

Another Structure of Weakly Left C-wrpp Semigroups

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Abstract—It is known that a left C-wrpp semigroup can be described as curler structure of a left band and a C-wrpp semigroup. In this paper, we introduce the class of weakly left C-wrpp semigroups which includes the class of weakly left C-rpp semigroups as a subclass. We shall particularly show that the spined product of a left C-wrpp semigroup and a right normal band is a weakly left C-wrpp semigroup. Some equivalent characterizations of weakly left C-wrpp semigroups are obtained. Our results extend that of left C-wrpp semigroups.

Keywords—Left C-wrpp semigroup, left quasi normal regular band, weakly left C-wrpp semigroup.

I. INTRODUCTION

THROUGHOUT this paper, we adopt the notation and terminologies given by Howei[1] and Du[2].

Tang[3] considered a Green-like right congruence relation \mathcal{L}^{**} on a semigroup S : for $a, b \in S$, $a\mathcal{L}^{**}b$ if and only if $ax\mathcal{R}ay \Leftrightarrow bx\mathcal{R}by$ for all $x, y \in S^1$. Moreover, Tang pointed out in [3] that a semigroup S is a wrpp semigroup if and only if each \mathcal{L}^{**} -class of S contains at least one idempotent.

Recall that a wrpp semigroup S is a C-wrpp semigroup if the idempotents of S are central. It is well known that a semigroup S is a C-wrpp semigroup if and only if S is a strong semilattice of left- \mathcal{R} cancellative monoids(see[3]). Because a Clifford semigroup can be expressed as a strong semilattice of groups and a C-rpp semigroup can be expressed as a strong semilattice of left cancellative monoids(see[4-9]), we see immediately that the concept of C-wrpp semigroups is a common generalization of Clifford semigroups and C-rpp semigroups.

For wrpp semigroups, Du-Shum [2] first introduced the concept of left C-wrpp semigroups, that is, a left C-wrpp semigroup whose satisfy the following conditions: (i) for all $e \in E(L_a^{**})$, $a = ae$, where $E(L_a^{**})$ is the set of idempotents in L_a^{**} ; (ii) for all $a \in S$, there exists a unique idempotent a^+ satisfying $a\mathcal{L}^{**}a^+$ and $a = a^+a$; (iii) for all $a \in S$, $aS \subseteq L^{**}(a)$, where $L^{**}(a)$ is the smallest left $**$ -ideal of S generated by a . For such semigroups, Du-Shum[2] gave a method of construction.

Zhang[10] showed that the spined product of a left C-wrpp semigroup and a right normal band which is a weakly left C-wrpp semigroup by virtue of left C-full Ehreman cybergroups. In this paper, we first define the concept of

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weakly left C-wrpp semigroups. A equivalent descriptions of weakly left C-wrpp semigroups is therefore obtained and our results generalize that of Cao on weakly left C-rpp in[5]. In view of the theorems given in this paper, one can easily observe that the results of weakly left C-wrpp semigroups are a common generalizations of weakly left C-semigroups and left C-wrpp semigroups in range of wrpp semigroups.

II. PRELIMINARIES

We first recall some known results used in the sequel. To start with, we introduce the concept of simi-spined product.

Let $T = \cup_{\alpha \in Y} T_\alpha$ and $I = \cup_{\alpha \in Y} I_\alpha$ be the semilattice decomposition of the semigroups T and I with respect to semilattice Y respectively. For all $\alpha \in Y$, we denote the direct product $I_\alpha \times T_\alpha$ by S_α . Let $S = \cup_{\alpha \in Y} S_\alpha$. we define the mapping η by the following rules:

$\eta: S \rightarrow T_l(I)$, $(i, a) \mapsto \eta(i, a)$, $\eta(i, a): I \rightarrow I$, $j \mapsto (i, a)^{\#}j$, where $T_l(I)$ is a left transformation semigroup on I . Suppose that the mapping η satisfies the following conditions:

(Q1) If $(i, a) \in S_\alpha, j \in I_\beta$, then $(i, a)^{\#}j \in I_{\alpha\beta}$;

(Q2) If $(i, a) \in S_\alpha, (j, b) \in S_\beta$ with $\alpha \leq \beta$, then $(i, a)^{\#}j = ij$, where ij is the semigroup product in the semigroup $I = \cup_{\alpha \in Y} I_\alpha$;

(Q3) If $(i, a) \in S_\alpha, (j, b) \in S_\beta$, then $\eta(i, a)\eta(j, b) = \eta((i, a)^{\#}j, ab)$, where ab is the semigroup product in the semigroup $T = \cup_{\alpha \in Y} T_\alpha$.

Then we define a multiplication " \circ " on $S = \cup_{\alpha \in Y} S_\alpha$ by $(i, a) \circ (j, b) = ((i, a)^{\#}j, ab)$. By a straightforward verification, we can prove that the multiplication " \circ " satisfies the associative law and hence (S, \circ) becomes a semigroup, denoted by $S = I \times_{\eta} T$. We call this semigroup the semi-spined product of I and T with respect to the structure mapping η .

Lemma 1[2] Let I be a left regular band which is expressed as a semilattice of left zero bands I_α (that is, $I = \cup_{\alpha \in Y} I_\alpha$) and let $T = \cup_{\alpha \in Y} T_\alpha$ be a C-wrpp semigroup(that is, T is a strong semilattice of left- \mathcal{R} cancellative monoids $[Y; T_\alpha, \phi_{\alpha, \beta}]$ (see[3]). If the structure mapping η satisfies the following condition:

(Q): $\ker \eta(i, a) = \ker \eta(j, b)$ for every $(i, a), (j, b) \in S_\alpha$.

Then S is a left C-wrpp semigroup. Conversely, every left C-wrpp semigroup S can be constructed in terms of above method.

Lemma 2[5] A semigroup S is a weakly left C-semigroup, that is, S is a regular semigroup and

$$(\forall e \in E(S))\eta'_e: S \rightarrow eS, x \mapsto ex$$

is a homomorphism if and only if S is a completely regular and $E(S)$ is a left quasi-normal band.

Lemma 3[2] If S is a left C-wrpp, then $\text{Reg}S$ is a left C-semigroup.

Lemma 4[7] A band B is a left normal band (that is, a band satisfies identity $efeg = efg$) if and only if Green relation \mathcal{L} and \mathcal{R} are congruence on B and B/R is a right normal band.

Definition 1 A monoid T is called a left- \mathcal{R} cancellative monoid if for $a, b, c \in T$, $(ab, ac) \in \mathcal{R}$ implies $(b, c) \in \mathcal{R}$. We call the direct product of a left- \mathcal{R} cancellative monoid T and a rectangular band I a left cancellative plank because the direct product looks like a two-dimensional plank. We denote the left- \mathcal{R} cancellative plank by $I \times T$.

Lemma 5[2] Let $I = \cup_{\alpha \in Y} I_\alpha$ be a semilattice of left zero bands, and $T = [Y; T_\alpha, \phi_{\alpha, \beta}]$ a strong semilattice of left- \mathcal{R} cancellative monoids T_α . Then $(i, a)\mathcal{R}(j, b)$ if and only if $a\mathcal{R}b$ and $i = j$ for any $(i, a), (j, b) \in S = \cup_{\alpha \in Y} (I_\alpha \times T_\alpha)$.

III. THE WEAKLY LEFT C-WRPP SEMIGROUPS

In this section, the concept of weakly left C-wrpp semigroups is introduced. We shall give equivalent characterization for the structure of weakly left C-wrpp semigroups. First, we introduce the concept of weakly left C-wrpp semigroups.

Definition 2 A semigroup S is called a weakly left C-wrpp semigroup, if S is isomorphic to a semilattice of left- \mathcal{R} cancellative planks, and

$$(\forall \in E(S))\eta'_e : S \rightarrow eS, x \mapsto ex$$

is a homomorphism.

We now characterize the weakly left C-wrpp semigroups.

Theorem 1 Let S be a semigroup. Then the following conditions are equivalent:

- (1) S is a weakly left C-wrpp semigroup;
- (2) S is a semilattice of left- \mathcal{R} cancellative planks, and $\text{Reg}S$ is a weakly left C-semigroup;
- (3) S is a semilattice of left- \mathcal{R} cancellative planks, and $E(S)$ is a left quasi-normal band;
- (4) S is a spined product of left C-wrpp semigroup and a right normal band.

Proof. (1) \Rightarrow (2). We only need show that $\text{Reg}S$ is a weakly left C-semigroup. Let $a, b \in \text{Reg}S$. Then there exists $x, y \in S$ such that $a = axa, x = xax, b = byb$. So $e = xa \in E(S)$. According to (1), we know that η'_e is a semigroup homomorphism from S to eS . Thus

$$\begin{aligned} ab &= axabyb = a\eta'_e[(by)b] = a\eta'_e(by)\eta'_e(b) \\ &= axabyxab = (ab)(yx)(ab) \end{aligned}$$

So $ab \in \text{Reg}S$. Therefore, $\text{Reg}S$ is a subsemigroup of S . Again $E(\text{Reg}S) = E(S)$, according to Lemma 3, we obtain $\text{Reg}S$ is a weakly left C-semigroup.

(2) \Rightarrow (3). Clearly, we omit it.

(3) \Rightarrow (4). Let $S = \cup_{\alpha \in Y} (I_\alpha \times T_\alpha \times \Lambda_\alpha)$ is a semilattice decomposition Y of left- \mathcal{R} cancellative planks, and $E(S)$ is a left quasi-normal band, and put $S_l = \cup_{\alpha \in Y} (I_\alpha \times T_\alpha)$, $\Lambda = \cup_{\alpha \in Y} \Lambda_\alpha$, $S_\alpha = I_\alpha \times T_\alpha \times \Lambda_\alpha$, where I_α, T_α and Λ_α are a left zero band, a left- \mathcal{R} cancellative monoid and a right zero band, respectively. Next, we verify that $S_l = \cup_{\alpha \in Y} (I_\alpha \times T_\alpha)$ is a

left C-wrpp semigroup, and $\Lambda = \cup_{\alpha \in Y} \Lambda_\alpha$ is a right normal band.

Step 1 Let $T = \cup_{\alpha \in Y} T_\alpha$. we shall show that T is a C-wrpp semigroup, and $\Lambda = \cup_{\alpha \in Y} \Lambda_\alpha$ is a right normal band. For this purpose, we only need to show that T is a strong semilattice of left- \mathcal{R} cancellative monoids T_α , and a strong semilattice of right zero bands Λ_α , respectively.

Identity in T_α denoted by I_α , obviously, we have $E(S) = \{(i, 1_\alpha, \lambda) | (i, \lambda) \in I_\alpha \times \Lambda_\alpha, \alpha \in Y\}$, and

$$(i, 1_\alpha, \lambda)\mathcal{L}^E(j, 1_\beta, \mu) \Leftrightarrow \alpha = \beta, \lambda = \mu, \quad (1)$$

$$(i, 1_\alpha, \lambda)\mathcal{R}^E(j, 1_\beta, \mu) \Leftrightarrow \alpha = \beta, i = j \quad (2)$$

where \mathcal{L}^E and \mathcal{R}^E are Green's relations on semigroup $E(S)$.

For all $\alpha \geq \beta$, let $a = (i, g, \lambda) \in S_\alpha$, if $(j, \mu) \in I_\beta \times \Lambda_\beta$, then there exists $(j_1, h_1, \mu_1) \in S_\beta$ such that $(j, 1_\beta, \mu)a = (j_1, h_1, \mu_1)$. Since $(j_1, h_1, \mu_1) = (j, 1_\beta, \mu)[(j, 1_\beta, \mu)a] = (j, h_1, \mu_1)$, we obtain $j_1 = j$. On the other hand, for all $j' \in I_\beta$, we have $(j', 1_\beta, \mu)a = (j', 1_\beta, \mu) = [(j, 1_\beta, \mu)a] = (j', h_1, \mu_1)$. So h_1, μ_1 do not depend on the choice of j in I_β . Let $h_1 = \mu(i, g, \lambda)\chi_{\alpha, \beta}, \mu_1 = \mu(i, g, \mu)\psi_{\alpha, \beta}$. Then we have

$$(j, 1_\beta, \mu)(i, g, \lambda) = (j, \mu(i, g, \lambda)\chi_{\alpha, \beta}, \mu(i, g, \mu)\psi_{\alpha, \beta}). \quad (3)$$

Similarly, we show that there exists $\phi_{\beta, \alpha}(i, g, \lambda)j \in I_\beta$, $\varphi_{\beta, \alpha}(i, g, \lambda)j \in T_\beta$ such that

$$(i, g, \lambda)(j, 1_\beta, \mu) = (\phi_{\beta, \alpha}(i, g, \lambda)j, \varphi_{\beta, \alpha}(i, g, \lambda)j, \mu). \quad (4)$$

For all $\lambda' \in \Lambda_\alpha$, we have obtain $(i, 1_\alpha, \lambda)\mathcal{R}^E(i, 1_\alpha, \lambda')$ by (2). According to lemma 4, we know that \mathcal{R}^E is a congruence on $E(S)$, it follows that $(i, 1_\alpha, \lambda)(j, 1_\beta, \mu)\mathcal{R}^E(i, 1_\alpha, \lambda')(j, 1_\beta, \mu)$. Since $E(S)$ is a band, Referring to (2) and (4), we can follow that $(i, 1_\alpha, \lambda)(j, 1_\beta, \mu) = (i, 1_\alpha, \lambda')(j, 1_\beta, \mu)$, multiplied with from left side of above formula's both sides, by (4), we obtain $\phi_{\beta, \alpha}(i, g, \lambda)j = \phi_{\beta, \alpha}(i, g, \lambda')j, \varphi_{\beta, \alpha}(i, g, \lambda)j = \varphi_{\beta, \alpha}(i, g, \lambda')j$. Therefore, $\phi_{\beta, \alpha}(i, g, \lambda)j$ and $\varphi_{\beta, \alpha}(i, g, \lambda)$ do not depend on the choice of λ , let

$$\phi_{\beta, \alpha}(i, g)j = \phi_{\beta, \alpha}(i, g, \lambda)j, \varphi_{\beta, \alpha}(i, g)j = \varphi_{\beta, \alpha}(i, g, \lambda)j, \quad (5)$$

where $\lambda \in \Lambda_\alpha, \alpha \geq \beta$. Similarly, by \mathcal{L}^E is a congruence on $E(S)$, we follow that $\mu(i, g, \lambda)\chi_{\alpha, \beta}$ and $\mu(i, g, \lambda)\psi_{\alpha, \beta}$ do not depend on the choice of i in I_α , let

$$\mu(g, \lambda)\chi_{\alpha, \beta} = \mu(i, g, \lambda)\chi_{\alpha, \beta}, \mu(g, \lambda) = \mu(i, g, \lambda)\psi_{\alpha, \beta} \quad (6)$$

where $i \in I_\alpha, \alpha \geq \beta$. It follows that $(j, \mu(g, \lambda)\chi_{\alpha, \beta}, \mu) = [(j, 1_\beta, \mu)(i, g, \lambda)](j, 1_\beta, \mu) = (j, 1_\beta, \mu)[(i, g, \lambda)(j, 1_\beta, \mu)] = (j, \varphi_{\beta, \alpha}(i, g)j, \mu)$. So $\mu(g, \lambda)\chi_{\alpha, \beta} = \varphi_{\beta, \alpha}(i, g)j$, write as c . Clearly, c is determined by g but does not depend on the choice of i, j, λ and μ . Let

$$g\sigma_{\alpha, \beta} = \mu(g, \lambda)\chi_{\alpha, \beta} = \varphi_{\beta, \alpha}(i, g)j, \quad (7)$$

where $i \in I_\alpha, j \in I_\beta, \lambda \in \Lambda_\alpha$ and $\mu \in \Lambda_\beta$. According to \mathcal{L}^E being a right normal band congruence on $E(S)$, for all $\mu, \mu' \in \Lambda_\beta$, we have $(j, 1_\beta, \mu')(j, 1_\beta, \mu)(i, 1_\alpha, \lambda)\mathcal{L}^E(j, 1_\beta, \mu)(j, 1_\beta, \mu')(i, 1_\alpha, \lambda)$, that is, $(j, 1_\beta, \mu)(i, 1_\alpha, \lambda)\mathcal{L}^E(j, 1_\beta, \mu')(i, 1_\alpha, \lambda)$. we can follow that $(j, 1_\beta, \mu)(i, 1_\alpha, \lambda) = (j, 1_\beta, \mu')(i, 1_\alpha, \lambda)$ in view

of (1) and (3), multiplied with (i, g, λ) from right side of above formula's both sides, referring to (3) and (6), we obtain $\mu(g, h)\psi_{\alpha, \beta} = \mu'(g, h)\psi_{\alpha, \beta}$. Therefore, $\mu(g, h)\psi_{\alpha, \beta}$ does not depend on the choice of μ in Λ_β , let

$$(g, \lambda)\psi_{\alpha, \beta} = \mu(g, \lambda)\psi_{\alpha, \beta} \quad (8)$$

where $\mu \in \Lambda_\beta, \alpha \geq \beta$, In view of (3)-(8), we have

$$\begin{aligned} & (j, g\sigma_{\alpha, \beta}, (g, \lambda)\psi_{\alpha, \beta}) \\ &= (j, 1_\beta, \mu)(i, g, \lambda) \\ &= [(j, 1_\beta, \mu)(i, g, \lambda)](i, 1_\alpha, \lambda) \\ &= (j, g\sigma_{\alpha, \beta}, (g, \lambda)\psi_{\alpha, \beta})(i, 1_\alpha, \lambda) \\ &= (j, g\sigma_{\alpha, \beta}, (g, \lambda)\psi_{\alpha, \beta})[(j, 1_\beta, (g, \lambda)\psi_{\alpha, \beta})(i, 1_\alpha, \lambda)] \\ &= (j, g\sigma_{\alpha, \beta}, (g, \lambda)\psi_{\alpha, \beta})(j, 1_\alpha\sigma_{\alpha, \beta}, (1_\alpha, \lambda)\psi_{\alpha, \beta}) \\ &= (j, (g\sigma_{\alpha, \beta})(1_\alpha\sigma_{\alpha, \beta}), (1_\alpha, \lambda)\psi_{\alpha, \beta}). \end{aligned}$$

Therefore

$$g\sigma_{\alpha, \beta} = (g\sigma_{\alpha, \beta})(1_\alpha\sigma_{\alpha, \beta}), (g, \lambda)\psi_{\alpha, \beta} = (1_\alpha, \lambda)\psi_{\alpha, \beta}.$$

Since T_β is a left- \mathcal{R} cancellative monoid,

$$1_\alpha\sigma_{\alpha, \beta}\mathcal{R}1_\beta(\alpha \geq \beta), \quad (9)$$

let

$$\lambda\theta_{\alpha, \beta} = (1_\alpha, \lambda)\psi_{\alpha, \beta} = (g, \lambda)\psi_{\alpha, \beta}, (g \in T_\alpha, \alpha \geq \beta). \quad (10)$$

Thus, summing up the above cases, we conclude that there exists the mapping: $\phi_{\beta, \alpha} : I_\alpha \times T_\alpha \rightarrow T_l(I_\beta), (i, g) \mapsto \phi_{\beta, \alpha}(i, g); \sigma_{\alpha, \beta} : T_\alpha \rightarrow T_\beta, g \mapsto g\sigma_{\alpha, \beta}; \theta_{\alpha, \beta} : \Lambda_\alpha \rightarrow \Lambda_\beta, \lambda \mapsto \lambda\theta_{\alpha, \beta}$ such that

$$(j, 1_\beta, \mu)(i, g, \lambda) = (i, g\sigma_{\alpha, \beta}, \lambda\theta_{\alpha, \beta}) \quad (11)$$

$$(i, g, \lambda)(j, 1_\beta, \mu) = (\phi_{\beta, \alpha}(i, g)j, g\sigma_{\alpha, \beta}, \mu) \quad (12)$$

for all $(i, g, \lambda) \in S_\alpha, (j, \mu) \in I_\beta \times \Lambda_\beta$.

The following we verify that $\sigma_{\alpha, \beta}$ and $\theta_{\alpha, \beta}$ are the structure homomorphism of strong semilattice on semigroups T_α and Λ_α , respectively. For all $\alpha, \beta \in Y, (i, g, \lambda) \in S_\alpha, (j, h, \mu) \in S_\beta$, let $(k, m, n) = (i, g, \lambda)(j, h, \mu) \in S_{\alpha\beta}$. Then for $\gamma \leq \alpha\beta$ and $(l, v) \in I_\gamma \times \Lambda_\gamma$, according to (11), we have

$$\begin{aligned} (l, m\sigma_{\alpha, \beta}, n\theta_{\alpha, \beta, \gamma}) &= (l, 1_\gamma, v)(k, m, n) \\ &= (l, 1_\lambda, v)(i, g, \lambda)(j, h, \mu) \\ &= (l, g\sigma_{\alpha, \gamma}, \lambda\theta_{\alpha, \gamma})(j, h, \mu) \\ &= (l, g\sigma_{\alpha, \gamma}, \lambda\theta_{\alpha, \gamma})(l, 1_\gamma, \lambda\theta_{\alpha, \gamma})(j, h, \mu) \\ &= (l, g\sigma_{\alpha, \gamma}, \lambda\theta_{\alpha, \gamma})(l, h\sigma_{\beta, \gamma}, \mu\theta_{\beta, \gamma}) \\ &= (l, (g\sigma_{\alpha, \gamma})(h\sigma_{\beta, \gamma}), (\lambda\theta_{\alpha, \gamma})(\mu\theta_{\beta, \gamma})) \end{aligned}$$

Therefore,

$$m\sigma_{\alpha, \beta, \gamma} = (g\sigma_{\alpha, \gamma})(h\sigma_{\beta, \gamma}), n\theta_{\alpha, \beta, \gamma} = (\lambda\theta_{\alpha, \gamma})(\mu\theta_{\beta, \gamma}), \quad (13)$$

$$(l, 1_\gamma, v)(i, g, \lambda)(j, h, \mu) = (l, (g\sigma_{\alpha, \gamma})(h\sigma_{\beta, \gamma}), (\lambda\theta_{\alpha, \gamma})(\mu\theta_{\beta, \gamma})). \quad (14)$$

(i) If $\beta = \alpha$, then $m = gh, n = \lambda\mu$. By (13), we have $(gh)\sigma_{\alpha, \gamma} = (g\sigma_{\alpha, \gamma})(h\sigma_{\alpha, \gamma}), (\lambda\mu)\theta_{\alpha, \gamma} = (\lambda\theta_{\alpha, \gamma})(\mu\theta_{\alpha, \gamma})$, where $g, h \in T_\alpha, \lambda, \mu \in \Lambda_\alpha$. So $\sigma_{\alpha, \gamma}$ and $\theta_{\alpha, \gamma}$ are semigroup

homomorphism of from T_α to T_β and from Λ_α to Λ_β , respectively, where $\alpha \geq \gamma$. Similarly, it follows that $\sigma_{\alpha, \beta}$ is also a semigroup homomorphism, by (9), we have

$$1_\alpha\sigma_{\alpha, \beta} = 1_\beta, (\alpha \geq \beta). \quad (15)$$

(ii) If $\beta = \alpha$, let $\gamma = \alpha, h = 1_\alpha, \mu = \lambda$. In view of (14) and (15), it follows that $g = g\sigma_{\alpha, \alpha}, \lambda = \lambda\theta_{\alpha, \alpha}$ for any $g \in T_\alpha, \lambda \in \Lambda_\alpha$. So $\sigma_{\alpha, \alpha}$ and $\theta_{\alpha, \alpha}$ are identical mapping on T_α and T_γ , respectively.

(iii) Let $\gamma = \alpha\beta, l = k$. According to (13), (14) and the results above (ii), we have

$$m = (g\sigma_{\alpha, \alpha\beta})(h\sigma_{\beta, \alpha\beta}), n = (\lambda\theta_{\alpha, \alpha\beta})(\mu\theta_{\beta, \alpha\beta}), \quad (16)$$

$$(i, g, \lambda)(j, h, \mu) = (k, (g\sigma_{\alpha, \alpha\beta})(h\sigma_{\beta, \alpha\beta}), (\lambda\theta_{\alpha, \alpha\beta})(\mu\theta_{\beta, \alpha\beta})). \quad (17)$$

(iv) If $\alpha \geq \beta \geq \gamma$, then $\alpha\beta = \beta$. Referring to (13), (16) and (17), we have $(g\sigma_{\alpha, \beta})\sigma_{\beta, \alpha} = [(g\sigma_{\alpha, \beta})(1_\beta)\sigma_{\beta, \alpha}]\sigma_{\beta, \gamma} = (g\sigma_{\alpha, \gamma})(1_\beta\sigma_{\beta, \gamma}) = (g\sigma_{\alpha, \gamma})1_\gamma = g\sigma_{\alpha, \gamma}, (\lambda\theta_{\alpha, \beta})\theta_{\beta, \gamma} = [(\lambda\theta_{\beta, \beta})(\lambda\theta_{\alpha, \beta})]\theta_{\beta, \gamma} = (\lambda\theta_{\beta, \gamma})(\lambda_{\alpha, \gamma}) = \lambda\theta_{\alpha, \gamma}$. This leads to $\sigma_{\alpha, \beta}\sigma_{\beta, \gamma} = \sigma_{\alpha, \gamma}, \theta_{\alpha, \beta}\theta_{\beta, \gamma} = \theta_{\alpha, \gamma}$.

Define multiplication operations on $T = \cup_{\alpha \in Y} T_\alpha$ and $\Lambda = \cup_{\alpha \in Y} \Lambda_\alpha$, as follows respectively:

$$g \circ h = (g\sigma_{\alpha, \alpha\beta})(h\sigma_{\beta, \alpha\beta}) (g \in T_\alpha, h \in T_\beta), \quad (18)$$

$$\lambda \circ \mu = (\lambda\theta_{\alpha, \alpha\beta})(\mu\theta_{\beta, \alpha\beta}) (\lambda \in \Lambda_\alpha, \mu \in \Lambda_\beta). \quad (19)$$

According to (i), (ii) and (iv), we know that $T = [Y; T_\alpha, \sigma_{\alpha, \beta}]$ is a strong semilattice of left- \mathcal{R} cancellative monoid T_α and $\Lambda = [Y; \Lambda_\alpha, \theta_{\alpha, \beta}]$ is a strong semilattice of right zero band Λ_α , that is, (T, \circ) is a C-wrpp semigroup and (Λ, \circ) is a right normal band. It follows that

$$(i, g, \lambda)(j, h, \mu) = (k, g \circ h, \lambda \circ \mu) \quad (20)$$

by (18)-(20).

Step 2 We shall show that $S_l = \cup_{\alpha \in Y} (I_\alpha \times T_\alpha)$ forms a left C-wrpp semigroup. Let $I = \cup_{\alpha \in Y} I_\alpha$. We wish to define a mapping $\eta : S_l \rightarrow T_l(I)$ so that S_l can be made into a semi-spined product. For all $k' \in I_{\alpha\beta}$, we have

$$\begin{aligned} (k, m, n) &= (k, m, n)(k', 1_{\alpha\beta}, n) = (i, g, \lambda)(j, h, \mu)(k', 1_{\alpha\beta}, n) \\ &= (i, g, \lambda)(\phi_{\alpha\beta}(j, h)k', \dots, \dots) \\ &= (\phi_{\alpha\beta, \alpha}(i, g)\phi_{\alpha\beta, \beta}(j, h)k', \dots, \dots). \end{aligned}$$

So $k = \phi_{\alpha\beta, \alpha}(i, g)\phi_{\alpha\beta, \beta}(j, h)k'$. Therefore, $\phi_{\alpha\beta, \alpha}(i, g)\phi_{\alpha\beta, \beta}(j, h)$ is a constant mapping on $I_{\alpha\beta}$, write as $k = \langle \phi_{\alpha\beta, \alpha}(i, g)\phi_{\alpha\beta, \beta}(j, h) \rangle$, we have

$$\begin{aligned} (k, m, n) &= (k, m, n)(j, 1_\beta, \mu)(j, h, \mu)(k', 1_{\alpha\beta}, n) \\ &= (i, g, \lambda)(j, 1_\beta, \mu)(\phi_{\alpha\beta, \beta}(j, h)k', \dots, \dots) \\ &= (\phi_{\alpha\beta, \alpha}(i, g)\phi_{\alpha\beta, \beta}(j, 1_\beta)[\phi_{\alpha\beta, \beta}(j, h)k'], \dots, \dots) \\ &= (\langle \phi_{\alpha\beta, \alpha}(i, g)\phi_{\alpha\beta, \beta}(j, 1_\beta) \rangle, \dots, \dots). \end{aligned}$$

Thus $k = \langle \phi_{\alpha\beta, \alpha}(i, g)\phi_{\alpha\beta, \beta}(j, 1_\beta) \rangle$ does not depend on the choice of h , let $k = \eta(i, g)j$. We define the mapping η by the following rules:

$$\eta(i, g) : S_l \rightarrow T_l(I), (i, g) \mapsto \eta(j, g);$$

$$\eta(i, g) : I \rightarrow I, j \mapsto \eta(i, g)j,$$

and such that

$$(i, g, \lambda)(j, h, \mu) = (\eta(i, g), g \circ h, \lambda \circ \mu)$$

for $(i, g, \lambda), (j, h, \mu) \in S$.

To see that η is a structure mapping defining a semi-spined product $I \times_{\eta} T$, we need to verify that η satisfies the required conditions (Q1)-(Q3). If $(i, g) \in I_{\alpha} \times T_{\alpha}, j \in I_{\beta}, \alpha \leq \beta$, then $\eta(i, g)j = \langle \phi_{\alpha\beta, \alpha}(i, g)\phi_{\alpha\beta, \beta}(j, 1_{\beta}) \rangle \in I_{\alpha\beta}$, (Q1) holds. To verify that (Q2) holds, we let $(i, g) \in I_{\alpha} \times T_{\alpha}, j \in I_{\beta}, \alpha \leq \beta$, then we obtain

$$\begin{aligned} (\eta(i, g)j, g \circ h, \lambda \circ \mu) &= (i, g, \lambda)[(i, 1_{\alpha}, \lambda)(j, h, \mu)] \\ &= (i, g, \lambda)(i, h\sigma_{\beta, \alpha}, \mu\theta_{\beta, \alpha}) \\ &= (i, \dots, \dots) \end{aligned}$$

by (11) and (20). Consequently, we have $\eta(i, g)j = i$. Thus, (Q2) holds. Finally, we let $(i, g) \in I_{\alpha} \times T_{\alpha}, (j, h) \in I_{\alpha} \times T_{\beta}$. For all $\gamma \in Y, l \in I_{\gamma}, v \in \Lambda_{\alpha}$, according to (20), we have

$$\begin{aligned} (\eta(\eta(i, g)j, g \circ h)l, (g \circ h) \circ 1_{\gamma}, \lambda \circ \mu) &= (i, g, \lambda)(j, h, \mu)(l, 1_{\gamma}, \nu) \\ &= (i, g, \lambda)(\eta(j, h)l, \dots, \dots) \\ &= (\eta(i, g)\eta(j, h)l, \dots, \dots). \end{aligned}$$

This leads to $\eta(\eta(i, g)j, g \circ h)l = \eta(i, g)\eta(j, h)l$, so $\eta(\eta(i, g)j, g \circ h) = \eta(i, g)\eta(j, h)$. In fact, we have shown that (Q3) holds. Thus, η satisfies (Q1)-(Q3) and we do have a semi-spined product $I \times_{\eta} T$.

Next we need to prove that the structure mapping η on this semispined product satisfies the condition (Q) in lemma 1. For this purpose, we let (i, a) and $(j, b) \in I_{\alpha} \times T_{\alpha}$. Take $k \in I_{\tau}$ and $l \in I_{\delta}$ for some τ and δ , and suppose that $\eta(i, a)k = \eta(i, a)l$, that is, $(i, a)^{\#}k = (i, a)^{\#}l$. By condition (Q1), we have $\delta\alpha = \tau\alpha$. Denote the identity elements of the monoids T_{δ} and T_{τ} by 1_{δ} and 1_{τ} , respectively. Since T is a strong semilattice of T_{α} , we have $a1_{\delta} = a1_{\tau}$. By invoking Lemma 5, we have $(i, a)(k, 1_{\tau})\mathcal{R}(i, a)(l, 1_{\delta})$. Since $i\mathcal{L}j$, we have $(i, a)\mathcal{L}^{**}(j, b)$ so that $(j, b)(k, 1_{\tau})\mathcal{R}(j, b)(l, 1_{\delta})$. Hence we have

$$((j, b)^{\#}k, b1_{\tau})\mathcal{R}((j, b)^{\#}l, b1_{\delta}) \Rightarrow (j, b)^{\#}k = (j, b)^{\#}l.$$

This shows that $\ker\eta(i, a) \subseteq \ker\eta(j, b)$. Analogously, we can also prove that $\ker\eta(j, b) \subseteq \ker\eta(i, a)$. Thus $\ker\eta(i, a) = \ker\eta(j, b)$ and so condition (Q) is satisfied. This shows that $S_l = \cup_{\alpha \in Y} (I_{\alpha} \times T_{\alpha})$ is indeed a left C-wrpp semigroup.

Summing up step1 and step2, we conclude that S is the spined product of a left C-wrpp semigroup S_l and a right normal band Λ .

(4) \Rightarrow (1). Let S be the spined product of a left C-wrpp semigroup $S_l = I \times_{Y, \eta} T$ and a right normal band $\Lambda = [Y; \Lambda_{\alpha}, \theta_{\alpha, \beta}]$. Clearly, S is a semilattice of left- \mathcal{R} cancellative planks, and for all $e = (i, 1_{\alpha}, \lambda) \in E(S) \cap (I_{\alpha} \times T_{\alpha} \times \Lambda_{\alpha}), x = (j, h, \mu) \in I_{\beta} \times T_{\beta} \times \Lambda_{\beta}, y = (k, m, n) \in I_{\gamma} \times T_{\gamma} \times \Lambda_{\gamma}$, let $(l, q) = (i, 1_{\alpha})(j, h) \in I_{\alpha\beta} \times T_{\alpha\beta}$. According to S_l is a left C-wrpp semigroup and Lemma 1, we have $(i, q)(i, 1_{\alpha}) = (\eta(l, g)i, (q\sigma_{\alpha\beta, \alpha\beta})(1_{\alpha}\sigma_{\alpha, \alpha\beta})) = (l, q) = (i, 1_{\alpha})(j, h) \in$

$I_{\alpha\beta} \times T_{\alpha\beta}$, so

$$\begin{aligned} \eta'_e(xy) &= exy = ((i, 1_{\alpha})(j, h)(k, m), \lambda\mu\nu) \\ &= ((l, q)(i, 1_{\alpha})(i, 1_{\alpha})(k, m), \lambda\mu\nu) \\ &= ((i, 1_{\alpha})(j, h)(i, 1_{\alpha})(k, m), \lambda\mu\nu) \\ &= exey = \eta'_e(x)\eta'_e(y). \end{aligned}$$

Consequently, η'_e is a semigroup homomorphism from S to eS , thus S is a weakly left C-wrpp semigroup.

Corollary 1 Let S be a semigroup. Then the following conditions are equivalent:

- (1) S is a weakly left C-rpp semigroup;
- (2) S is a semilattice of left cancellative monoids, and $\text{Reg}S$ is a weakly left C-semigroup;
- (3) S is a semilattice of left cancellative monoids, and S is a left quasi-normal band;
- (4) S is a spined product of left C-rpp semigroup and a right normal band.

Corollary 2 A weakly left C-wrpp semigroup is a wrpp semigroup.

Proof. According to theorem 1, a weakly left C-wrpp semigroup is a spined product of a left C-wrpp semigroup and right normal band, but a left C-wrpp semigroup and a right normal band are wrpp semigroups, it follows that a weakly left C-wrpp emigroup is a wrpp semigroup.

By above corollary, we have the following results:

Corollary 3 A weakly left C-rpp semigroup is a rpp semigroup.

Corollary 4 A semigroup S is a weakly left C-semigroup if and only if S is a spined product of left C-semigroup and a right normal band.

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REFERENCES

- [1] J. M. Howie, *An introduction to semigroup theory*, London: London Acaemic Press, 1976.
- [2] L. Du and K. P. Shum, *On left C-wrpp semigroups*, Semigroup Forum, Vol. 67, pp. 373-387, 2003.
- [3] X. D. Tang, *On a theorem of C-wrpp semigroups*, Comm. Algebra, Vol. 25, pp. 1499-1504, 1997.
- [4] J. B. Fountain, *Right pp monoids with central idempotents*, Semigroup Forum, Vol. 13: 229-237, 1977.
- [5] Y. L. Cao, *The structure of weakly left C-rpp semigroups*, J..Zibo University, Vol. 2, pp. 3-8, 2000.
- [6] P. Y. Zhu, Y. Q. Guo and K. P. Shum, *Structure and characterization of left Clifford semigroups*, Sci.China, Ser., Vol.(A)35, pp. 791-805, 1991.
- [7] Y. Q. Guo, *Structure of weakly left C-semigroups*, Chinese Sci.Bull., Vol. 41, pp. 462-467, 1996.
- [8] Y. Q. Guo, K. P. Shum and P. Y. Zhu, *The structure of left C-rpp semigroups*, Semigroup Forum, Vol.50, pp. 9-23, 1995.
- [9] X. M. Ren and K. P. Shum, *Structure theorem for right pp semigroups with left central idempotents*, Discuss.Math.Gen.Algebra Appl., Vol., 20, pp. 63-75, 2000.
- [10] X. M. Zhang, *The Structure of Weakly Left C-wrpp semigroups*, International Journal of Computational and Mathematical Sciences, Vol. 2, pp. 170-172, 2008.