiang Province, China, in 1989. He received the B.S. degree from Anhui Polytechnic University, Wuhu, in 2011. He is currently pursuing the M.S. degree with UESTC. His research interests include neural networks, switch and delay dynamic systems.