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Anti-Homomorphism in Fuzzy Ideals

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Abstract—The anti-homomorphic image of fuzzy ideals, fuzzy ideals of near-rings and anti ideals are discussed in this note. A necessary and sufficient condition has been established for near-ring anti ideal to be characteristic.

Keywords—Fuzzy Ideals, Anti fuzzy subgroup, Anti fuzzy ideals, Anti homomorphism, Lower α level cut.

I. Introduction

N 1971, Rosenfeld [11] constituted the elementary concepts L of fuzzy subgroupoid, fuzzy ideals and fuzzy subgroups. Biswas [3] introduced the notion of anti fuzzy subgroups. Fuzzy subnear-rings are introduced by Abou-Zaid [1]. He studied fuzzy left (resp. right) ideals of a near-ring and gave some properties of fuzzy prime ideals of a near-ring. In [7], it has been established that homomorphic image of a fuzzy left (resp. right) ideal which has "sup property" is a fuzzy left (resp. right) ideal. In the year 1998, Sung M.H. et al. [12] proved the same result using the level fuzzy subsets and obtained some properties based on near-ring homomorphism. Properties of anti-homomorphic images of near-rings are discussed in [5]. Homomorphic images and pre images of anti fuzzy ideals are investigated by K.H. Kim et al. [9]. The notion of anti homomorphic image and pre image of fuzzy and anti fuzzy ideals are investigated in this paper. Also, near-ring anti homomorphic image and pre image of ideals are obtained.

A. Preliminaries

In this section, review of fuzzy set theoretic concepts are given briefly (for details one can refer [4], [11] and [10]). A fuzzy set μ of a set N is a function $\mu: N \to [0,1]$.

 μ will be called a fuzzy left ideal [11], if $\mu(xy) \geq \mu(y)$; a fuzzy right ideal, if $\mu(xy) \geq \mu(x)$; anti fuzzy left ideal [3] if $\mu(xy) \leq \mu(y)$; anti fuzzy right ideal, if $\mu(xy) \leq \mu(x)$;.

Let $f:N\to N'$ be a function and let μ and ν be fuzzy sets in N and N' respectively. Then $f(\mu)$ [11], the image of μ under f is a fuzzy set in N' defined by

$$f(\mu)(y) = \left\{ \begin{array}{ll} \sup \left(\mu(x) : x \in f^{-1}(y) \right) & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{Otherwise} \end{array} \right.$$

for all $y\in N'.$ $f^{-1}(\nu)$ [11], the preimage of ν under f is a fuzzy set in N given by

$$f^{-1}(\nu)(x) = \nu \left(f(x) \right)$$

for all $x \in N$.

Similar to an α level cut [4], we have lower level cut [9] as follows:

Let μ be a fuzzy set in a set N. For $\alpha \in [0, 1]$, the lower α level cut of μ is denoted by αN_{μ} and is given by

$$_{\alpha}N_{\mu} = \{n \in N : \mu(n) \leq \alpha\}.$$

Definition 1.1: [1], [7] Let N be a left near-ring and μ be a non empty fuzzy sub set of N. μ is said to be a fuzzy left N-ideal if

- I-1. $\mu(x-y) \ge \min\{\mu(x), \mu(y)\},\$
- I-2. $\mu(xy) \ge \min\{\mu(x), \mu(y)\}\$, for all $x, y \in N$,
- I-3. $\mu(y+x-y) \ge \mu(x)$ and
- I-4. $\mu(xy) \ge \mu(y)$ where $x, y \in N$

are satisfied. If axioms (I-1), (I-2), (I-3) with

I-5.
$$\mu((x+z)y - xy) \ge \mu(z)$$

holds, μ is a fuzzy right N-ideal.

From the definition of right near-ring [6], ideals can be defined as follows:

Definition 1.2: Let N be a right near-ring and μ be a non empty fuzzy sub set of N. μ is said to be a fuzzy left N-ideal if (I-1), (I-2), (I-3) and

I-6.
$$\mu(xy) \ge \mu(y)$$
 where $x, y \in N$

are satisfied. If (I-7) is postulated instead of (I-5) of fuzzy N-ideal of left near-ring, μ is a fuzzy right N-ideal where

I-7.
$$\mu(y(x+z) - yx) \ge \mu(z)$$
.

Definition 1.3: [9] Let N be a left near-ring and μ be a non empty fuzzy sub set of N. μ is said to be an anti fuzzy left N-ideal if

- AI-1. $\mu(x y) \le max(\mu(x), \mu(y)),$
- AI-2. $\mu(xy) \le max(\mu(x), \mu(y)),$
- AI-3. $\mu(y+x-y) \leq \mu(x)$ and
- AI-4. $\mu(xy) \leq \mu(x)$ where $x, y \in N$.

If axioms (AI-1), (AI-2), (AI-3) with the following (AI-5) are satisfied then μ is an anti fuzzy right N-ideal;

AI-5.
$$\mu((x+z)y - xy) \le \mu(z)$$
.

Definition 1.4: Let N be a right near-ring and μ be a non empty fuzzy sub set of N. μ is said to be an anti fuzzy left N-ideal when AI-1 to AI-3 along with

AI-6.
$$\mu(xy) \le \mu(y)$$
 where $x, y \in N$

are postulated. If axioms (AI-1), (AI-2), (AI-3) with (AI-7) are satisfied then μ is an anti fuzzy right N-ideal;

AI-7.
$$\mu(y(x+z) - yx) \le \mu(z)$$
.

Recall that, a function $f: N \to N'$ of near-rings is called an *anti-homomorphism* [5] when

- 1. f(n+m) = f(m) + f(n)
- 2. f(nm) = f(m)f(n), for all $n, m \in N$.

A surjective anti-homomorphism is called an **anti-epimorphism** (\cong).

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II. MAIN RESULTS

A. Fuzzy Ideals

It is to be noted that the anti homomorphic image pre-image of a fuzzy groupoid is again a fuzzy groupoid. Where as,

Result 2.1: An anti homomorphic pre-image of a right (left) ideal is a left (right) ideal.

Proof: Let ν be a fuzzy left ideal. Then

$$\mu(xy) = \nu(f(xy))$$

$$= \nu(f(y)f(x))$$

$$\geq \nu(f(x))$$

$$= \mu(x),$$

a right ideal. When ν is a fuzzy right ideal,

$$\mu(xy) = \nu(f(xy))$$

$$= \nu(f(y)f(x))$$

$$\geq \nu(f(y))$$

$$= \mu(y),$$

a left ideal.

Result 2.2: Anti-homomorphic image of a fuzzy left (right) ideal, with spermium property, is a fuzzy right (left) ideal.

Proof: Let μ be a fuzzy left ideal with sup property. Given f(x), f(y) in f(N), let $x_0 \in f^{-1}[f(x)], y_0 \in f^{-1}[f(y)]$ be such that

$$\mu(x_0) = \sup_{t \in f^{-1}[f(x)]} \mu(t), \ \mu(y_0) = \sup_{t \in f^{-1}[f(y)]} \mu(t)$$

respectively. Then

$$\nu[f(x)f(y)] = \nu[f(yx)]
= \sup_{t \in f^{-1}[f(yx)]} \mu(t)
\ge \mu(y_0x_0)
\ge \mu(x_0)
= \sup_{t \in f^{-1}[f(x)]} \mu(t)
= \nu[f(x)],$$

implies ν is a fuzzy right ideal. Similarly, when μ is a fuzzy right ideal with above property, we have ν is a fuzzy left ideal. \square

In sequel to the above results, the following also can be established for the case of anti fuzzy left (right) ideals.

Result 2.3: An anti homomorphic pre-image of an anti fuzzy right (left) ideal is an anti fuzzy left (right) ideal.

Result 2.4: An anti homomorphic image of an anti fuzzy right (left) ideal with sup property, is an anti fuzzy left (right) ideal.

B. Fuzzy Ideals in Near-rings

Result 2.5: ([5] Theorem 2.2) Anti homomorphic image of a right near-ring (left near-ring) is a left near-ring (right near-ring).

Result 2.6: Let $f: N \to N'$ be an anti-epimorphism of near-rings. If ν is a fuzzy (left/right) ideal in the right (left) near-ring N', then μ , which is $f^{-1}(\nu)$ is a fuzzy (left/right) ideal in the left (right) near-ring N.

Proof:

Let ν be a fuzzy left ideal of right near-ring N'. The proof of conditions (I-1), (I-2) and (I-3) of definition 1.1 are similar to that of proof of [7] Theorem 2.12. For any $x,y\in N$, we have

$$\mu(xy) = \nu(f(xy))$$

$$= \nu(f(y)f(x))$$

$$\geq \nu(f(y))$$

$$= \mu(y).$$

Thus μ is a fuzzy left ideal of the left near-ring N. When ν is a right ideal of right near-ring N', for any $x,y,z\in N$ we have.

$$\begin{array}{lcl} \mu((x+z)y-xy) & = & \nu(f((x+z)y-xy)) \\ & = & \nu(f(y)f(x+z)-f(xy)) \\ & = & \nu(f(y)(f(x)+f(z))-f(y)f(x)) \\ & \geq & \nu(f(z)) \\ & = & \mu(z), \end{array}$$

 $\boldsymbol{\mu}$ is a fuzzy right ideal of left near-ring N.

Let ν be a right ideal of left near-ring N'. For any $x,y,z\in N,$ we have

$$\begin{array}{lll} \mu(y(x+z)-yx) & = & \nu(f((x+z)y-xy)) \\ & = & \nu(f(x+z)f(y)-f(yx)) \\ & = & \nu((f(x)+f(z))f(y)-f(x)f(y)) \\ & \geq & \nu(f(z)) \\ & = & \mu(z). \end{array}$$

Thus μ is a fuzzy left ideal of right near-ring N. Let ν is a fuzzy left ideal of left near-ring N'. Then

$$\mu(xy) = \nu(f(xy))$$

$$= \nu(f(y)f(x))$$

$$\geq \nu(f(x))$$

$$= \mu(x)$$

for all $x,y\in N$, implies μ is a left ideal of right near ring N. \square

Result 2.7: Let $f: N \to N'$ be an anti-epimorphism of near-rings. If μ is a fuzzy (left/right) ideal in the left (right) near-ring N with sup property, then $\nu = f(\mu)$ is a fuzzy (left/right) ideal in the right (left) near-ring N'.

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Proof: Let μ be a fuzzy left ideal of the left near-ring N with sup property and ν be the image of μ under f. Let $x_0 \in f^{-1}[f(x)], y_0 \in f^{-1}[f(y)]$ such that

$$\mu(x_0) = \sup_{t \in f^{-1}[f(x)]} \mu(t), \quad \mu(y_0) = \sup_{t \in f^{-1}[f(y)]} \mu(t).$$

$$\nu[f(x)f(y)] = \sup_{t \in f^{-1}[f(yx)]} \mu(t)$$

$$\geq \mu(y_0x_0)$$

$$\geq \mu(x_0)$$

$$= \sup_{t \in f^{-1}[f(x)]} \mu(t)$$

$$= \nu[f(x)]$$

That is ν is a fuzzy left ideal of right near-ring N'.

If μ is a fuzzy right ideal of left near-ring. For any $f(z) \in f(N)$, let $z_0 \in f^{-1}[f(z)]$ such that

$$\mu(z_0) = \sup_{t \in f^{-1}[f(z)]} \mu(z).$$

Now,

$$\begin{split} &\nu[f\{(x+z)y-(xy)\}]\\ = &\nu[f(y)f(x+z)-f(xy)]\\ = &\nu[f(y)\{f(x)+f(z)\}-f(y)f(x)]\\ = &\sup_{t\in f^{-1}\{f[f(y)\{f(x)+f(z)\}-f(y)f(x)]\}}\mu(t)\\ \geq &\mu[y_0\{x_0+z_0\}-y_0x_0]\\ \geq &\mu[z_0]\\ = &\sup_{t\in f^{-1}[f(z)]}\mu(t)\\ = &\nu[f(z)]. \end{split}$$

That is, ν is a fuzzy right ideal of right near-ring.

The image and pre-image of the fuzzy ideal of a fuzzy right near-ring N can be proved to be the fuzzy ideal of a left near-ring N'.

C. Anti Fuzzy Ideals in Near-rings

Definition 2.8: A left N-ideal A of a near-ring is said to be characteristic [2], [8] if

$$f(A) = A \ \forall f \in Aut(N)$$

where Aut(N) is set of all automorphism of N.

Anti fuzzy left N-ideal of μ of a near-ring N is said to be anti fuzzy characteristic if

$$\mu f_{(x)} = \mu(x) \quad \forall x \in N, f \in Aut(N).$$

Lemma 2.9: Let μ be an anti fuzzy left N-ideal of a near-ring N and let $x \in N$. Then $\mu(x) = s$ if and only if $x \in {}_sN_{\mu}$ and $x \notin {}_tN_{\mu} \ \forall s > t$.

Proof is obvious.

The proof of the following theorem is analogous to the proof of theorem 3.9 [2], [8].

Theorem 2.10: Let μ be an anti fuzzy N-ideal of a near-ring N. Then each lower α level left N-ideal of μ is characteristic iff μ is an anti fuzzy characteristic of N.

K. H. Kim et al. [9] proved the following theorems:

Result 2.11 ([9], Theorem 3.19 (1)): Let $f: N \to N'$ be an epimorphism of near-rings. Let ν be an anti-fuzzy left N'-ideal and μ be the pre-image of ν under f. Then μ is an anti-fuzzy left N-ideal.

Result 2.12 ([9], Theorem 3.19 (2)): Let $f: N \to N'$ be a surjective homomorphism of near-rings. If μ is an anti fuzzy left N'-ideal then $f^{-1}(\mu)$ is an anti fuzzy left N-ideal.

Result 2.13: Let $f: N \to N'$ be an anti-epimorphism of near-rings. Let ν be an anti-fuzzy (left/right) ideal of right (left) near-ring N' then μ , the pre-image of ν under f, is an anti-fuzzy (left/right) ideal of left (right) near-ring N.

The proof of AI-4 and AI-5 are similar to the proof of result 2.6.

Result 2.14: Let $f: N \to N'$ be an anti epimorphism. If μ is an anti fuzzy (left/right) ideal of left (right) near-ring N with sup property, $f(\mu)$ is an anti fuzzy (left/right) ideal of right (left) near-ring N'.

The proof of AI-6 and AI-7 are similar to that of result 2.7.

REFERENCES

- [1] S. Abou-Zaid, On fuzzy subnear-rings and ideals, Fuzzy Sets and Systems 44, 139 - 146, 1991.
- [2] M. Akram and K. H. Dar, On Anti Fuzzy Left h-ideals in Hemirings, International Mathematical Forum, 2, no. 46, 2295 - 2304, 2007.
- [3] R. Biswas, Fuzzy subgroups and anti fuzzy subgroups, Fuzzy Sets Systems 44, 121 - 124, 1990.
- [4] P. S. Das, Fuzzzy groups and level subgroups, J. Math. Anal. and Appl. 84, 264 - 269, 1981.
- [5] K. Chandrasekhara Rao and V. Swaminathan, Anti-homomorphisms in Near-Rings Jour. of Inst. of Maths. Comp. Sciences (Maths. Ser.) Vol. 21, No. 2, 83 - 88, 2008.
- [6] Günter Pilz, Near-rings, The Theory and applications, North-Holland Publishing company, Amsterdam, New York, Oxford, 1983.
- [7] S. D. Kim and H. S. Kim, On fuzzy ideals of near-rings, Bull. Korean Math. Soc. 33, 593 - 601, 1996.
- [8] K. H. Kim and Y. B. Jun, A note on fuzzy R-subgroups of near-rings, Soochow Journal of Mathematics, Vol. 28, no. 4, 330 - 346, 2002.
- [9] K. H. Kim and Y. B. Jun, On Anti Fuzzy Ideals in Near-Rings, Iranian Jr. of Fuzzy Systems, Vol. 2, No. 2, 71 - 80, 2005.
- [10] Mohamed Asaad, Groups and fuzzy subgroups, Fuzzy Sets and Systems 39, 323 - 328, 1991.
- [11] A. Rosenfeld, Fuzzy groups, J. Math. Anal. Appl. 35, 512 517, 1971.
- [12] Sung Min Hong, Young Bae Jun and Hee Sik Kim, Fuzzy Ideals in Near-rings, Bull. Korean Math. Soc. 35, pp 455 - 464, 1998.
- [13] H. Kopka and P. W. Daly, A Guide to BTEX, 3rd ed. Harlow, England: Addison-Wesley, 1999.

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