

# Analytical Solutions of Three Dimensional Steady-State Heat Transfer in Rectangular Ribs

Tao Nie, Weiqiang Liu

**Abstract**—In order to obtain an accurate result of the heat transfer of the rib in the internal cooling Rectangular channel, using separation of variables, analytical solutions of three dimensional steady-state heat conduction in rectangular ribs are given by solving three dimensional steady-state function of the rectangular ribs. Therefore, we can get solution of three dimensional temperature field in the rib. Based on the solution, we can get how the Bi number affected on heat transfer. Furthermore, comparisons of the analytical and numerical results indicate agreement on temperature field in the rib.

**Keywords**—variable separation method, analytical solution, rib, heat transfer

## I. INTRODUCTION

INTERNAL convection cooling is a common technique used in the blade and thrust chamber. Ribbed channel is usual used for this technique which can heat exchange because it can intensify turbulence of the coolant and enlarge area of heat transfer; accordingly, it can take away the heat quantity which hot gas transfers to the wall, which can effectively reduce the temperature of the wall. For the moment it mostly has two means to investigate internal flow and heat transfer in internal cooling duct. One is experimental investigation, the other one is numerical simulation. But if no experimental data, it can not ensure the veracity of numerical simulation [1]. In addition, the cost of experimental investigation is too high, which sometimes go with danger. Furthermore, sometimes the experimental investigation conditions can not reach the real operational environment, for example, physical dimension. Therefore the analytical Solutions of some subject fundamental equations have importance theory significance, which can comprehensive and downright illuminate physical phenomena. Furthermore, it can be criterion which can accelerate the development of numerical solution [2]. Many scholars have studied analytical Solutions of two dimension steady-state heat conduction [3-5]. Xiao qibo, et al analyzed pool boiling, finding that the heat flux of pool boiling is the function of degree of wall superheat, activate cave, maximum dimension and fluid physical characteristics.

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In this paper considering the real operation environment and establishing mathematic model, we can find that the analytical solution of three dimension steady-state heat conduction in rectangular rib using variable separation method.

## II. PHYSICAL MODEL

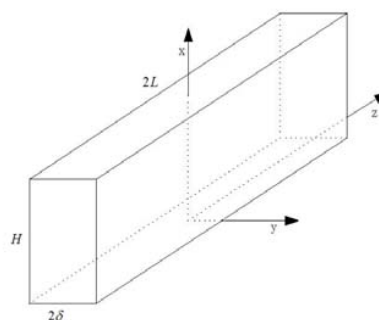


Fig. 1 rectangular rib at Cartesian coordinate system

Rectangular rib in internal cooling duct has been shown in Fig.1. Supposing thermal conductivity is constant, no internal heat source and steady state. The bottom of the rib is constant temperature. The length of the rib is  $2L$ , whose width is  $2\delta$  and height is  $H$ . The boundary condition of Both ends and the top of the rib are adiabatic, in addition both side are convection. Considering symmetry of the model, the differential equation of the rib heat conduction can be written as follows in cartesian coordinate system:

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} = 0 \quad (1)$$

Boundary conditions:

$$\left. \begin{aligned} x=0, t=t_0 \\ x=H, \frac{\partial t}{\partial x} = 0 \\ y=0, \frac{\partial t}{\partial y} = 0 \\ y=\delta, \lambda \frac{\partial t}{\partial y} + h(t-t_f) = 0 \\ z=0, \frac{\partial t}{\partial z} = 0 \\ z=L, \frac{\partial t}{\partial z} = 0 \end{aligned} \right\} \quad (2)$$

Where,  $t$  is temperature,  $t_0$  is temperature at the bottom of the rib,  $h$  is coefficient of heat transfer on the surface,  $\lambda$  is thermal conductivity,  $t_f$  is ambient temperature.

### III. ANALYTICAL SOLUTIONS

Over temperature  $\theta = t - t_f$ , for the sake of more easily analyzing, we take the differential equation and boundary conditions to be dimensionless.

$$\frac{\partial^2 \Theta}{\partial X^2} + \frac{\partial^2 \Theta}{\partial Y^2} + \frac{\partial^2 \Theta}{\partial Z^2} = 0 \quad (3)$$

$$\left. \begin{aligned} X=0, \Theta &= 1 \\ X=K_H, \frac{\partial \Theta}{\partial X} &= 0 \\ Y=0, \frac{\partial \Theta}{\partial Y} &= 0 \\ Y=1, \frac{\partial \Theta}{\partial Y} + Bi\Theta &= 0 \\ Z=0, \frac{\partial \Theta}{\partial Z} &= 0 \\ Z=K_L, \frac{\partial \Theta}{\partial Z} &= 0 \end{aligned} \right\} \quad (4)$$

Where,  $\Theta = (t - t_f) / (t_0 - t_f)$ ,  $Bi = h\delta / \lambda$ , and  $K_L = L / \delta$ ,  $K_H = H / \delta$ .

Supposing  $\Theta(X, Y, Z) = f(X)g(Y)j(Z)$ , which can be substituted into Eq.(3). three ordinary differential equations can be obtained as follows:

$$f'' - (\beta^2 + \chi^2)f = 0 \quad (5)$$

$$g'' + \beta^2 g = 0 \quad (6)$$

$$j'' + \chi^2 j = 0 \quad (7)$$

Where,  $\beta$ ,  $\chi$  are independent of  $X$ ,  $Y$  and  $Z$ .

#### A. Analytical solution of $g(Y)$

The general solution of the Eq.(6):

$$g(Y) = C_1 \cos(\beta Y) + C_2 \sin(\beta Y) \quad (8)$$

Boundary conditions:

$$Y=0, \frac{\partial g(Y)}{\partial Y} = 0 \quad (9)$$

$$Y=1, \frac{\partial g(Y)}{\partial Y} + Bi g(Y) = 0 \quad (10)$$

This gives the final form of the solution

$$g(Y) = C_1 \cos(\beta Y) \quad (11)$$

With

$$\beta \tan(\beta) = Bi \quad (12)$$

$\beta$  satisfying transcendental equation (12) is characteristic value. From Eq.(12) we can find the number of  $\beta$  is infinite. Omitting constant  $C$ , we can get corresponding eigenfunction.

$$g(Y) = \cos(\beta_m Y) \quad (m=1, 2, 3, \dots) \quad (13)$$

#### B. Analytical solution of $j(Z)$

The general solution of the Eq.(7):

$$j(Z) = C_3 \cos(\chi Z) + C_4 \sin(\chi Z) \quad (14)$$

Boundary conditions:

$$Z=0, j' = 0 \quad (15)$$

$$Z=K_L, j' = 0 \quad (16)$$

The final form of the solution

$$j(Z) = C_3 \cos(\chi Z) \quad (17)$$

Where,

$$\sin(\chi K_L) = 0 \quad (18)$$

The characteristic value of Eq.(18)

$$\chi_n = \frac{n\pi}{K_L} \quad (n=0, 1, 2, \dots) \quad (20)$$

Omitting constant  $C$ , we can get corresponding eigenfunction.

$$j(Z) = \cos(\chi_n Z) \quad (21)$$

#### C. Analytical solution of $f(X)$

For different  $\chi_n$  and  $\beta_m$ , The general solution of the Eq.(5)

$$f(X) = C_5 \cosh(\eta X) + C_6 \sinh(\eta X) \quad (22)$$

Where,  $\eta^2 = \beta^2 + \chi^2$ .

Boundary conditions:

$$X=K_H, f' = 0 \quad (23)$$

The final form of the solution

$$f(X) = C_5 \left[ \cosh(\eta X) - \frac{\sinh(\eta K_H)}{\cosh(\eta K_H)} \sinh(\eta X) \right] \quad (24)$$

Omitting constant  $C_5$ , we can gain corresponding eigenfunction.

$$f(X) = \cosh(\eta X) - \frac{\sinh(\eta K_H)}{\cosh(\eta K_H)} \sinh(\eta X) \quad (25)$$

#### D. Analytical solution of $\Theta(X, Y, Z)$

The linear sum of the basic solution for different characteristic value is the analytical solution of the temperature field.

$$\theta(X, Y, Z) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} C_{mn} f(X) g(Y) j(Z) \quad (26)$$

Using inhomogeneous boundary conditions at  $X=0$ , we can find:

$$\sum_{m=1}^{\infty} \sum_{n=0}^{\infty} C_{mn} f(0) g(Y) j(Z) = 1 \quad (27)$$

Using the orthogonality of the characteristic function, termwise integration using arithmetic operators at both side of Eq.(9):  $\int_0^1 g_m(Y) dY$  and  $\int_0^{K_L} j_n(Z) dZ$ , we can get:

$$C_{mn} = \frac{\int_0^1 g(\beta_m, Y) dY \cdot \int_0^{K_L} j(\chi_n, Z) dZ}{N^2(\beta_m) \cdot N^2(\chi_n) \cdot f(0)} \quad (28)$$

Where:

$$\int_0^1 g(\beta_m, Y) dY = \frac{1}{\beta_m} \sin(\beta_m) \quad (29)$$

$$N^2(\beta_m) = \frac{(\beta_m^2 + Bi^2 + Bi)}{2(\beta_m^2 + Bi^2)} \quad (30)$$

when  $\chi_n \neq 0$  ( $n=1, 2, 3, \dots$ ):

$$N^2(\chi_n) = \frac{K_L}{2} \quad (31)$$

$$\int_0^{K_L} j(\chi_n, Z) dz = \frac{1}{\chi_n} \sin(\chi_n K_L) = 0 \quad (32)$$

while  $\chi_n = 0$  ( $n=0$ ):

$$N^2(\chi_n) = K_L \quad (33)$$

$$\int_0^{K_L} j(\chi_n, Z) dz = K_L \quad (34)$$

#### IV. RESULTS AND DISCUSSION

The computational conditions are  $Bi=1$ ,  $K_H=3$ ,  $K_L=10$ ,  $t_0=300K$  and  $t_f=200K$ . The 3d temperature field of rib has been shown in Fig.2. From Fig.2, we can find the temperature reduces along the y direction, and the same rule can be found at x direction, which is coincident with the data measured in tests. Fig.3 shows the dimensionless parameter  $\Theta$  distribution of the rib. From Fig.3 we know that the dimensionless parameter  $\Theta$  distribution of the rib is coincident with the temperature distribution, so the  $\Theta$  is corking parameter that can represent temperature distribution in the rib.

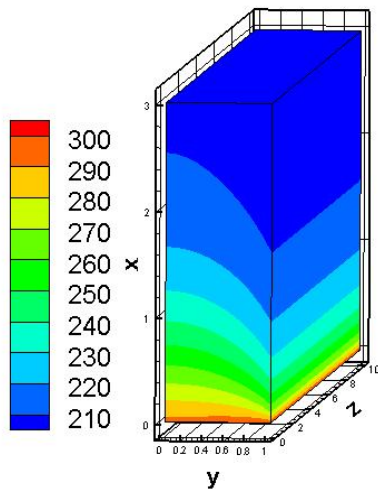


Fig. 2 3d temperature field of the rib

The 2d temperature field along different direction has been shown in Fig.4. From Fig.4, we can obtain temperature is constant along z direction. The reason is the adiabatic at  $Z=0$ ,  $Z=K_L$  and  $Y=1$ . In addition, the boundary condition is isothermal wall at  $Y=0$ .

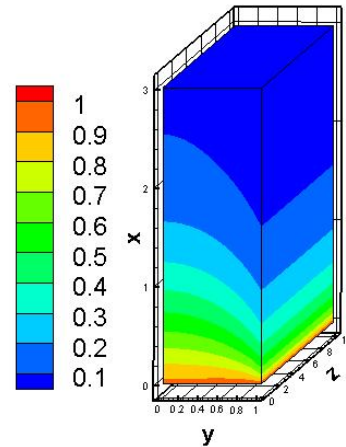


Fig. 3 3d temperature field of the rib

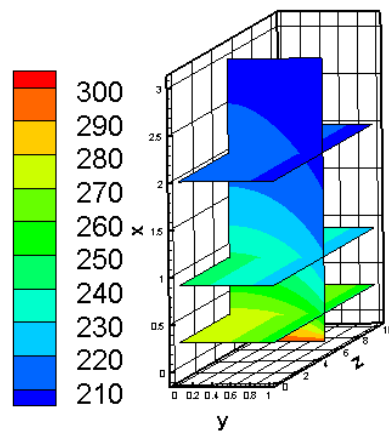


Fig. 4 temperature field along different direction

The comparison between analytical solution and numerical solution are shown in Fig.5. From Fig.5, we can find analytical solution is consistent with numerical solution.

The  $\Theta$  distributions with different  $Bi$  are shown in Fig.6. From Fig.6, we can find the high  $\Theta$  field distribution with different  $Bi$  is different. And the high  $\Theta$  field extends when  $Bi$  increases.

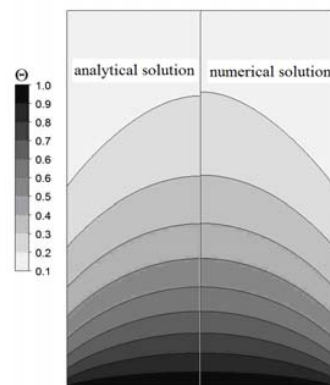
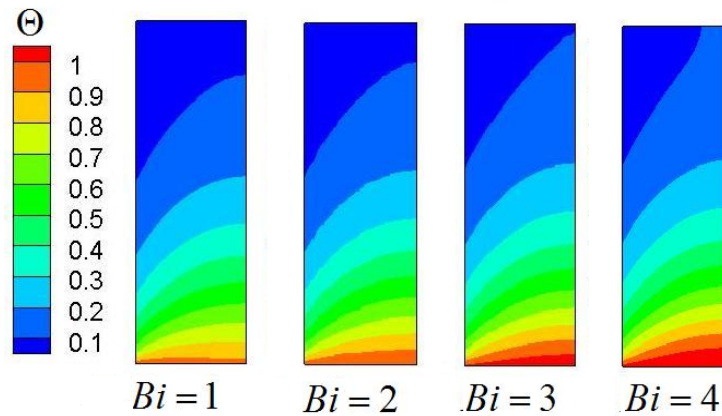


Fig. 5 comparison between result of analysis and simulation

Fig. 5  $\Theta$  distribution with different  $Bi$ 

#### V. CONCLUSIONS

Internal cooling is a common technique used for enhanced heat transfer by using ribs, which can be usually used in heat exchange equipments. It can evidently enlarge heat exchange area and turbulence. In this paper, we can find analytical solution of rectangular rib Laplace equation in internal cooling channel. Then according to the analytical solution, we get three dimensional temperature field of Rectangular Rib.

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