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# Approximating Fixed Points by a Two-Step Iterative Algorithm

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***Abstract*—**In this paper, we introduce a two-step iterative algorithm to prove a strong convergence result for approximating common fixed points of three contractive-like operators. Our algorithm basically generalizes an existing algorithm..Our iterative algorithm also contains two famous iterative algorithms: Mann iterative algorithm and Ishikawa iterative algorithm. Thus our result generalizes the corresponding results proved for the above three iterative algorithms to a class of more general operators. At the end, we remark that nothing prevents us to extend our result to the case of the iterative algorithm with error terms.

***Keywords*—**Contractive-like operator, iterative algorithm, fixed point, strong convergence.

## I. Introduction

LET $C$ be a nonempty convex subset of a normed space $E$ and $T : C \to C$ a mapping. Throughout this paper, $\mathrm{N}$ denotes the set of all positive integers, $I$ the identity mapping on $C$ and $F(T)$ the set of all fixed points of $T$.

The Mann iterative process [12] is defined by the sequence $\{v_n\}$ :

$$\begin{cases} v_1 = v \in C, \\ v_{n+1} = (1-\alpha_n)v_n + \alpha_n T v_n, \ n \in \mathrm{N} \end{cases} \tag{1}$$

where $\{\alpha_n\}$ is in $(0,1)$.

The sequence $\{z_n\}$ defined by

$$\begin{cases} z_1 = z \in C, \\ z_{n+1} = (1-\alpha_n)z_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)z_n + \beta_n T z_n, \ n \in \mathrm{N} \end{cases} \tag{2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0,1)$, is known as the Ishikawa iterative process [5].

Yao and Chen [14] considered the following extension (but with error terms) of the above mentioned Mann iteration process for asymptotically nonexpansive mappings in uniformly convex Banach spaces:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = \alpha_n x_n + \beta_n T x_n + \gamma_n S x_n, \ n \in \mathbf{N} \end{cases} \tag{3}$$

where $\{\alpha_n\},\{\beta_n\}$ and $\{\gamma_n\}$ are in $(0,1)$ with $\alpha_n + \beta_n + \gamma_n = 1$. Note that this algorithm reduces to Mann iterative algorithm (1) but not to Ishikawa Algorithm (2).

On the other hand, Berinde [1] introduced a new class of quasi-contractive type operators on a normed space $E$ satisfying

$$\|Tx - Ty\| \le \delta\|x - y\| + L\|Tx - x\| \tag{4}$$

for any $x, y \in E, 0 < \delta < 1$ and $L \ge 0$.

To appreciate this class of operators, we have to go through some definitions in a metric space $(X,d)$ .

A mapping $T : X \to X$ is called an $a$-contraction if

$$d(Tx,Ty) \le a d(x,y) \text{ for all } x, y \in X,$$

where $0 < a < 1$.

The map $T$ is called Kannan mapping [7] if there exists $b \in (0,\frac{1}{2})$ such that

$$d(Tx,Ty) \le b[d(x,Tx) + d(y,Ty)] \text{ for all } x, y \in X.$$

A similar definition is due to Chatterjea [2]: there exists $c \in (0,\frac{1}{2})$ such that

$$d(Tx,Ty) \le c[d(x,Ty) + d(y,Tx)] \text{ for all } x, y \in X.$$

Combining the above three definitions, Zamfirescu [15] proved the following important result.

**Theorem 1.** Let $(X,d)$ be a complete metric space and $T : X \to X$ a mapping for which there exist real numbers $a,b$ and $c$ satisfying $0 < a < 1$, $b \in (0,\frac{1}{2})$, $c \in (0,\frac{1}{2})$, such that for each pair $x, y \in X$, at least one of the following conditions holds:

$$(z_1)\ d(Tx,Ty) \le a d(x,y) \text{ for all } x, y \in X$$

$$(z_2)\ d(Tx,Ty) \le b[d(x,Tx) + d(y,Ty)] \text{ for all } x, y \in X$$

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$(z_3)\ d(Tx,Ty) \le c[d(x,Ty)+d(y,Tx)]$ for all $x, y \in X$.

Then $T$ has a unique fixed point $p$ and the Picard iterative sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n, n \in \mathrm{N}$ converges to $p$ for any arbitrary but fixed $x_1 \in X$.

An operator $T$ satisfying the contractive conditions $(z_1)$, $(z_2)$ and $(z_3)$ in the above theorem is called Zamfirescu operator. The class of Zamfirescu operators is one of the most studied classes of quasi-contractive type operators. In this class, Mann and Ishikawa iterative processes are known to converge to a unique fixed point of $T$.

This class of mappings is larger than not only contractions but also Kannan mappings and Zamfirescu operators. Berinde [1] used the Ishikawa iterative process (2) to approximate fixed points of this class of operators in a normed space. Actually, the following was his main theorem:

**Theorem 2.** [1] Let $C$ be a nonempty closed convex subset of a normed space $E$. Let $T : C \to C$ be an operator satisfying (4). Let $\{z_n\}$ be defined by the iterative process (3). If $F(T) \neq \phi$ and $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$ then $\{x_n\}$ converges strongly to a fixed point of $T$.

This kind of operators was further studied by Khan [9], [10], for example.

Imoru and Olatinwo [4] gave a more general definition: An operator $T$ is called a contractive-like operator if there exists a constant $\delta \in [0,1)$ and a strictly increasing and continuous function $\phi : [0,\infty) \to [0,\infty)$, with $\phi(0) = 0$, such that for each $x, y \in E$,

$$\|Tx - Ty\| \le \delta \|x - y\| + \phi(\|Tx - x\|) \tag{5}$$

In this paper, we introduce a two-step iterative algorithm to prove a strong convergence result for contractive-like operators (to be defined later) as follows:

$$\begin{cases} x_1 = x \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n) Q x_n, \\ x_{n+1} = a_n x_n + b_n T y_n + c_n S x_n, \ n \in \mathbf{N} \end{cases} \tag{6}$$

where $\{\alpha_n\}, \{a_n\}, \{b_n\}$ and $\{c_n\}$ are in $(0,1)$ with $a_n + b_n + c_n = 1$.

Note that this iterative algorithm reduces to

- Mann iterative algorithm (1) when $Q=I$.
- Ishikawa iterative algorithm (2) (as opposed to (3)) when $S=I$and $Q=T$.
- Yao-Chen iterative algorithm (3) when $T=I$.

This algorithm does not reduce to the one used by Khan [8] because we cannot allow $c_n = 0$, for all $n$, in view of the condition $\sum_{n=1}^{\infty} c_n = \infty$ imposed in the theorem.

Our purpose in this paper is to prove a convergence result for approximating common fixed points of three contractive-like operators $Q$, $S$ and $T$ as defined in (5) using the iterative process (6) in the setting of normed spaces. Thus our result generalizes the corresponding results proved for the Mann algorithm (1), Ishikawa algorithm (2) and Yao-Chen algorithm (3) to a class of more general operators.

## II. Main Result

We now prove our main theorem which deals with the iterative process (4) for the mappings defined in (6) in normed spaces.

**Theorem 3.** Let $C$ be a nonempty closed convex subset of a normed space $E$. Let $T : C \to C$ be an operator satisfying (5) and $F(T) \neq \phi$. Let $\{x_n\}$ be defined by the iterative process (6). Suppose that either $\sum_{n=1}^{\infty} b_n = \infty$ or $\sum_{n=1}^{\infty} c_n = \infty$, then $\{x_n\}$ converges strongly to a point of $F(T)$.

**Proof.** Let $w \in F(T)$. Then taking $T$=$S$, $x$=$w$, y= $x_n$ in (5), we have

$$\|Sx_n - w\| \le \delta \|x_n - w\|.$$

Similarly, $\|Qx_n - w\| \le \delta \|x_n - w\|$.

Next,

$$\begin{aligned} \|y_n - w\| &\le (1-\alpha_n)\|x_n - w\| + \alpha_n \|Qx_n - w\| \\ &\le (1-\alpha_n)\|x_n - w\| + \alpha_n \delta \|x_n - w\| \\ &\le (1-\alpha_n)\|x_n - w\| + \alpha_n \|x_n - w\| \\ &\le \|x_n - w\|. \end{aligned}$$

By choosing $x$=$w$and $y = y_n$ in (5), we have

$$\begin{aligned} \|Ty_n - w\| &\le \delta \|y_n - w\| \\ &\le \delta \|x_n - w\|. \end{aligned}$$

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Thus

$$\begin{aligned}\|x_{n+1}-w\| &= \|a_n x_n + b_n T y_n + c_n S x_n - w\| \\ &\le a_n\|x_n - w\| + b_n\|Ty_n - w\| \\ &\quad + c_n\|Sx_n - w\| \\ &\le a_n\|x_n - w\| + b_n\delta\|x_n - w\| \\ &\quad + \delta c_n\|x_n - w\| \\ &\le [a_n + (b_n + c_n)\delta]\|x_n - w\| \\ &= [1-(b_n + c_n) + (b_n + c_n)\delta]\|x_n - w\| \\ &= [(1-(b_n + c_n)(1-\delta)]\|x_n - w\|\end{aligned}$$

for all $n \in \mathrm{N}$.

By induction,

$$\begin{aligned}\|x_{n+1}-w\| &\le \prod_{k=1}^{n}[1-(1-\delta)(b_k + c_k)]\|x_1 - w\| \\ &= \|x_1 - w\|\exp\left(\sum_{k=1}^{n} -(1-\delta)(b_k + c_k)\right) \\ &= \|x_1 - w\|\exp\left(-(1-\delta)\sum_{k=1}^{n}(b_k + c_k)\right)\end{aligned}$$

for all $n \in \mathbf{N}$.

Since $b_n, c_n \in (0,1)$ and either $\sum_{n=1}^{\infty} b_n = \infty$ or $\sum_{n=1}^{\infty} c_n = \infty$, therefore $\sum_{n=1}^{\infty}(b_n + c_n) = \infty$ so that

$$\begin{aligned}\limsup_{n\to\infty}\|x_n - w\| &\le \limsup_{n\to\infty}\|x_1 - w\| \\ &\times \exp\left(-(1-\delta)\sum_{k=1}^{n}(b_k + c_k)\right) = 0.\end{aligned}$$

Hence $\lim_{n\to\infty}\|x_n - w\| = 0.$

Consequently $x_n \to w \in F.$

This completes the proof.

Our result may be compared with Khan and Fukhar-ud-din [11] though proved for two families of mapping but with using only one-step iterative algorithm which does not reduce to Ishikawa iterative algorithm.

Liu [6] introduced iterative processes with error terms. Later on, Xu [13] improved these processes by giving more satisfactory error terms. Both processes constitute generalizations of Mann and Ishikawa iterative processes. Our iterative process with error terms in the sense of Xu looks like:

$$\begin{cases} x_1 = x \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ x_{n+1} = a_n x_n + b_n T y_n + c_n S x_n + c_n u_n, \ n \in \mathbf{N} \end{cases}$$

where $\{u_n\}$ is a bounded sequence in $C$. Numerous papers have been produced on Ishikawa and Mann iterative processes with errors and follow a similar computational techniques as those without errors, see, for example, [3]. To avoid repetition, we omit the proof here.

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## REFERENCES

[1] V. Berinde, *A convergence theorem for some mean value fixed point iterations procedures*, Dem. Math., 38(1)2005, 177-184.

[2] S.K. Chatterjea, *Fixed point theorems*, C.R. Acad. Bulgare Sci., 25 (1972), 727-730.

[3] H. Fukhar-ud-din and S. H. Khan, *Convergence of iterates with errors of asymptotically quasi-nonexpansive mappings and applications*, J. Math. Anal. Appl. 328 (2007), 821-829.

[4] C.O. Imoru, M.O. Olatinwo, *On the stability of Picard and Mann iteration processes*, Carpathian Journal of Mathematics, 19, 155-160 (2003).

[5] S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc., 44 (1974), 147-150.

[6] L. S. Liu, *Ishikawa and Mann Iteration process with errors for nonlinear strongly accretive mappings in Banach spaces*, J. Math. Anal. Appl. 194(1) (1995), 114-125.

[7] R. Kannan, *Some results on fixed points*,Bull. Calcutta Math. Soc., 10(1968), 71-76.

[8] S.H. Khan, *A Picard-Mann hybrid iterative process*, Fixed Point Theory and Applications 2013, 2013:69.

[9] S.H. Khan, *Approximating Common fixed points by an iterative process involving two steps and three mappings*, Journal of Concrete and Applicable Mathematics, Vol 8, number 3, 407-415, 2010.

[10] S.H. Khan, *Fixed points of quasi-contractive type operators in normed spaces by a three-step iteration process*, Proceedings of the World Congress on Engineering 2011 Vol I, WCE 2011, July 6 - 8, 2011, London, U.K, pp 144-147.

[11] Safeer Hussain Khan, Hafiz Fukhar-ud-din, *Common fixed points of two finite families of quasi-contractive type operators in normed spaces,* 2nd Annual International Conference on Computational Mathematics, Computational Geometry & Statistics (CMCGS 2013), Organized by Global Science & Technology Forum (GSTF), held in Singapore 4-5 February, 2013.

[12] W.R. Mann, *Mean value methods in iterations*, Proc. Amer. Math. Soc., 4 (1953), 506-510.

[13] Y. Xu, *Ishikawa and Mann Iteration process with errors for nonlinear strongly accretive operator equations*, J. Math. Anal. Appl. 224 (1998), 91-101.

[14] Y. Yao, Y.Chen,*Weak and strong convergence of a modified Manniteration for asymptotically nonexpansive mappings*, Nonlin. Funct. Anal. Appl., 12(2007), 307-315.

[15] T. Zamfirescu, *Fix point theorems in metric spaces*, Arch. Math. (Basel), **23**(1972), 292-298.