

# Analysis of the Secondary Stationary Flow Around an Oscillating Circular Cylinder

Artem Nuriev, Olga Zaitseva

**Abstract**—This paper is devoted to the study of a viscous incompressible flow around a circular cylinder performing harmonic oscillations, especially the steady streaming phenomenon. The research methodology is based on the asymptotic explanation method combined with the computational bifurcation analysis. The research approach develops Schlichting and Wang decomposition method. Present studies allow to identify several regimes of the secondary streaming with different flow structures. The results of the research are in good agreement with experimental and numerical simulation data.

**Keywords**—Oscillating cylinder, Secondary Streaming, Flow Regimes, Asymptotic and Bifurcation Analysis.

## I. INTRODUCTION

**T**HE problem of a viscous incompressible flow around a circular cylinder performing harmonic oscillations is a well known classical fluid mechanics problem. Although it has a long history of research, first studies was made by Stokes [1] in 1851, it still retains the theoretical and practical relevance today. Marine and civil engineering, aerospace engineering, robotics - these are just some of the areas in which the problem has a practical application (see [2]–[6]). From a theoretical point of view the study of complex physical mechanisms of vortex formation, structural features of the flow, the analysis of the integral characteristics (such as the hydrodynamic forces acting on the cylinder), the questions of stability and bifurcations of solutions are of the great interest.

Analysing the recent works devoted to the study of the problem, one can distinguish the following perspective areas of research: structure of the flow regimes around the oscillating circular cylinder and phenomenon of a steady streaming. These areas formed the research field of the current work.

The present research is based on the asymptotic explanation method combined with the computational bifurcation analysis. The investigations is carried out in the region of small amplitude and high-frequency oscillations of the cylinder. This approach develops Schlichting-Wang [7], [8] asymptotic expansions method in unsteady Stokes boundary layer and in the outer region. The complex flow model is considered, in which the secondary stationary flow (steady streaming) in the outer region is governed by the full system of Navier-Stokes equations. To solve this problem a computational

bifurcation analysis is used. Analysis is performed according to the classical approach (that was presented for example in [9]) for the analysis of one-parameter non-linear systems. Bifurcation analysis allows to identify several regimes of secondary streaming.

## II. GOVERNING EQUATIONS

Let's consider a flow around a circular cylinder of the radius  $R$  performing small amplitude, high-frequency harmonic oscillations with the velocity

$$u_0 = U_0 \cos(\omega t),$$

where  $U_0$ ,  $\omega$  – the velocity amplitude and the oscillation frequency, respectively.

We write the governing equations in the moving polar coordinate system  $(r, \theta)$  associated with the cylinder. Normalizing the spatial coordinates, time and velocity by  $R$ ,  $\omega^{-1}$ ,  $U_0$  respectively, we get the dimensionless stream-function formulation of the governing system in the following form:

$$\frac{\partial}{\partial t} \Delta \psi + \frac{\gamma \varepsilon}{r} \frac{\partial(\psi, \Delta \psi)}{\partial(r, \theta)} - \varepsilon^2 \Delta^2 \psi = 0 \quad (1)$$

$$\frac{\partial(\psi, \phi)}{\partial(r, \theta)} = \left[ \frac{\partial \psi}{\partial r} \frac{\partial \phi}{\partial \theta} - \frac{\partial \psi}{\partial \theta} \frac{\partial \phi}{\partial r} \right]$$

Control parameters of the problem are defined as

$$\varepsilon = \frac{d_{st}^2}{R^2}, \quad \gamma = \frac{U_0 d_{st}}{\nu}, \quad d_{st} = \sqrt{\frac{\nu}{\omega}},$$

where  $\nu$  – kinematic viscosity of fluid,  $d_{st}$  – the thickness of the Stokes unsteady boundary layer. Under the conditions that the values of  $d_{st}$  and  $\nu$  are small enough, we can assume that  $\varepsilon$  is a small number and  $\gamma$  has a value of order unity.

On the surface of the cylinder, we use the no-slip boundary conditions:

$$r = 1 : \quad \psi = \frac{\partial \psi}{\partial r} = 0 \quad (2)$$

At infinity the boundary conditions correspond to an oscillating irrotational flow:

$$r \rightarrow \infty : \quad \psi = r \sin \theta \cos t + o(r) \quad (3)$$

The order of term  $o(r)$  in (3) will be refined in the next section.

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III. ASYMPTOTIC REPRESENTATION

We use the approach of Schlichting and Wang [7], [8] decomposing the solution into oscillating  $\tilde{\psi}$  and stationary  $\bar{\psi}$  parts:

$$\psi = \tilde{\psi} + \bar{\psi}.$$

Substituting this decomposition to the governing system, we get the equations describing stationary and oscillatory flows:

$$\frac{\partial}{\partial t} \Delta \tilde{\psi} + \frac{\gamma \varepsilon}{r} \left[ \frac{\partial (\bar{\psi}, \Delta \tilde{\psi})}{\partial (r, \theta)} + \frac{\partial (\tilde{\psi}, \Delta \bar{\psi})}{\partial (r, \theta)} \right] + \quad (4)$$

$$\frac{\gamma \varepsilon}{r} \left[ \frac{\partial (\tilde{\psi}, \Delta \tilde{\psi})}{\partial (r, \theta)} \right]_u = \varepsilon^2 \Delta^2 \tilde{\psi},$$

$$\frac{\gamma \varepsilon}{r} \left[ \frac{\partial (\bar{\psi}, \Delta \bar{\psi})}{\partial (r, \theta)} \right] + \frac{\gamma \varepsilon}{r} \left[ \frac{\partial (\tilde{\psi}, \Delta \tilde{\psi})}{\partial (r, \theta)} \right]_s = \varepsilon^2 \Delta^2 \bar{\psi}. \quad (5)$$

Here subscripts "u" and "s" denote the unsteady part and the steady part of the product, respectively. To solve the problem we use the method of asymptotic expansions. We represent oscillatory and stationary components in the form of expansions in outer and inner regions.

In the outer region ( $r \sim 1$ ) we expand the stream function as a power series, in the small parameter  $\varepsilon$ :

$$\begin{aligned} \tilde{\psi} &= \tilde{\Psi}_0(r, \theta, t) + \varepsilon \tilde{\Psi}_1(r, \theta, t) + \varepsilon^2 \tilde{\Psi}_2(r, \theta, t) + \dots \\ \bar{\psi} &= \bar{\Psi}_1(r, \theta) + \varepsilon^2 \bar{\Psi}_2(r, \theta) + \dots \end{aligned} \quad (6)$$

Internal expansion is constructed in the unsteady boundary layer ( $r \sim \varepsilon$ ). Introducing the boundary layer coordinate

$$\eta = \frac{r-1}{\varepsilon}$$

we write a formal internal expansion in the form:

$$\begin{aligned} \tilde{\psi} &= \varepsilon \tilde{\psi}_0(\eta, \theta, t) + \varepsilon^2 \tilde{\psi}_1(\eta, \theta, t) + \varepsilon^3 \tilde{\psi}_2(\eta, \theta, t) + \dots \\ \bar{\psi} &= \varepsilon^2 \bar{\psi}_1(\eta, \theta) + \varepsilon^3 \bar{\psi}_2(\eta, \theta) + \dots \end{aligned} \quad (7)$$

If we substitute (6) and (7) into (4) and equate coefficients of powers of  $\varepsilon$ , we get the number of subproblems for expansion terms.

The priority of subproblems solving is determined by the matching conditions:

$$\begin{aligned} \tilde{\psi}_0 \Big|_{\eta \rightarrow \infty} &\sim \tilde{\Psi}_1 \Big|_{r \rightarrow 1} + \eta \frac{\partial \tilde{\Psi}_0}{\partial r} \Big|_{r \rightarrow 1} \\ \tilde{\psi}_1 \Big|_{\eta \rightarrow \infty} &\sim \tilde{\Psi}_2 \Big|_{r \rightarrow 1} + \eta \frac{\partial \tilde{\Psi}_1}{\partial r} \Big|_{r \rightarrow 1} + \frac{\eta^2}{2} \frac{\partial^2 \tilde{\Psi}_0}{\partial r^2} \Big|_{r \rightarrow 1} \\ \bar{\psi}_1 \Big|_{\eta \rightarrow \infty} &\sim \bar{\Psi}_2 \Big|_{r \rightarrow 1} + \eta \frac{\partial \bar{\Psi}_1}{\partial r} \Big|_{r \rightarrow 1} \\ \bar{\psi}_2 \Big|_{\eta \rightarrow \infty} &\sim \bar{\Psi}_3 \Big|_{r \rightarrow 1} + \eta \frac{\partial \bar{\Psi}_2}{\partial r} \Big|_{r \rightarrow 1} + \frac{\eta^2}{2} \frac{\partial^2 \bar{\Psi}_1}{\partial r^2} \Big|_{r \rightarrow 1} \end{aligned} \quad (8)$$

The resulting asymptotic expansion scheme is shown in Fig. 1.

Let us discuss the behaviour of  $\tilde{\Psi}_k$  and  $\bar{\Psi}_k$  at infinity. In fact we must determine the order of term  $o(r)$  in (3).

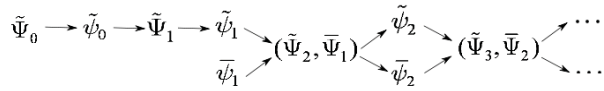


Fig. 1. Asymptotic expansion scheme

The original condition requires damping of all components of the velocity at infinity except the one that corresponds to the stream-function  $\Psi = r \sin \theta \cos t$ . It does not exclude the existence (when  $r \rightarrow \infty$ ) of a potential circulation flow with  $\Psi = a(t) \ln r$ . However, it is necessary to take into account that this flow will have the infinite kinetic energy. Infinite variation of the kinetic energy in a finite time is impossible. Therefore, the existence for the oscillatory terms of expansion the circulating flow at infinity should be prohibited, thus we have

$$r \rightarrow \infty : \quad \tilde{\Psi}_0 = r \sin \theta \cos t + O(1), \quad \tilde{\Psi}_k = O(1), k > 1$$

In contrast, for emerging endlessly stationary components  $\bar{\Psi}$ , an infinite kinetic energy, i.e. the circulation flow at infinity of the form  $\bar{\Psi} = a \ln r$  is permissible.

IV. STEADY STREAMING

The low-order terms of the expansion can be found analytically (their expressions are presented in [8]). The first term that is responsible for the nonlinear behaviour of the flow is  $\bar{\Psi}_1$ . This term is representing the steady streaming around the cylinder. We rewrite the governing equations for  $\bar{\Psi}_1$  in terms of the stream-function ( $\psi = \bar{\Psi}_1/\gamma$ ) and vorticity  $\omega = -\Delta \psi$  as

$$\frac{\text{Re}_s}{r} \left( \frac{\partial \psi}{\partial \theta} \frac{\partial \omega}{\partial r} - \frac{\partial \psi}{\partial r} \frac{\partial \omega}{\partial \theta} \right) = \Delta \omega, \quad (9)$$

$$\Delta \psi = -\omega. \quad (10)$$

Here  $\text{Re}_s = \gamma^2$  is analogue of the Reynolds number for a stationary secondary flow. The boundary conditions on the cylinder surface are determined by matching with the analytical solution in the boundary layer, at infinity we use the damping conditions:

$$r = 1 : \quad \psi = 0, \quad \frac{\partial \psi}{\partial r} = \frac{3}{2} \sin 2\theta. \quad (11)$$

$$r \rightarrow \infty : \quad \frac{\partial}{\partial r} r \frac{\partial \psi}{\partial r} = 0, \quad \frac{\partial \omega}{\partial r} = 0. \quad (12)$$

To solve this problem a computational bifurcation analysis is used.

V. BIFURCATION ANALYSIS

The secondary streaming problem is solved in the range [0, 200] of Reynolds numbers. We discretize the problem with the second order of accuracy in the limited area  $z \in [0, 3.4]$  (it is corresponding the physical area  $r \in [1, 30]$ ) divided into  $n = 256 \times 256$  grid cells. Coordinate transformation  $z = \ln r$  leads to concentration of the grid nodes around the cylinder in the physical domain (see Fig. 2).

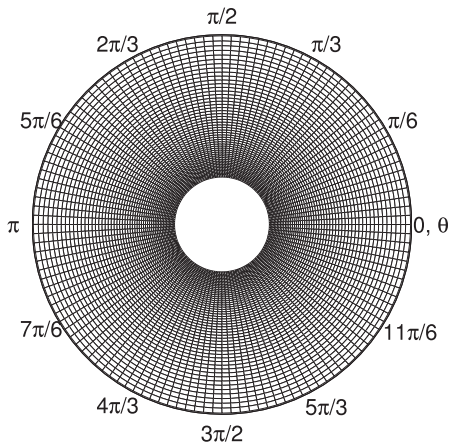


Fig. 2. The structure of the computational grid in the vicinity of the cylinder.

For the solution of the discretized problem we use a classical BA approach for the study of one-parameter nonlinear systems (see [9]). The main components of the analysis are:

- 1) Localization problem. Construction of solutions belonging to different branches. To solve this problem FPN (Fixed point – Newton) homotopy method is used.
- 2) Continuation problem. Tracking the evolution of the solution branches with the Reynolds number variation. Basic elements of the continuation method:
  - Predictor — method of tangents.
  - Corrector — Moore-Penrose method.
  - Step size correction according the Reynolds number.
- 3) Identification of the bifurcation points for the final branches stratification. Localization of bifurcation points is carried out as the solution of the spectral problem.

More detailed description of the solution techniques can be found in the work [14].

VI. RESULTS

In the studied range three different solution branches of the problem were found. In Fig. 3 the projection of the bifurcation diagram of the phase-parametric space on the  $K - Re_s$  plane is shown, where  $K$  is the kinetic energy:

$$K = \frac{1}{2} \int_{\Omega} \left( \left( \frac{\partial \psi}{\partial r} \right)^2 + \left( \frac{\partial \psi}{\partial \theta} \right)^2 \right) dr d\theta.$$

Solid lines are solution branches, markers are bifurcation points.

The lowest line on a diagram corresponds to the main solution. It is unique for Reynolds numbers less than 16. For  $Re_s > 16$  after the first fold bifurcation two additional solutions appear. The second fold bifurcation was localized at  $Re_s \approx 87$ , thus for the larger Reynolds numbers there are five different solutions of the problem.

The characteristic flow patterns for each solution branch is shown in Fig. 4. The regime of the first type has two symmetries: it is symmetric about the  $y$ -axis and  $\pi$ -periodic about the  $\theta$  angle. The second type regime is symmetric about

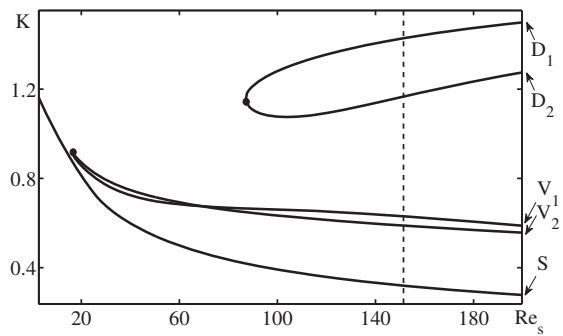


Fig. 3. The bifurcation diagram of the phase-parametric space

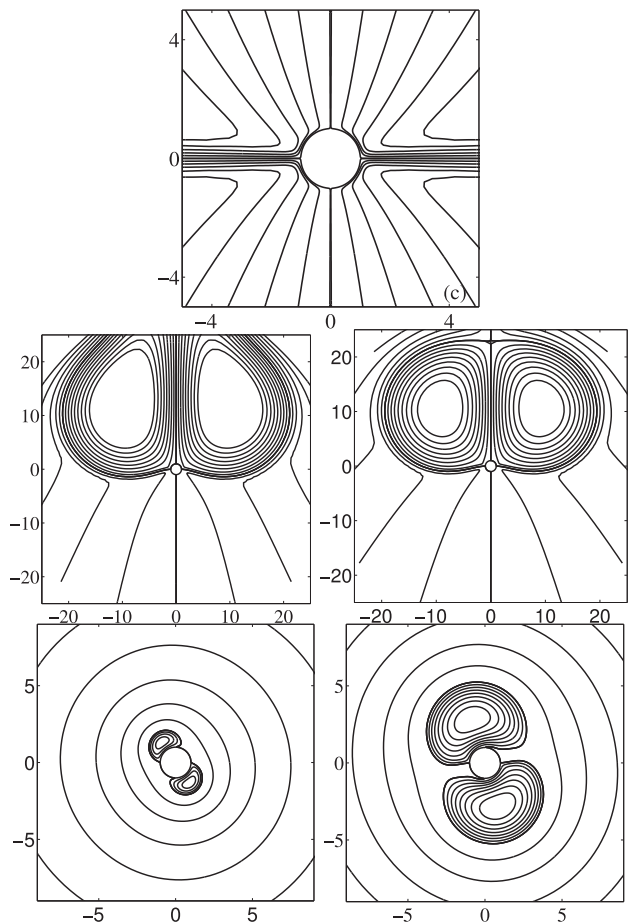


Fig. 4. Stream function for  $Re_s = 150$ .

the  $y$ -axis, but not  $\pi$ -periodic. The last one is  $\pi$ -periodic, but not symmetric about the  $y$ -axis.

These regimes structures are in good agreement with one observed in experimental and direct numerical simulation studies [10]–[13].

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