

Approximate Solution to Non-Linear Schrödinger Equation with Harmonic Oscillator by Elzaki Decomposition Method

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Abstract—Nonlinear Schrödinger equations are regularly experienced in numerous parts of science and designing. Varieties of analytical methods have been proposed for solving these equations. In this work, we construct an approximate solution for the nonlinear Schrödinger equations, with harmonic oscillator potential, by Elzaki Decomposition Method (EDM). To illustrate the effects of harmonic oscillator on the behavior wave function, nonlinear Schrödinger equation in one and two dimensions is provided. The results show that, it is more perfectly convenient and easy to apply the EDM in one- and two-dimensional Schrödinger equation.

Keywords—Non-linear Schrödinger equation, Elzaki decomposition method, harmonic oscillator, one and two-dimensional Schrödinger equation.

I. INTRODUCTION

THE nonlinear Schrödinger equation frequently emerges in various branches of physics and engineering science such as in quantum mechanics, optics, heat plus in solids, plasma and material science among other. The investigation of this equation and their solutions has happened of great interest to various researchers because of its different applications. Different methods have been proposed to manage Schrödinger equation: Aboodh decomposition method [3], [4], Natural decomposition method [1], [2], Adomain decomposition method [6]-[8] and EDM [5], [9]. In this study we are applying EDM to study the behavior of the wave function of Non-Linear Schrödinger Equation.

II. ELZAKI DECOMPOSITION METHOD

The EDM is a method to solve partial differential equation. Elzaki introduces a successive solution to the linear and nonlinear partial differential equation [9]. He presented a new integral transform defined by the following integral equation [13].

$$E[f(t)] = v^2 \int_0^\infty f(vt)e^{-t} dt = f(t)$$

Then, $f(t)$ is called inverse Elzaki transform of $E[f(t)]$ denoted $f(t) = E^{-1}[E[f(t)]]$.

To illustrate the basic idea of EDM, we consider the following nonlinear Schrödinger equation with the initial condition as the form [5]:

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$$L\Psi - iR\Psi - iN\Psi = 0, \quad \Psi(x, 0) = f(x) \quad (1)$$

where $L = \frac{\partial}{\partial t}$ and $R = \frac{\partial^2}{\partial x^2}$, L and R are Linear operators, and $N\Psi$ is a nonlinear term.

Applying the Elzaki transform to both sides of the (1), we get

$$E[L\Psi] - iE[R\Psi] - iE[N\Psi] = 0$$

From Elzaki transform of first and derivative and substituting the initial condition, we get:

$$\frac{1}{v}E[\Psi] - vf(x) = iE[R\Psi] + iE[N\Psi] \\ E[\Psi] = v^2f(x) + ivE[R\Psi] + ivE[N\Psi] \quad (2)$$

Next step is replacing the wave function Ψ by an infinite series to obtain:

$$\Psi(x, t) = \sum_{n=0}^{\infty} \Psi_n(x, t) \quad (3)$$

And we replace the nonlinear terms by the series:

$$N\Psi = \sum_{n=0}^{\infty} A_n(\Psi_0, \Psi_1 \dots \Psi_n) \quad (4)$$

where $A_n(\Psi_0, \Psi_1 \dots)$'s are Adomain Polynomials [6]-[8] defined by:

$$A_n = \frac{1}{n!} \left[\frac{d^n}{d\lambda^n} N(\sum_{i=0}^{\infty} \lambda^i \Psi_i) \right] \Big|_{\lambda=0}, \quad n = 0, 1, 2 \dots \quad (5)$$

Substituting (3) and (4) into (2); we get:

$$E[\sum_{n=0}^{\infty} \Psi_n(x, t)] = E[v^2f(x)] + ivE[R \sum_{n=0}^{\infty} \Psi_n(x, t)] + ivE[\sum_{n=0}^{\infty} A_n] \quad (6)$$

Taking inverse Elzaki transform to (6), we have:

$$\sum_{n=0}^{\infty} \Psi_n(x, t) = E^{-1}[E[v^2f(x)]] + E^{-1}[ivE[R \sum_{n=0}^{\infty} \Psi_n(x, t)]] + E^{-1}[ivE[\sum_{n=0}^{\infty} A_n]] \quad (7)$$

Thus, on comparing both sides of (7), we obtain the general solution of (1):

$$\Psi_0(x, t) = f(x) \\ \Psi_1(x, t) = E^{-1}[ivE[R\Psi_0] + E^{-1}[ivE[A_0]]] \\ \Psi_2(x, t) = E^{-1}[ivE[R\Psi_1] + E^{-1}[ivE[A_1]]]$$

$$\Psi_n(x, t) = iE^{-1}[vE[R\Psi_{n-1}] + iE^{-1}[vE[A_{n-1}]]$$

III. ONE- DIMENSIONAL NON-LINEAR SCHRÖDINGER EQUATION WITH HARMONIC OSCILLATOR

Consider that the particle of mass (m) moves in one dimension then, the nonlinear Schrödinger equation with harmonic oscillator and the initial condition can be expressed as [10]:

$$\frac{\partial \Psi}{\partial t} - \frac{i}{2m} \frac{\partial^2 \Psi}{\partial x^2} + \frac{ikx^2\Psi}{2} + i|\Psi|^2\Psi = 0 \quad (8)$$

$$\Psi(x, 0) = e^{ix}$$

where Ψ is the wave function; m is the mass of the particle, i the imaginary unit to describe motion and k spring constant.

Applying Elzaki Transform method to (8); we get:

$$E \left[\frac{\partial \Psi}{\partial t} \right] - E \left[\frac{i}{2m} \left(\frac{\partial^2 \Psi}{\partial x^2} \right) \right] + E \left[\frac{i}{2} kx^2 \Psi \right] + E [i|\Psi|^2\Psi] = 0 \quad (9)$$

Using the differentiation property of Elzaki transform on (9) and inserting the initial condition, we have

$$\frac{1}{v} E[\Psi(x, t)] - v\Psi(x, 0) - E \left[\frac{i}{2m} \frac{\partial^2 \Psi}{\partial x^2} \right] + E \left[\frac{i}{2} kx^2 \Psi \right] + E [i|\Psi|^2\Psi]$$

$$E[\Psi(x, t)] = v^2 e^{ix} + vE \left[\frac{i}{2m} \left(\frac{\partial^2 \Psi}{\partial x^2} \right) \right] - vE \left[\frac{ikx^2}{2} \Psi \right] - vE [i|\Psi|^2\Psi] \quad (10)$$

Inserting (3) and (4) into (10), we have:

$$E[\sum_{n=0}^{\infty} \Psi_n(x, t)] = v^2 e^{ix} + vE \left[\frac{i}{2m} \frac{\partial^2}{\partial x^2} (\sum_{n=0}^{\infty} \Psi_n(x, t)) \right] - vE \left[\frac{ikx^2}{2} \sum_{n=0}^{\infty} \Psi_n(x, t) \right] - vE [i \sum_{n=0}^{\infty} A_n] \quad (11)$$

By applying inverse Elzaki transform on both sides to (11), we have:

$$E^{-1}[E[\Psi_0(x, t)]] = E^{-1}[v^2 e^{ix}]$$

$$E^{-1}[E[\Psi_1(x, t)]] = E^{-1} \left[vE \left[\frac{i}{2m} \left(\frac{\partial^2 \Psi_0}{\partial x^2} \right) \right] \right] - E^{-1} \left[vE \left[\frac{ikx^2 \Psi_0}{2} \right] \right] - E^{-1}[vE[iA_0]]$$

The general solution can be written

$$\Psi_n(x, t) = E^{-1} \left[vE \left[\frac{i}{2m} \left(\frac{\partial^2 \Psi_{n-1}}{\partial x^2} \right) \right] \right] - E^{-1} \left[vE \left[\frac{ikx^2 \Psi_{n-1}}{2} \right] \right] - E^{-1}[vE[iA_{n-1}]] \quad (12)$$

where A_n are the so called Adomian Polynomials of ($\Psi_0, \Psi_1, \dots, \Psi_n$) to replace $N\Psi = |\Psi|^2\Psi = \Psi^2\bar{\Psi}$ and $\bar{\Psi}$ is the conjugate of Ψ .

The Adomian Polynomials A_n 's are determined in (5), the first four Adomian Polynomials expressed as:

$$A_0 = \Psi_0^2 \bar{\Psi}_0$$

$$A_1 = 2\Psi_0 \Psi_1 \bar{\Psi}_0 + \Psi_0^2 \bar{\Psi}_1$$

$$A_2 = 2\Psi_0 \Psi_2 \bar{\Psi}_0 + \Psi_1^2 \bar{\Psi}_0 + 2\Psi_0 \Psi_1 \bar{\Psi}_1 + \Psi_0^2 \bar{\Psi}_2$$

$$A_3 = 2\Psi_0 \Psi_3 \bar{\Psi}_0 + 2\bar{\Psi}_0 \Psi_2 \Psi_1 + 2\Psi_0 \Psi_2 \bar{\Psi}_1 + \bar{\Psi}_1 \Psi_1^2 + 2\Psi_0 \Psi_1 \bar{\Psi}_2 + \Psi_0^2 \bar{\Psi}_3.$$

By using to (11), the solution reads:

$$\Psi_0(x, t) = e^{ix}$$

$$\Psi_1(x, t) = -\frac{it e^{ix}}{2m} - \frac{ikx^2 t e^{ix}}{2} - it e^{ix}$$

$$\Psi_2(x, t) = \frac{t^2}{2!} \left(\frac{-e^{ix}}{4m^2} + \frac{ke^{ix}}{2m} + \frac{ikxe^{ix}}{m} - \frac{e^{ix}}{m} - \frac{kx^2 e^{ix}}{2m} - \frac{k^2 x^4 e^{ix}}{4} - kx^2 e^{ix} - e^{ix} \right)$$

$$\Psi_3(x, t) = \frac{t^3}{3!} \left(\frac{3ie^{ix}}{4m^2} + \frac{3ikx^2 e^{ix}}{8m^2} + \frac{ie^{ix}}{8m^3} + \frac{3kxe^{ix}}{2m^2} - \frac{7ike^{ix}}{4m^2} - \frac{3ike^{ix}}{2m} + \frac{3ikx^2 e^{ix}}{2m} + \frac{3ie^{ix}}{2m} + \frac{ik^3 x^6 e^{ix}}{8} + \frac{i3k^2 x^4 e^{ix}}{4} + ie^{ix} + \frac{3kxe^{ix}}{m} + \frac{3ik^2 x^4 e^{ix}}{8m} - \frac{7ik^2 x^2 e^{ix}}{4m} + \frac{3k^2 x^3 e^{ix}}{2m} + \frac{3ikx^2 e^{ix}}{2} \right)$$

Therefore, the solution $\Psi(x, t)$ is given by:

$$\Psi(x, t) = \Psi_0 + \Psi_1 + \Psi_2 + \dots \Psi(x, t) = e^{ix} \left[1 + \left(-\frac{it}{2m} - \frac{ikx^2 t}{2} - it \right) + \frac{1}{2!} \left(\frac{-it}{2m} - \frac{ikx^2 t}{2} - it \right)^2 + \frac{t^2}{2!} \left(\frac{k}{2m} + \frac{ikx}{m} \right) + \frac{1}{3!} \left(\frac{-it}{2m} - \frac{ikx^2 t}{2} - it \right)^3 + \frac{t^3}{3!} \left(\frac{3kx}{2m^2} - \frac{7ik}{4m^2} - \frac{3ik}{2m} + \frac{3kx}{m} - \frac{7ik^2 x^2}{4m} + \frac{3k^2 x^3}{2m} \right) + \dots \right] \quad (13)$$

Fig. 1 shows the absolute approximate solution of the wave function only tends to wards $\Psi(x) = 0$ in one direction of x and increasing exponentially in the other one. Fig. 2 presents the real and imaginary approximate solution, therefore the graphs illustrates wave's reflection. We use $k = m = 1, t=0.1$ in all calculation.

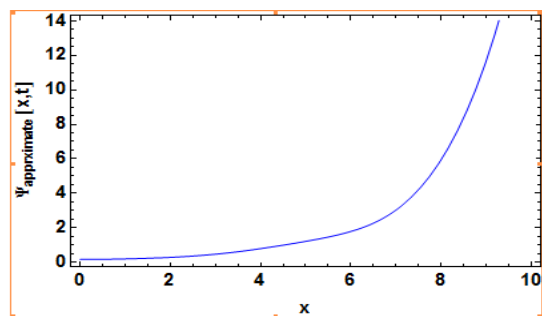


Fig. 1 Absolute approximate solution of the one- dimensional wave function by EDM

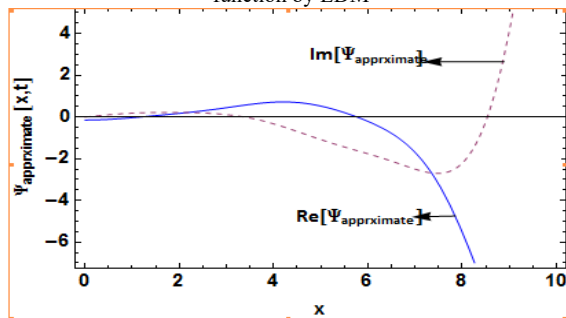


Fig. 2 The real and imaginary approximate solution of the one- dimensional wave function by EDM

IV. TWO DIMENSIONAL NON-LINEAR SCHRÖDINGER EQUATIONS WITH HARMONIC OSCILLATOR

The nonlinear Schrödinger equation with harmonic oscillator when a particle moves in two dimensions with the initial condition can be written as [10]-[12]:

$$\frac{\partial \Psi}{\partial t} - \frac{i}{2m} \left(\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} \right) + \frac{ik}{2} (x^2 + y^2) \Psi + i|\Psi|^2 \Psi = 0 \quad (14)$$

$$\Psi(x, y, 0) = e^{i(x+y)}$$

First, applying Elzaki Transform to (14), we have:

$$\left[\frac{\partial \Psi}{\partial t} \right] - E \left[\frac{i}{2m} \left(\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} \right) \right] + E \left[\frac{ik}{2} (x^2 + y^2) \Psi \right] + E[i|\Psi|^2 \Psi] = 0 \quad (15)$$

By using the differentiation property of Elzaki transform and inserting the initial conditions, we get

$$\frac{1}{v} E[\Psi(x, y, t)] - v\Psi(x, y, 0) - E \left[\frac{i}{2m} \left(\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} \right) \right] + E \left[\frac{ik}{2} (x^2 + y^2) \Psi \right] + E[i|\Psi|^2 \Psi] = 0 \quad (16)$$

$$E[\Psi(x, y, t)] = v^2 e^{i(x+y)} + vE \left[\frac{i}{2m} \left(\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} \right) \right] - vE \left[\frac{ik}{2} (x^2 + y^2) \Psi \right] - vE[i|\Psi|^2 \Psi] \quad (17)$$

In the next step, we replace the wave function Ψ by an infinite series given by

$$\Psi = \sum_{n=0}^{\infty} \Psi_n(x, y, t) \quad (18)$$

Then, we replace the Non-Linear terms $N\Psi = |\Psi|^2 \Psi$ by the series

$$|\Psi|^2 \Psi = \sum_{n=0}^{\infty} A_n(\Psi_0, \Psi_1, \dots, \Psi_n) \quad (19)$$

Substituting (18) and (19) into (17) we have:

$$E \left[\sum_{n=0}^{\infty} \Psi_n(x, y, t) \right] = \frac{v^2 e^{i(x+y)}}{v} + vE \left[\frac{i}{2m} \left(\frac{\partial^2}{\partial x^2} \left(\sum_{n=0}^{\infty} \Psi_n(x, y, t) \right) + \frac{\partial^2}{\partial y^2} \left(\sum_{n=0}^{\infty} \Psi_n(x, y, t) \right) \right) \right] - vE \left[\frac{ik}{2} (x^2 + y^2) \sum_{n=0}^{\infty} \Psi_n(x, y, t) \right] - vE \left[i \sum_{n=0}^{\infty} A_n \right] \quad (20)$$

From matching the both sides and taking inverse Elzaki transform to (20), we have

$$E^{-1} [E[\Psi_0(x, y, t)]] = E^{-1} [v^2 e^{i(x+y)}] \quad (21)$$

$$\Psi_1(x, y, t) = E^{-1} \left[vE \left[\frac{i}{2m} \left(\frac{\partial^2 \Psi_0(x, y, t)}{\partial x^2} + \frac{\partial^2 \Psi_0(x, y, t)}{\partial y^2} \right) \right] \right] - E^{-1} \left[vE \left[\frac{ik}{2} (x^2 + y^2) \Psi_0(x, y, t) \right] \right] - E^{-1} [vE[iA_0]] \quad (22)$$

The general recursion relation is:

$$\Psi_n(x, y, t) = E^{-1} \left[vE \left[\frac{i}{2m} \left(\frac{\partial^2 \Psi_{n-1}(x, y, t)}{\partial x^2} + \frac{\partial^2 \Psi_{n-1}(x, y, t)}{\partial y^2} \right) \right] \right] - E^{-1} \left[vE \left[\frac{ik}{2} (x^2 + y^2) \Psi_{n-1}(x, y, t) \right] \right] - E^{-1} [vE[iA_{n-1}]] \quad (23)$$

The first four Adomain polynomials are calculated in (5) to obtain:

$$A_0 = \Psi_0, A_1 = \Psi_1, A_2 = \Psi_2, A_3 = \Psi_3 \dots A_n = \Psi_n \quad (24)$$

The solution reads:

$$\Psi_0(x, y, t) = e^{i(x+y)} \quad (25)$$

$$\Psi_1(x, y, t) = \frac{-it}{m} (e^{i(x+y)}) - \frac{ikt e^{i(x+y)}}{2} (x^2 + y^2) - it e^{i(x+y)}$$

$$\Psi_2(x, y, t) = \frac{-t^2 e^{i(x+y)}}{(2!)m^2} - \frac{kt^2 e^{i(x+y)}}{(2!)m} (x^2 + y^2) - \frac{2t^2 e^{i(x+y)}}{(2!)m} + \frac{kt^2 e^{i(x+y)}}{(2!)m} + \frac{ikt^2 e^{i(x+y)}}{(2!)m} (x+y) - \frac{k^2 e^{i(x+y)}}{4(2!)} (x^2 + y^2)^2 t^2 - \frac{t^2 e^{i(x+y)}}{(2!)} - \frac{ke^{i(x+y)} t^2 (x^2 + y^2)}{(2!)} \quad (26)$$

$$\Psi_3(x, y, t) = \frac{t^3}{3!} \left(\frac{ie^{i(x+y)}}{m^3} + \frac{3k(x+y)e^{i(x+y)}}{m^2} - \frac{5ike^{i(x+y)}}{m^2} + \frac{3k(x+y)e^{i(x+y)}}{m} + \frac{3k^2(x^2+y^2)(x+y)e^{i(x+y)}}{2m} + \frac{3ik(x^2+y^2)e^{i(x+y)}}{2m} - \frac{5ik^2(x^2+y^2)^2 e^{i(x+y)}}{2m} + \frac{3ik^2(x^2+y^2)^2 e^{i(x+y)}}{4m} + \frac{2ie^{i(x+y)}}{m^2} - \frac{3ike^{i(x+y)}}{3ik^3(x^2+y^2)^3 e^{i(x+y)}} + \frac{3ik(x^2+y^2)e^{i(x+y)}}{m} + \frac{3ie^{i(x+y)}}{m} + \frac{ik^3(x^2+y^2)^3 e^{i(x+y)}}{8} + \frac{3ik(x^2+y^2)e^{i(x+y)}}{2} + \frac{3ik^2(x^2+y^2)^2 e^{i(x+y)}}{4} + ie^{i(x+y)} \right) \quad (27)$$

The approximate solution can be written as:

$$\Psi = \Psi_0 + \Psi_1 + \Psi_2 + \Psi_3 + \dots \Psi = e^{i(x+y)} \left[1 + \left(\frac{-it}{m} - \frac{ikt}{2} (x^2 + y^2) - it \right) + \frac{1}{2!} \left(\frac{-it}{m} - \frac{ikt}{2} (x^2 + y^2) - it \right)^2 + \frac{t^2}{2!} \left(\frac{ik(x+y)}{m} + \frac{k}{m} \right) + \frac{1}{3!} \left(\frac{-it}{m} - \frac{ikt}{2} (x^2 + y^2) - it \right)^3 + \frac{t^3}{3!} \left(\frac{3k(x+y)}{m^2} - \frac{5ik}{m^2} + \frac{3k(x+y)}{m} + \frac{3k^2(x^2+y^2)(x+y)}{2m} - \frac{5ik^2(x^2+y^2)}{2m} - \frac{3ik}{m} \right) + \dots \right] \quad (28)$$

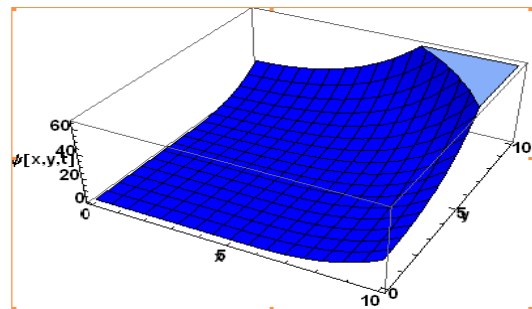
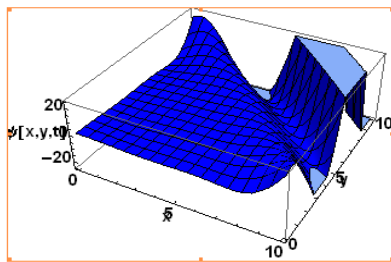


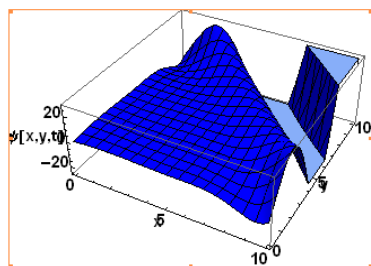
Fig. 3 Absolute approximate solution of the wave function in three-dimensional by EDM

Fig. 3 shows the surface absolute approximate solutions in three-dimensional with $t = 0.1$, $k = m = 1$, while, in Fig. 4,

it represents the surface real and imaginary approximate solution with $t = 0.1$, $k = m = 1$.



(a)



(b)

Fig. 4 (a) Real approximate solution of the three- dimensional wave function. (b) Imaginary approximate solution of the three-dimensional wave function

V. CONCLUSION

In this paper, we applied nonlinear Schrödinger equation with harmonic oscillator by EDM, and the EDM is more reliable and useful tool for obtaining the analytical solution and it presented the behavior of real, imaginary and absolute wave function in one and two dimensions.

REFERENCES

- [1] Rawashdeh M. S. and Maitama S. Solving coupled system of nonlinear PDEs using the Natural decomposition method, *Int J Pure Appl Math*, 92(2014) 757-776.
- [2] Nuruddeen R. I. and Nass A. M, Aboodh decomposition method and its application in solving linear and nonlinear heat equations, *European Journal of Advances in Engineering and Technology*, 3(7)(2016) 34-37.
- [3] Khuri SA. A Laplace decomposition algorithm applied to a class of nonlinear deferential equations. *Journal of Applied Mathematics*.2001;1(4):141-155.
- [4] Emad K. Jaradat WA Amer D Aloqali. Using Laplace Decomposition Method to Solve Non-linear Klein- Gordon Equation. *UPB Sci Bull*.2018;80(2): Series D.
- [5] Nuruddeen RI. Elzaki decomposition method and its applications in solving linear and nonlinear Schrodinger equations. *Sohag Journal of Mathematics*.2017; 4(2):1-5.
- [6] Adomain G. Solution of physical problem by decomposition, *Computers & Mathematics with Applications*.1994; 27(9-10):145-154.
- [7] Adomain G. A review of the decomposition method in applied mathematics. *Journal of mathematical analysis and applications*. 1988;135(2):501-544.
- [8] Biazar, J., et al." Solution of the linear and non-linear Schrödinger equations using homotopy perturbation and Adomain Decomposition methods." *International Mathematical Forum*. Vol.3. No.38. 2008.
- [9] Elzaki T. M., The new integral transform " Elzaki Transform" ,*Global Journal of Pure and Applied Mathematics*, 7(1)(2011) 57-64.
- [10] Zheng L, Wang T, Zhang X, Na L. The nonlinear Schrödinger harmonic

oscillator problem with small odd or even disturbances. *Applied Mathematics Letters*. 2013; 26(4):463-468.

- [11] Cresser J. *Quantum Physics Notes*. Department of Physics, Macquarie University, Australia.
- [12] Griffiths DJ, Schroeter DF. *Introduction to quantum mechanics*. Cambridge University Press; 2018.
- [13] Tariq M. Elzaki, The New Integral Transform " Elzaki Transform" *Global Journal of Pure and Applied Mathematics*, ISSN 0973-1768, Number 1(2011),PP.57-64.