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Application of Double Side Approach Method on Super Elliptical Winkler Plate

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Abstract**—**In this study, the static behavior of super elliptical Winkler plate is analyzed by applying the double side approach method. The lack of information about super elliptical Winkler plates is the motivation of this study and we use the double side approach method to solve this problem because of its superior ability on efficiently treating problems with complex boundary shape. The double side approach method has the advantages of high accuracy, easy calculation procedure and less calculation load required. Most important of all, it can give the error bound of the approximate solution. The numerical results not only show that the double side approach method works well on this problem but also provide us the knowledge of static behavior of super elliptical Winkler plate in practical use.

Keywords**—**Super elliptical Winkler Plate, double side approach method, error bound.

I. Introduction

THE double side approach method originates from the method of weighted Residual (MWR). MWR is a kind of mathematical methods that can find the approximate solutions of differential equations. This method has unified principles and wide applications [1]-[3]. The first step in MWR is to assume a trial function as the approximate solution to control the differential equation and this trial function has either the terms that already known or undetermined coefficients and functions. Generally speaking, if we substitute this trial function into the governing differential equation, the trial function won't satisfy the differential equation and residuals will come into existence. The notion of MWR is to force residuals to be zero in some average sense over the region. Therefore, a system of equations is formed to eliminate residuals. In order to eliminate residuals, residuals are multiplied by weighting functions that are introduced to realize the concept of eliminating residuals in an average sense. Equations for eliminating residuals are a series of linear or nonlinear algebraic equations. By solving these equations, undetermined coefficients in the trial function can be finally attained - that is, the desired approximate solution which makes the residual be the smallest or even zero is obtained.

According to the choices for weighting functions, MWR can be classified into five basic sub-methods: Collocation method; Subdomain method; Least Square Method; Galerkin's Method; Method of moments.

The double side approach method can be treated as the advanced MWR. In the double side approach method, the sub-methods of MWR are merely used to form the residuals, unlike the traditional MWR to force them to be zero.

In the double side approach method, there is still another important tool, that is-optimization method. In this study, we choose to use Genetic Algorithms (GAs), a class of probabilistic search algorithms for optimization problems.

In the study of the interaction and coordination between the system itself and the outer environment, Holland proposed the concept involved in evolutionary algorithms [4]. Bagley adopted several steps such as reproduction, crossover and mutation to probe into the strategy of international chess in his thesis of game theory and this is the first time that the noun "Genetic Algorithms" was formally used [5]. Holland developed the mode theorem which was the main theorem of genetic algorithms and wrote a book - "Adaptation in Nature and Artificial System." in which he introduced GAs completely in details [6].

From the late 70s to the early 80s, many scholars were engaged in the researches of GAs contributed to significant advancements in GAs [7]-[9]. Especially in the middle of 80s, GAs boomed in the artificial researches in America. Lawrence summarized the experiences of numerous scholars' engagement in GAs; he published a book "Genetic Algorithms and Simulated Annealing" and adopted a lot of practical examples to illustrate the use of GAs [10].

After the 90s, development of GAs was more extensive. In 1991, D. Lawrence published a book - "Handbook of Genetic Algorithms."[11] The book not only introduced the principles and practical examples of GAs, but also explained how to compile a C language to put GAs into realization. GAs make up for the weakness of traditional optimization methods by its inherent characteristics.

GAs start with a population of randomly generated candidates and evolve toward better solutions by applying genetic operators, just like the genetic processes occurring in the natural environment. GAs proceed the repeated iterations continuously and approach the optimal solution gradually. This kind of searching technique has the following advantages: Intellectual searching; Gradual optimization; Global optimum; Black-box structure; Strong universality; Parallel calculation.

II. Double Side Approach Method

The main concept of the double side approach method is on the basis of the maximum principle of differential equations [12]-[14]. By introducing the maximum principle, the

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monotonic relation between residuals and the solution can be proved to exist and the double side approach method just develops on this relation which can be briefly shown as when

$$R\breve{u} \le 0 \le R\hat{u} \tag{1}$$

then

$$\breve{u} \le u \le \hat{u} \tag{2}$$

Based on the relation in (1), (2), we can utilize the Mathematical Programming to demonstrate the minimal larger approximate solution (upper bound) and maximal smaller approximate solution (lower bound).

When we desire to find the upper and lower bounds of the solution, we can make use of the residual relation of approximate solutions to establish the constraints. When the residuals of approximate solutions $R\hat{u} \ge 0$, and all the approximate solutions are larger than the exact solution, the Mathematical Programming can be adopted to determine the minimal larger approximate solution. On the contrary, when the residuals of approximate solutions $R\hat{u} \le 0$, and all the approximate solutions are smaller than the exact solution, the Mathematical Programming can be adopted to determine the maximal smaller approximate solution. On the whole, the double side approach method is to solve the following two sets of Mathematical Programming problem.

$$\begin{cases} R\hat{u} \ge 0 \Rightarrow \min\ \hat{u} \\ R\breve{u} \le 0 \Rightarrow \max\ \breve{u} \end{cases} \tag{3}$$

and the best approximate solution of the problem is set as

$$\tilde{u} = \frac{\hat{u} + \breve{u}}{2} \tag{4}$$

The above is the outline of the solution procedure of double side approach method. In this study, we use Subdomain method in MWR to construct the inequality constraints of approximate solution and then use GAs to find the optimal solutions from both sides to determine the upper and lower bound solutions.

III. Problem Formulation

In the Cartesian coordinate, the shape equation of a super ellipse can be shown as

$$\frac{x^{2n}}{a^{2n}} + \frac{y^{2n}}{b^{2n}} = 1 \tag{5}$$

where n is the power of the super ellipse which dominates the shape of the super ellipse, ellipse (n=1) and rectangle (n=∞) are two limiting cases of the super ellipse. a and b are called the semi-diameters of the super ellipse, and all of them are positive numbers.

In practical applications, plates on elastic foundation are very important, such as concrete pavements of highways, airport runways, foundation of buildings and so on. However, most former studies presented results on rectangular plates on elastic foundation, lacking information about elliptical or super elliptical plates on the elastic foundation.

Winkler-type foundation which assumes the supporting medium is isotropic, homogeneous, and linear elastic is a kind of simplified model of plates on elastic foundation. The reaction can be written as

$$p_z(x, y) = kw \tag{6}$$

Then the governing equation of the plate with Winkler-type foundation under uniform loading for the deflection in rectangular coordinate is given as [15]

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{kw}{D} = \frac{q}{D} \tag{7}$$

where k is the foundation stiffness. Equation (7) can be also represented by bilaplacian as

$$\nabla^4 w + \frac{kw - q}{D} = 0 \tag{8}$$

In this study, the super elliptical plates are assumed to be clamped here, therefore, on the boundary S, the boundary conditions can be written as:

$$w_S = 0,\ \frac{\partial w_S}{\partial n_i} = w_x \times \cos\theta + w_y \times \sin\theta = 0 \tag{9}$$

where n_i is the outward normal of the boundary S.

There are two parts in this section. Because of the lack of information about the super elliptical plates on the elastic foundation, so in the first part, Galerkin's method is first conducted as the reference and then in the second part, the double side approach method based on the Subdomain method is utilized.

A. Galerkin's Method

Galerkin's method has been used to solve numerous engineering problems, either linear or nonlinear. Galerkin's method doesn't need a mesh generation and direct use of the differential equation and it's a powerful tool for solving plate problems.

To satisfy the clamped boundary conditions (9), the trial function should include the term $(x^{2n}/a^{2n} + y^{2n}/b^{2n} - 1)^2$. The trial function can be constructed as below:

$$Z(x, y) = \sum_i^r \sum_j^r C_{ij} \left(\frac{x^{2n}}{a^{2n}} + \frac{y^{2n}}{b^{2n}} - 1 \right)^2 x^i y^j \tag{10}$$

In Galerkin's method, the weighting function is chosen from the same family as the trial function, so the basis functions $\phi_{ij}(x, y)$ can be assumed as

$$\phi_{ij}(x, y) = \left(\frac{x^{2n}}{a^{2n}} + \frac{y^{2n}}{b^{2n}} - 1 \right)^2 x^i y^j \tag{11}$$

Equations having the same number as the undetermined coefficients C_{ij} can be obtained then to solve these algebraic equations to get the approximate solutions.

B. Double Side Approach Method

The residual of (8) can be presented as

$$R[w] = \nabla^4 w + \frac{kw-q}{D} \tag{12}$$

The double side approach method can only be applied when monotonicity holds. The monotonicity between the residual and the solution can be represented as

$$\frac{\partial R[w]}{\partial w} > 0 \tag{13}$$

The monotonicity of the biharmonic operator has been successfully proved in the former literatures [16], [17].And for the remaining part of the residual, the monotonicity can also be presented as in (14)

$$\frac{\partial}{\partial w}\left(\frac{kw-q}{D}\right) = \frac{k}{D} > 0 \tag{14}$$

Since (12) satisfies the monotonicity, this kind of problem can be solved by the double side approach method. Here q and D are set as the value of 1 for convenience and let the semi-diameter b (in y direction) be equal to 1.

One-dimensional Subdomain method is applied to transfer this differential equation into a mathematical programming problem. It should be noted that the number of constrained equations must be greater than the number of undetermined coefficients, so divide the y-axis (from (0,0) to (0,1)) into i+2 parts as the subdomains, and let the number of constrained equations be bigger than the number of undetermined coefficients by two. Then, do the integration toward each subdomain to get the inequalities. Here, we let i to be six.

Similarly, the mid-point (x=0 and y=0) of the plate is chosen as the index of the objective function, so the objective of this optimal problem under inequalities is to find the coefficients of the trial function by minimizing $Z(0,0,C_j)$ when

$$\begin{cases} \int_0^{1/i+2} R[Z]dy\Big|_{x=0} \geq 0 \\ \int_{1/i+2}^{2/i+2} R[Z]dy\Big|_{x=0} \geq 0 \\ \vdots \\ \int_{i+1/i+2}^{1} R[Z]dy\Big|_{x=0} \geq 0 \end{cases} \tag{15}$$

and on the other side, by maximizing $Z(0,0,C_j)$ when

$$\begin{cases} \int_0^{1/i+2} R[Z]dy\Big|_{x=0} \leq 0 \\ \int_{1/i+2}^{2/i+2} R[Z]dy\Big|_{x=0} \leq 0 \\ \vdots \\ \int_{i+1/i+2}^{1} R[Z]dy\Big|_{x=0} \leq 0 \end{cases} \tag{16}$$

The trial function which contains the shape equation of the super elliptic boundary is established to satisfy either the geometric or the kinematic boundary conditions as the form of

$$Z(x,y,c) = \sum_{j=1}^{i} C_j \left(\frac{x^{2n}}{a^{2n}} + \frac{y^{2n}}{b^{2n}} - 1\right)^{2j} \tag{17}$$

IV. Results and Discussions

Tables I-III present the deflection at the center of the clamped super elliptical Winkler plate under various values of a (a=1, 2, 4), n (n=2, 4, 6) and k (k=10, 20, 30, 40 50), and the results listed in these three tables are plotted in Figs. 1-3. Comparing the calculation results with those obtained by Galerkin's method, they support each other. It can be seen from Figs. 1-3 that the results obtained by Galerkin's method always locate between upper and lower bounds no matter how the parameters a, n, and k vary. Without solving a large set of algebraic equations as in Galerkin's method, the computing time of the double side approach method is much less. Besides, the double side approach method can find the upper and lower bounds of approximate solutions, that is-the error bound can be determined to provide the reliability of the approximate solution.

TABLE I
Solutions of Deflection at the Central Point (0, 0)

		a=1	a=2	a=4
k=10	upper	0.01904	0.03275	0.03295
	lower	0.01748	0.02995	0.03052
	approximate	0.01826	0.03135	0.03174
		(0.01786)	(0.03114)	(0.03148)
k=20	upper	0.01748	0.02639	0.02618
	lower	0.01515	0.02460	0.02402
	approximate	0.01632	0.02550	0.02510
		(0.01599)	(0.02535)	(0.02524)
k=30	upper	0.01592	0.02256	0.02302
	lower	0.01395	0.02071	0.01987
	approximate	0.01494	0.02164	0.02145
		(0.01446)	(0.02131)	(0.02145)
k=40	upper	0.01518	0.01998	0.02053
	lower	0.01256	0.01714	0.01670
	approximate	0.01387	0.01856	0.01862
		(0.01319)	(0.01833)	(0.01803)
k=50	upper	0.01348	0.01855	0.01704
	lower	0.01155	0.01513	0.01372
	approximate	0.01252	0.01684	0.01538
		(0.01211)	(0.01605)	(0.01575)

(): results obtained by Galerkin's method, n=2

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TABLE II
SOLUTIONS OF DEFLECTION AT THE CENTRAL POINT (0, 0)

		a=1	*a*=2	*a*=4
k=10	upper	0.02000	0.03265	0.03266
	lower	0.01798	0.03047	0.02995
	approximate	0.01899	0.03156	0.03131
		(0.01826)	(0.03171)	(0.03148)
k=20	upper	0.01817	0.02744	0.02710
	lower	0.01482	0.02428	0.02451
	approximate	0.01650	0.02586	0.02581
		(0.01631)	(0.02562)	(0.02524)
k=30	upper	0.01604	0.02296	0.02265
	lower	0.01317	0.01947	0.02008
	approximate	0.01461	0.02122	0.02137
		(0.01473)	(0.02143)	(0.02131)
k=40	upper	0.01492	0.01992	0.01965
	lower	0.01210	0.01728	0.01682
	approximate	0.01351	0.01860	0.01824
		(0.01341)	(0.01838)	(0.01802)
k=50	upper	0.01446	0.01773	0.01702
	lower	0.01115	0.01450	0.01431
	approximate	0.01281	0.01612	0.01567
		(0.01230)	(0.01616)	(0.01574)

(): results obtained by Galerkin's method, *n*=4

TABLE III
SOLUTIONS OF DEFLECTION AT THE CENTRAL POINT (0, 0)

		a=1	*a*=2	*a*=4
k=10	upper	0.01993	0.03295	0.03315
	lower	0.01726	0.02955	0.02962
	approximate	0.01860	0.03125	0.03139
		(0.01811)	(0.03106)	(0.03147)
k=20	upper	0.01845	0.02791	0.02715
	lower	0.01502	0.02386	0.02360
	approximate	0.01674	0.02589	0.02538
		(0.01617)	(0.02531)	(0.02522)
k=30	upper	0.01608	0.02265	0.02210
	lower	0.01276	0.01956	0.01928
	approximate	0.01442	0.02111	0.02069
		(0.01456)	(0.02130)	(0.02064)
k=40	upper	0.01512	0.01995	0.01986
	lower	0.01215	0.01747	0.01700
	approximate	0.01364	0.01871	0.01843
		(0.01330)	(0.01835)	(0.01801)
k=50	upper	0.01412	0.01839	0.01715
	lower	0.01053	0.01486	0.01390
	approximate	0.01233	0.01663	0.01553
		(0.01220)	(0.01608)	(0.01574)

(): results obtained by Galerkin's method, *n*=6

The calculation results presented in this section demonstrate that the double side approach method is definitely an efficiency and reliable tool to solve this kind of boundary value problems.

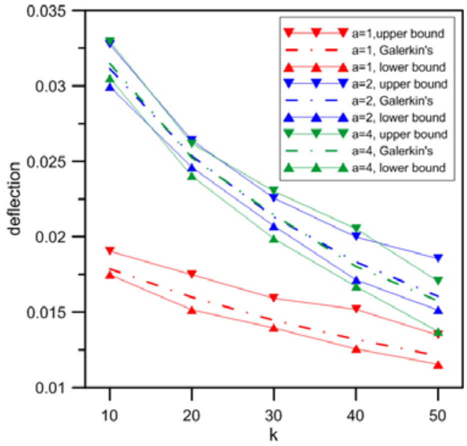

Fig. 1 Upper, lower bounds and Galerkin's solution under *n*=2

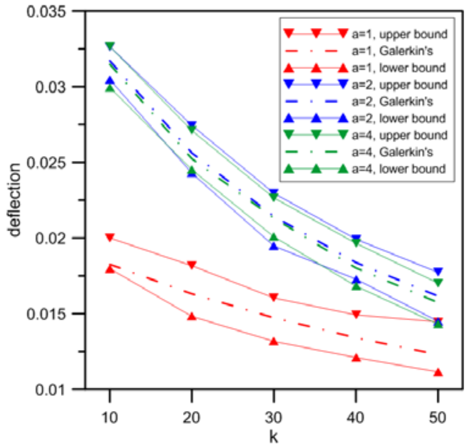

Fig. 2 Upper, lower bounds and Galerkin's solution under *n*=4

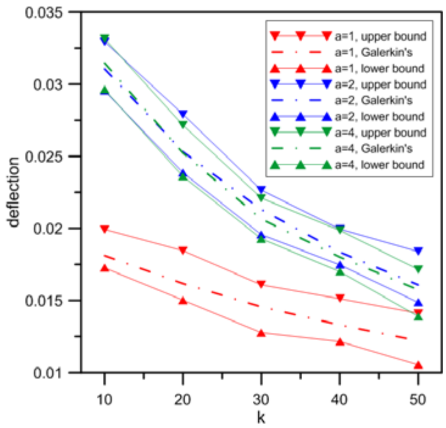

Fig. 3 Upper, lower bounds and Galerkin's solution under *n*=6

V. CONCLUSION

The static behavior of super elliptical plate is analyzed in this study and both Galerkin's method and the double side approach method are used and their numerical results support each other. In this study, the advantage of the double side approach method on dealing with boundary value problems is presented and the detailed information of static behavior of clamped super elliptical plates are given. Comparing to other numerical methods of finding approximate solutions, the double side approach method not only has high efficiency but also directly gives the upper and lower bounds of the approximate solution and this is what other numerical methods are hard to do.

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