Analytical Based Truncation Principle of Higher-Order Solution for a $x^{1/3}$ Force Nonlinear Oscillator

Md. Alal Hosen

Abstract—In this paper, a modified harmonic balance method based an analytical technique has been developed to determine higher-order approximate periodic solutions of a conservative nonlinear oscillator for which the elastic force term is proportional to $x^{1/3}$. Usually, a set of nonlinear algebraic equations is solved in this method. However, analytical solutions of these algebraic equations are not always possible, especially in the case of a large oscillation. In this article, different parameters of the same nonlinear problems are found, for which the power series produces desired results even for the large oscillation. We find a modified harmonic balance method works very well for the whole range of initial amplitudes, and the excellent agreement of the approximate frequencies and periodic solutions with the exact ones has been demonstrated and discussed. Besides these, a suitable truncation formula is found in which the solution measures better results than existing solutions. The method is mainly illustrated by the $x^{1/3}$ force nonlinear oscillator but it is also useful for many other nonlinear problems.

Keywords—Approximate solutions, Harmonic balance method, Nonlinear oscillator, Perturbation.

I. INTRODUCTION

MANY complex real world problems in nature are due to nonlinear phenomena. Nonlinear processes are one of the biggest challenges and not easy to control because the nonlinear characteristic of the system abruptly changes due to some small changes of valid parameters including time. Thus the issue becomes more complicated and hence needs ultimate solution. Therefore, the studies of approximate solutions of nonlinear differential equations (N.D.Es.) play a crucial role to understand the internal mechanism of nonlinear phenomena. Advance nonlinear techniques are significant to solve inherent nonlinear problems, particularly those involving differential equations, dynamical systems and related areas. In recent years, both the mathematicians and physicists have made significant improvement in finding a new mathematical tool would be related to nonlinear differential equations and dynamical systems whose understanding will rely not only on analytic techniques but also on numerical and asymptotic methods. They establish many effective and powerful methods to handle the N.D.Es.

The study of given nonlinear problems is of crucial importance not only in all areas of physics but also in engineering and other disciplines, since most phenomena in our world are essential nonlinear and are described by nonlinear equations.

It is very difficult to solve nonlinear problems and in general it is often more difficult to get an analytic approximation than a numerical one for a given nonlinear problem. There are many analytical approaches to solve nonlinear differential equations. One of the widely used techniques is perturbation [1]-[4], whereby the solution is expanded in powers of a small parameter. However, for the nonlinear conservative systems, generalizations of some of the standard perturbation techniques overcome this limitation. In particular, generalization of Lindsted-Poincare method and He's homotopy perturbation method yield desired results for strongly nonlinear oscillators [5]-[12].

The harmonic balance method (HBM) [13]-[23] is another technique for solving strongly nonlinear systems. Usually, a set of difficult nonlinear algebraic equations appears when HBM is formulated. In article [23], such nonlinear algebraic equations are solved in powers of a small parameter. Sometimes, higher approximations also fail to measure the desired results when $a_0 >> 1$. In this article this limitation is removed. Approximate solutions of the same equations are found in which the nonlinear algebraic equations are solved by a new parameter. The higher order approximations (mainly third approximation) have been obtained for mentioned nonlinear oscillator. However, a suitable truncation of these algebraic equations takes the solution very close to the previous one but it saves a lot of calculation. This is the main advantage of the method presented in this article.

II. THE METHOD

Let us consider a nonlinear differential equation

$$\ddot{x} + \omega_0^2 x = -\varepsilon f(x, \dot{x}), \quad [x(0) = a_0, \ \dot{x}(0) = 0] \tag{1}$$

where $f(x, \dot{x})$ is a nonlinear function such that $f(-x, -\dot{x}) = -f(x, \dot{x})$, $\omega_0 \geq 0$ and ε is a constant.

A periodic solution of (1) is taken in the form as

$$x = a_0(\rho \cos(\omega t) + u \cos(3\omega t) + v \cos(5 \ t) \ w \cos(7 \ t) \ \cdots)\omega(2)+$$

where a_0, ρ and $\dot{\varphi}$ are constants. If $\rho = 1 - u - v - \cdots$ and the initial phase $\varphi_0 = 0$, solution (2) readily satisfies the initial conditions $[x(0) = a_0, \dot{x}(0) = 0]$.

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Substituting (2) into (1) and expanding $f(x,\dot{x})$ in a Fourier series, it is easily takes to an algebraic identity

$$a_0[\rho(\omega_0^2-\omega^2)\cos(\omega t)+u(\omega_0^2-9\omega^2)\cos(3\omega t)+\cdots]$$
$$=-\varepsilon[F_1(a_0,u,\cdots)\cos(\omega t)+F_3(a_0,u,\cdots)\cos(3\omega t)+\cdots] \qquad (3)$$

By comparing the coefficients of equal harmonics of (3), the following nonlinear algebraic equations are found

$$\rho(\omega_0^2-\omega^2)=-\varepsilon F_1, \quad u(\omega_0^2-9\omega^2)=-\varepsilon F_3,$$
$$v(\omega_0^2-25\omega^2)=-\varepsilon F_5,\cdots \qquad (4)$$

With help of the first equation, $\dot{\varphi}$ is eliminated from all the rest of (4). Thus (4) takes the following form

$$\rho\omega^2=\rho\omega_0^2+\varepsilon F_1, \quad 8\omega_0^2 u\rho=\varepsilon(\rho F_3-9uF_1),$$
$$24\omega_0^2 v\rho=\varepsilon(\rho F_5-25vF_1), \cdots \qquad (5)$$

Substitution $\rho=1-u-v-\cdots$, and simplification, second-, third- equations of (5) take the following form

$$u=G_1(\omega_0,\varepsilon,a_0,u,v,\cdots,\lambda_0),$$
$$v=G_2(\omega_0,\varepsilon,a_0,u,v,\cdots,\lambda_0),\cdots, \qquad (6)$$

where $G_1, G_2, \cdots$ exclude respectively the linear terms of $u, v, \cdots$.

Whatever the values of ω_0 and a_0, there exists a parameter $\lambda_0(\omega_0,\varepsilon,a_0)\ll 1$, such that $u, v, \cdots$ are expandable in following series

$$u=U_1\lambda_0+U_2\lambda_0^2+\cdots,$$
$$v=V_1\lambda_0+V_2\lambda_0^2+\cdots, \quad \cdots \qquad (7)$$

where $U_1, U_2, \cdots, V_1, V_2, \cdots$ are constants.

Finally, substituting the values of $u, v, \cdots$ from (7) into the first equation of (5), ω is determined. This completes the determination of all related functions for the proposed periodic solution as given in (2).

III. EXAMPLE

Let us consider the following nonlinear oscillator which was first studied in detail by Mickens [18]

$$\ddot{x}+x^{1/3}=0 \qquad (8)$$

This is a conservative system and the solution of (8) is periodic. The classical harmonic balance method [23] cannot be applied directly. From (2) the first-order approximation solution of (8) is

$$x=a_0\cos(\omega t) \qquad (9)$$

where the second term of (8) *i.e.* $x^{1/3}$ can be expanded in a Fourier series as

$$x^{1/3}=\sum_{n=0}^{\infty}b_{2n+1}x^{1/3}=b_1\cos(\omega t)+b_3\cos(3\omega t)+\cdots \qquad (10)$$

Herein b_1, b_3, $\cdots$ are evaluated by the relation

$$b_{2n+1}=\frac{4}{\pi}\int_0^{\pi/2}x^{1/3}\cos[(2n+1)\varphi]d\varphi \qquad (11)$$

where $\varphi=\omega t$. From (11) and using (9) we obtained b_1, b_3,$\cdots$,

$$b_1=\frac{3a_0^{1/3}I_0}{\sqrt{\pi}}, \qquad (12)$$

$$b_3=-\frac{3a_0^{1/3}I_0}{5\sqrt{\pi}}, \text{ where } \quad I_0=\frac{Gamma\left[\dfrac{7}{6}\right]}{Gamma\left[\dfrac{2}{3}\right]} \qquad (13)$$

and so on.

Now substituting (9) and (10) along with (12)-(13) into (8) and setting the coefficient of $\cos\varphi$ the following algebraic equation is obtained

$$-a_0\omega^2+\frac{3a_0^{1/3}I_0}{\sqrt{\pi}}=0 \qquad (14)$$

Solving (14) the first-order approximate frequency is

$$\omega=\sqrt{\frac{3a_0^{1/3}I_0}{a_0\sqrt{\pi}}}=\frac{1.07685}{a_0^{1/3}} \qquad (15)$$

Therefore the first-order approximation solution of (8) is (9) *i.e.* $x=a_0\cos(\omega t)$ where ω is given by (15).

Let us consider a second-order approximation solution, *i.e.*

$$x=a_0\cos(\omega t)+a_0u(\cos(3\omega t)-\cos(\omega t)) \qquad (16)$$

Substituting (16) along with using (10)-(11) into (8) and then equating the coefficients of $\cos(\omega t)$ and $\cos(3\omega t)$, the following equations are

$$-(1-u)a_0\omega^2-\frac{3a_0^{1/3}(-110+44u+33u^2+50u^3+100u^4)I_0}{110\sqrt{\pi}}=0 \qquad (17)$$

$$-9ua_0\omega^2+\frac{3a_0^{1/3}(-374+935u+969u^2+1700u^3+3700u^4)I_0}{1870\sqrt{\pi}}=0 \qquad (18)$$

where I_0 is defined as (13) and b_1, b_3, $\cdots$ are evaluated as

$$b_1 = -\frac{3a_0^{1/3}(-110+44u+33u^2+50u^3+100u^4)I_0}{110\sqrt{\pi}} \qquad (19)$$

$$b_3 = \frac{3a_0^{1/3}(-374+935u+969u^2+1700u^3+3700u^4)I_0}{1870\sqrt{\pi}} \qquad (20)$$

and so on.

After simplification, (17) takes the form

$$\omega^2 = \left(-\frac{3a_0^{1/3}(-110+44u+33u^2+50u^3+100u^4)I_0}{110a_0(1-u)\sqrt{\pi}}\right) \qquad (21)$$

By elimination of ω^2 from (18) with the help of (21) and simplification, the following nonlinear algebraic equation of u is obtained as

$$u = \lambda_0\left(-1+\frac{199u^2}{11}+\frac{170u^3}{11}+\frac{4825u^4}{187}+\frac{5800u^5}{187}\right)$$

where $\lambda_0 = \frac{2}{83}$ $\qquad (22)$

The power series solution of (22) in terms of λ_0 is

$$u = -\lambda_0+\frac{199\lambda_0^3}{11}-\frac{170\lambda_0^4}{11}-\frac{1293359\lambda_0^5}{2057}+\frac{2811750\lambda_0^6}{2057}+\cdots \quad (23)$$

Now substituting the value of u from (23) into (21), the second-order approximate frequency is

$$\omega = \sqrt{\left(-\frac{3a_0^{1/3}(-110+44u+33u^2+50u^3+100u^4)I_0}{110a_0(1-u)\sqrt{\pi}}\right)} = \frac{1.069204}{a_0^{1/3}} \qquad (24)$$

Thus the second-order approximation solution of (8) is $x = a_0\cos(\omega t)+a_0u(\cos(3\omega t)-\cos(\omega t))$ where u and ω are respectively given by (23) and (24).

It is observed that solution (16) measures better result when (17)-(18) is truncated as

$$-(1-u)a_0\omega^2-\frac{3a_0^{1/3}(-110+44u+33u^2/2)I_0}{110\sqrt{\pi}} = 0 \qquad (25)$$

And

$$-9\,ua_0\,\omega^2+\frac{3a_0^{1/3}(-374+935u+969u^2/2)I_0}{1870\sqrt{\pi}} = 0 \qquad (26)$$

Seeing that (25)-(26), it is clear that the higher order terms of u (more than second) are ignored; but half of the second order terms are considered. Now from (25) we can easily obtain

$$\omega^2 = \left(-\frac{3a_0^{1/3}(-110+44u+33u^2/2)I_0}{110a_0(1-u)\sqrt{\pi}}\right) \qquad (27)$$

Substituting (22) into (26) and then simplifying we get

$$u = \lambda_0\left(-1+\frac{739u^2}{44}+\frac{60u^3}{11}\right) \qquad (28)$$

where λ_0 is defined as (22). The power series solution of (28) is

$$u = -\lambda_0+\frac{739}{44}\lambda_0^3-\frac{60}{11}\lambda_0^4-\frac{546121}{968}\lambda_0^5+\frac{55425}{121}\lambda_0^5+\cdots \qquad (29)$$

Substituting the value of u from (29) into (27) the second-order frequency in truncation form is

$$\omega_{trun2} = \sqrt{\left(-\frac{3a_0^{1/3}(-110+44u+33u^2/2)I_0}{110a_0(1-u)\sqrt{\pi}}\right)} = \frac{1.069242}{a_0^{1/3}} \qquad (30)$$

In a similar way, the method can be used to determine higher order approximations. In this article, a third-order approximate solution of (8) is

$$x = a_0\cos(\omega t)+a_0u(\cos(3\omega t)-\cos(\omega t))+a_0v(\cos(5\omega t)-\cos(\omega t)) \, (31)$$

Substituting (31) together with (10) and (11) into (8) and equating the coefficients of $\cos\varphi$, $\cos3\varphi$ and $\cos5\varphi$ the related functions are obtained from the following equations

$$-(1-u-v)a_0\omega^2-\frac{3a_0^{1/3}(-110+44u+33u^2+50u^3+33v+18uv+27v^2)I_0}{110\sqrt{\pi}} = 0 \qquad (32)$$

$$-9\,ua_0\omega^2-\frac{3a_0^{1/3}(374-935u-969u^2-1700u^3+102v+306uv+\cdots)I_0}{1870\sqrt{\pi}} = 0 \qquad (33)$$

$$-25va_0\omega^2+\frac{3a_0^{1/3}(4301-6347u-12903u^2-29900u^3+18768v+30222uv+\cdots)I_0}{43010\sqrt{\pi}} = 0 \qquad (34)$$

where I_0 is defined as (13).

Using (32) we can easily written as

$$\omega^2 = \left(-\frac{3a_0^{1/3}(-110+44u+33u^2+50u^3+33v+18uv+27v^2)I_0}{110a_0(1-u-v)\sqrt{\pi}}\right) (35)$$

By eliminating ω^2 from (33) and (34) with the help of (35) and simplification, the following nonlinear algebraic equations of u and v are obtained

$$u = \lambda_0 \left(-1 + \frac{199u^2}{11} + \frac{170\,u^3}{11} + \frac{175u^4}{11} + \frac{8v}{11} + \frac{115uv}{11} + \frac{741\,u^2v}{374} + \cdots \right) \tag{36}$$

$$v = \mu_0 \left(1 - \frac{28u}{11} - \frac{16u^2}{11} - \frac{739u^3}{187} + \frac{1300u^4}{187} + \frac{19487\,uv}{187} + \frac{16512\,u^2v}{187} + \cdots \right) \tag{37}$$

where λ_0 is defined as (22) and $\mu_0 = \dfrac{11}{2713}$.

The algebraic relation between λ_0 and μ_0 is

$$\mu_0 = \frac{913\lambda_0}{5426} \tag{38}$$

Therefore, (37) takes the form

$$v = \frac{913\lambda_0}{5426} \left(1 - \frac{28u}{11} - \frac{16u^2}{11} - \frac{739u^3}{187} + \frac{1300u^4}{187} + \frac{19487\,uv}{187} + \frac{16512\,u^2v}{187} + \cdots \right) \tag{39}$$

The power series solutions of (36) and (39) are obtained in terms of λ_0

$$u = -\lambda_0 + 0.122373\lambda_0^2 + 16.677576\lambda_0^3 \\ - 26.067224\lambda_0^4 - 527.273375\lambda_0^5 + \cdots , \tag{40}$$

$$v = 0.168263\lambda_0 + 0.428308\lambda_0^2 - 2.910391\lambda_0^3 \\ - 9.462411\lambda_0^4 + 112.788176\lambda_0^5 + \cdots . \tag{41}$$

Now substituting the values of u and v from (40)-(41) into (35), the third-order approximate frequency is

$$\omega = \sqrt{ \left(-\frac{3a_0^{1/3}(-110 + 44u + 33u^2 + 50u^3 + 33v + 18uv + 27v^2)\,I_0}{110a_0(1-u-v)\sqrt{\pi}} \right) } = \frac{1.070779}{a_0^{1/3}} \tag{42}$$

Therefore a third-order approximation periodic solution of (8) is defined as (31) where u, v and ω are respectively given by (40)-(42).

The third approximate solution (30) measures almost similar result when (32)-(34) are truncated as

$$-(1-u-v)a_0\omega^2 - \frac{3a_0^{1/3}(-110 + 44u + 33u^2 + 50u^3/2 + 33v + 18uv/2)\,I_0}{110\sqrt{\pi}} = 0 \tag{43}$$

$$-9\,u\,a_0\omega^2 - \frac{3a_0^{1/3}(374 - 935u - 969u^2 - 1700u^3/2 + 102v + 306uv/2 + \cdots)\,I_0}{1870\,\sqrt{\pi}} = 0 \tag{44}$$

$$-25va_0\omega^2 + \frac{3a_0^{1/3}(4301 - 6347u - 12903u^2 - 29900u^3/2 + 18768v + 30222uv/2 + \cdots)\,I_0}{43010\,\sqrt{\pi}} = 0 \tag{45}$$

Here, using (43) it can be written as

$$\omega^2 = \left(-\frac{3a_0^{1/3}(-110 + 44u + 33u^2 + 50u^3/2 + 33v + 18uv/2)\,I_0}{110a_0(1-u)\sqrt{\pi}} \right) \tag{46}$$

By eliminating ω^2 from (44)-(45) with the help of (46) and simplification, the following nonlinear algebraic equations of u and v are obtained

$$u = \lambda_0 \left(-1 + \frac{199u^2}{11} + \frac{145u^3}{11} + \frac{175u^4}{22} + \frac{8v}{11} + \frac{239uv}{22} + \frac{3u^2v}{2} + \cdots \right) \tag{47}$$

$$v = \mu_0 \left(1 - \frac{28u}{11} - \frac{16u^2}{11} - \frac{89u^3}{187} + \frac{650u^4}{187} + \frac{18830\,uv}{187} + \frac{13929\,u^2v}{187} + \cdots \right) \tag{48}$$

where λ_0 and μ_0 are defined as (22) and (37).

In similar way, the power series solutions of (47)-(48) in terms of λ_0 are

$$u = -\lambda_0 + 0.122373\lambda_0^2 + 16.582169\lambda_0^3 \\ - 23.803493\lambda_0^4 - 532.152105\lambda_0^5 + \cdots \tag{49}$$

$$v = 0.168263\lambda_0 + 0.428308\lambda_0^2 - 2.811605\lambda_0^3 \\ - 10.129025\lambda_0^4 + 108.343497\lambda_0^5 + \cdots \tag{50}$$

Substituting the values of u and v from (49)-(50) into (46), and simplifying we get the third-order approximate frequency in truncated form is

$$\omega_{trun3} = \sqrt{ \left(-\frac{3a_0^{1/3}(-110 + 44u + 33u^2 + 50u^3/2 + 33v + 18uv/2)\,I_0}{110a_0(1-u)\sqrt{\pi}} \right) } = \frac{1.070776}{a_0^{1/3}} \tag{51}$$

Therefore a third-order approximation periodic solution of (8) is defined as (31) where u, v and ω are respectively given by (49)-(51).

IV. RESULTS AND DISCUSSIONS

We illustrate the accuracy of a modified harmonic balance method by comparing the approximate solutions previously obtained with the exact frequency ω_{ex}. In particular, we will consider the solution of (8) using the classical harmonic balance method [25]. This method is a procedure for the determining analytical approximations to the periodic solutions of the differential equations using a truncated Fourier series representation [22]. Like, the homotopy purterbation method, the harmonic balance method can be applied to nonlinear oscillatory problems where a linear terms does not exist, the nonlinear terms are not small, and there is no perturbation parameter. For this nonlinear problem, the exact value [24] of the frequency is

$$\omega_{ex}(a_0) = \frac{1.070451}{a_0^{1/3}} \tag{52}$$

The first, second and third-order approximate frequencies and their Relatives Errors $(R.E.)$ or percentage errors obtained in this article by applying a modified harmonic balance technique to mentioned conservative nonlinear oscillator are the following

$$\omega_1(a_0) = \frac{1.07685}{a_0^{1/3}}, \qquad R.E. = 0.59\% \tag{53}$$

$$\omega_{trun2}(a_0) = \frac{1.069204}{a_0^{1/3}}, \qquad R.E. = 0.11\% \tag{54}$$

$$\omega_{trun3}(a_0) = \frac{1.070779}{a_0^{1/3}}, \qquad R.E. = 0.03\% \tag{55}$$

where $R.E.$ were calculated by using the following equation

$$R.E. = 100 \times \left| \frac{\omega_i(a_0) - \omega_e(a_0)}{\omega_e(a_0)} \right| \qquad i = 1, 2, 3 \tag{56}$$

In [25] approximately solved (8) using a classical harmonic balance method. They determined the following results for the first and second-order approximate angular frequencies in orders

$$\omega_1(a_0) = \frac{1.049115}{a_0^{1/3}}, \qquad R.E. = 2.00\% \tag{57}$$

$$\omega_2(a_0) = \frac{1.063410}{a_0^{1/3}}, \qquad R.E. = 0.7\% \tag{58}$$

Also in [21] approximately solved (8) using another improved harmonic balance method that incorporates salient feathers of both Newton's method and the harmonic balance method. They calculated the following results for the first, second and third-order approximate angular frequencies in orders

$$\omega_1(a_0) = \frac{1.07685}{a_0^{1/3}}, \qquad R.E. = 0.6\% \tag{59}$$

$$\omega_2(a_0) = \frac{1.06922}{a_0^{1/3}}, \qquad R.E. = 0.12\% \tag{60}$$

$$\omega_3(a_0) = \frac{1.07078}{a_0^{1/3}}, \qquad R.E. = 0.031\% \tag{61}$$

In [10] also approximately solved (8) using modified He's homotopy perturbation method. They achieved the following results for the first, second and third-order approximate angular frequencies are as follows

$$\omega_1(a_0) = \frac{1.07685}{a_0^{1/3}}, \qquad R.E. = 0.6\% \tag{62}$$

$$\omega_2(a_0) = \frac{1.06861}{a_0^{1/3}}, \qquad R.E. = 0.17\% \tag{63}$$

$$\omega_3(a_0) = \frac{1.07019}{a_0^{1/3}}, \qquad R.E. = 0.24\% \tag{64}$$

Comparing all the approximate frequencies the accuracy of the result obtained in this paper is better than those obtained previously by [25] and [10] and is almost similar those obtained by [21]. It has been mentioned that the procedure of [21] and [10] is laborious especially for obtaining the higher approximations.

V. CONCLUSION

Based on a modified harmonic balance method, an analytical technique has been presented to determine approximate periodic solutions of conservative nonlinear oscillator. In compared with the previously published methods, determination of solutions is straightforward and simple. The approximate angular frequency presented in this article using third-order approximate principle with relative error is 0.03%. The advantages of this method include its analytical simplicity and computational efficiency, and the ability to objectively better agreement in third-order approximate solution. To sum up, we can say that the method presented in this article for solving nonlinear oscillator can be considered as an efficient alternative of the previously proposed methods.

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