International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:6, No:8, 2012

# An Optimization of Orbital Transfer for Spacecrafts with Finite-thrust Based on Legendre Pseudospectral Method

Yanan Yang, Zhigang Wang, Xiang Chen

***Abstract*—**This paper presents the use of Legendre pseudospectral method for the optimization of finite-thrust orbital transfer for spacecrafts. In order to get an accurate solution, the System's dynamics equations were normalized through a dimensionless method. The Legendre pseudospectral method is based on interpolating functions on Legendre-Gauss-Lobatto (LGL) quadrature nodes. This is used to transform the optimal control problem into a constrained parameter optimization problem. The developed novel optimization algorithm can be used to solve similar optimization problems of spacecraft finite-thrust orbital transfer. The results of a numerical simulation verified the validity of the proposed optimization method. The simulation results reveal that pseudospectral optimization method is a promising method for real-time trajectory optimization and provides good accuracy and fast convergence.

***Keywords*—**Finite-thrust, Orbital transfer, Legendre pseudospectral method

## I. Introduction

FINITE-thrust propulsion is now widely used in space missions, such as lunar or mars descent, interplanetary transfer, spacecraft rendezvous, etc. The finite-thrust optimal control problem is qualitatively different from the impulsive case as there are now no integrable arcs and the control itself, must be modeled and determined. Optimizing finite thrust trajectory is a challenging problem due to the existence of long powered arcs [1].

Optimal control problems are generally nonlinear and therefore, do not have analytic solutions (e.g., like the linear-quadratic optimal control problem). As a result, it is necessary to employ numerical methods to solve optimal control problems [2]. Numerical solutions for Finite-thrust trajectory optimal control are broadly obtained by using Direct and Indirect optimization methods. In an indirect method, the calculus of variations (Pontryagin's minimum principle) is employed to obtain the first-order optimality conditions. Subsequently, these conditions result in a two-point (or a multi point, in the case of a complex problem) boundary-value problem. This boundary-value problem has a special structure because it arises from taking the derivative of a Hamiltonian. The same is termed as Hamiltonian Boundary Value Problem (HBVP) which is then solved to determine candidate optimal trajectories also called Extremal Trajectories.

Yanan Yang is a Master student in college of astronautics in Northwestern Polytechnical University. Youyi Xilu, Xi'an Shaanxi 710072, P.R. China (phone: 008615991716774; e-mail: yanan301415@gmail.com).

Zhigang Wang is a Professor in college of astronautics in Northwestern Polytechnical University. Youyi Xilu, Xi'an Shaanxi 710072, P.R. China (phone: 008688495783).

Xiang Chen is a Master student in college of astronautics in Northwestern Polytechnical University. Youyi Xilu, Xi'an Shaanxi 710072, P.R. China (phone: 008613916960943)

However, the main disadvantage of indirect methods is that HBVPs are hyper sensitive and is often extremely difficult to solve (particularly for problems that span large time intervals or problems with interior point constraints) and hence not suited to obtain an extremal trajectory. The approach that has risen to prominence in numerical optimal control over the past two decades is that of so called direct methods. In a direct method, the state and/or control are approximated using an appropriate function approximation (e.g., polynomial approximation or piecewise constant parameterization) [2]. Direct optimization method can transform the optimal control problem into a nonlinear planning problem [3].

An optimal control problem solution requires the approximation of following three types of mathematical objects:

1) The integration in the cost function
2) The differential equation of the control system
3) The state-control constraints

An ideal approximation method should be efficient for all of the above approximation tasks. These requirements make pseudospectral methods ideal because they are efficient for the approximation of all three mathematical objects [4], [5] & [6].Recently in Pseudospectral optimal control, Legendre and Ch-ebyshev polynomials are most commonly used. Mathematically, quadrature nodes are able to achieve high accuracy with less number of points whereas the interpolating polynomial of any smooth function at LGL nodes converges in $L^2$ sense at the so-called spectral rate, i.e., faster than any polynomial rate [5].

In short, Legendre pseudospectral method, comparing with other collocation methods, has the advantage of fast convergence rate, high accuracy and insensitive in prediction of the initial value. Therefore, we have used pseudospectral Legendre method to solve the finite-thrust orbit transfer optimization control problem.

## II. Optimal Control Problem Statement

Finite thrust model is considered here in the geocentric equatorial inertial coordinates, which establishes a spacecraft orbital transfer model. The problem is defined as follows:

### A. Dynamics Equations

System's Dynamics equations for the given optimal control problem are given below:

$$\begin{cases} \dot{x} = v_x \\ \dot{y} = v_y \\ \dot{z} = v_z \\ \dot{v}_x = -\mu x/r^3 + F/m \cos\psi \cos\theta \\ \dot{v}_y = -\mu y/r^3 + F/m \cos\psi \sin\theta \\ \dot{v}_z = -\mu z/r^3 + F/m \sin\psi \\ \dot{m} = -F/v_e \end{cases} \quad (1)$$

Where $x, y, z, v_x, v_y, v_z$ are the positions and velocity vectors of the spacecraft respectively.

$r$ is the geocentric distance of the spacecraft,
$F$ is the thrust,
$\theta$ is the azimuth angle ,
$\psi$ is the angle of site.
$m$ is the mass of spacecraft.
$v_e$ is the engine fuel injection speed

In order to improve the accuracy of system's dynamics variables and to calculate the convergence rate, we have introduced the following reference variables:

1) Reference distance $U_d = R_e, R_e$ is radius of the Earth;
2) Reference time $U_t = \sqrt{R_e/g_0}$ where, $g_0$ is gravitational acceleration on the Earth surface ( $9.81 m/s^2$ );
3) The reference speed is $U_v = \sqrt{R_e g_0}$ and reference mass is initial mass of the spacecraft : $U_m = m_0$ ;
4) The reference gravitational acceleration is $U_g = \mu/R_e^2$ , where $\mu$ is gravity constant of the Earth.

The dimensionless equations of motion can be written as follow:

$$\begin{cases} \dot{\bar{x}} = \bar{v}_x \\ \dot{\bar{y}} = \bar{v}_y \\ \dot{\bar{z}} = \bar{v}_z \\ \dot{\bar{v}}_x = -\bar{x}/\bar{r}^3 + G/\bar{m} \cos\psi \cos\theta \\ \dot{\bar{v}}_y = -\bar{y}/\bar{r}^3 + G/\bar{m} \cos\psi \sin\theta \\ \dot{\bar{v}}_z = -\bar{z}/\bar{r}^3 + G/\bar{m} \sin\psi \\ \dot{\bar{m}} = -G/\bar{v}_e \end{cases} \quad (2)$$

Where; $G = F / U_m U_g$

*B. The selection of performance indicator*

The aim of optimal trajectory design is to minimize the amount of fuel required to perform a free end-time descent from the given initial state to the given terminal state. The performance indicator J can be expressed as:

$$J = -m(t_f)$$

*C. Constraints*

The Following constraints are to be taken into account:

1) Control constrains

$$-\pi/2 \le \psi \le \pi/2$$
$$-\pi \le \theta \le \pi$$

2) Boundary constrains:

The boundary constraints are also descritized at the LGL points as

$$e(\mathbf{x}(t_0), \mathbf{x}(t_f), t_0, t_f) = 0$$

## III. Legendre Pseudospectral Methodology

Let $L_N(t)$ be the Legendre polynomial of degree N on the interval [-1, 1]. Let $t_m (m = 1, 2, 3, ..., N-1)$ be the LGL points given by $t_0 = -1,\ t_N = 1$ for the range $1 \le m \le N-1$, whereas $t_m$ are the zeros of the derivative of the Legendre polynomial $\dot{L}_N(t)$ . We use the following transformation to express the problem for t from $[\tau_0, \tau_f]$ to [-1, 1]

$$\tau(t) = 2t/(\tau_f - \tau_0) - (\tau_f + \tau_0)/(\tau_f - \tau_0)$$

For approximating the continuous equations, we seek a polynomial approximation of the form:

$$X^N(t) = \sum_{m=0}^{N} X(t_m)\phi_m(t) \quad (3)$$

$$U^N(t) = \sum_{m=0}^{N} U(t_m)\phi_m(t) \quad (4)$$

Where m= 0, 1, 2……N, and

$$\phi_m(t_j) = \frac{1}{N(N-1)L_N(t)} \cdot \frac{(t^2-1)\dot{L}_N(t)}{t - t_m} \quad (5)$$
$$m = 0, 1, 2, ..., N$$

are the Lagrange polynomials of order $N$ . It can be shown that:

$$\phi_m(t_j) = \begin{cases} 1 & m = j \\ 0 & m \ne j \end{cases} \quad (6)$$

From this property of $\phi_m$ it follows that

$$X^N(t_m) = X(t_m) \quad (7)$$
$$U^N(t_m) = U(t_m) \quad (8)$$

Generally the approximations are expressed as:

$$X^{\mathrm{N}}(t) \approx X(t) \tag{9}$$

$$U^{\mathrm{N}}(t) \approx U(t) \tag{10}$$

To express the derivative $\dot{X}^{\mathrm{N}}(t)$ in terms of $X^{\mathrm{N}}(t)$ at the discrete points $t_m$, we differentiate (9) which results in a matrix multiplication of the following form:

$$\dot{X}^{\mathrm{N}}(t_{\mathrm{m}}) = \sum_{l=0}^{N} D_{\mathrm{m}\times\mathrm{l}} X(t_l) \tag{11}$$

Where $D = (D_{\mathrm{m}\times\mathrm{l}})$ are entries of the $(N+1)\times(N+1)$ differentiation matrix $D$

$$D = (D_{\mathrm{m}\times\mathrm{l}})_{(\mathrm{N}+1)\times(\mathrm{N}+1)} = \begin{cases} \dfrac{L_{\mathrm{N}}(t_m)}{(t_m - t_l) L_{\mathrm{N}}(t_l)} & m \neq l \\ \dfrac{-N(N+1)}{4} & m = l = 0 \\ \dfrac{N(N+1)}{4} & m = l = N \\ 0 & otherwise \end{cases} \tag{12}$$

Re-writing (9)-(10) in the following form:

$$\begin{cases} \mathbf{x}^{\mathrm{N}}(t) = \sum_{m=0}^{N} \mathbf{a}_m \phi_m(t) \\ \mathbf{u}^{\mathrm{N}}(t) = \sum_{m=0}^{N} \mathbf{b}_m \phi_m(t) \end{cases} \tag{13}$$

For the derivative of the state vector $X^{\mathrm{N}}(t)$, collocated at the points $t_{\mathrm{m}}$, we re-write ( 11 ) as

$$c_k \approx \dot{X}^{N}(t_m) = \sum_{m=0}^{N} D_{ml} a_l \tag{14}$$

Then the differential equation is approximated by the following nonlinear algebraic inequalities:

$$\left\| \sum_{m=0}^{N} D_{ml} a_l - \frac{(\tau_{\mathrm{f}} - \tau_0)}{2} f(x_m, u_m) \right\|_{\infty} \leq \delta \tag{15}$$

Where; $\delta > 0$ is a small number which represents accuracy range.

Hence the new boundary conditions are:

$$\begin{aligned} &\mathbf{x}^{\mathrm{N}}(-1) = \mathbf{x}_0 \\ &\mathbf{x}^{\mathrm{N}}(1) = \mathbf{x}_{\mathrm{f}} \end{aligned} \tag{16}$$

Orbital condition is:

$$\begin{aligned} &C_j(\mathbf{x}^{\mathrm{N}}(t_{\mathrm{k}}), \mathbf{u}^{\mathrm{N}}(t_{\mathrm{k}})) \leq 0 \\ &j = 1, 2, ..., c \end{aligned} \tag{17}$$

Finally, the cost function is approximated by the Gauss-Lobatto integration rule:

$$\begin{aligned} J^{\mathrm{N}} &= h(\mathbf{x}^{\mathrm{N}}(1)) + \int_{-1}^{1} g(\mathbf{x}^{\mathrm{N}}(t), \mathbf{u}^{\mathrm{N}}(t)) dt \\ &\approx h(\mathbf{a}_{\mathrm{N}}) + \sum_{k=0}^{\mathrm{N}} g(\mathbf{x}^{\mathrm{N}}(t_{\mathrm{k}}), \mathbf{u}^{\mathrm{N}}(t_{\mathrm{k}})) w_{\mathrm{k}} \\ &= h(\mathbf{a}_{\mathrm{N}}) + \sum_{k=0}^{\mathrm{N}} g(\mathbf{a}_{\mathrm{k}}, \mathbf{b}_{\mathrm{k}}) w_{\mathrm{k}} \end{aligned} \tag{18}$$

Where;

$$\begin{aligned} &w_{\mathrm{k}} = -\frac{2}{N(N+1)} \cdot \frac{1}{[L_N(t_{\mathrm{k}})]^2} \\ &k = 0, 1, ..., N \end{aligned}$$

Finally, the optimal control problem is approximated by the discretized optimization problem by finding a= [$a_0$, $a_1$…$a_N$] & b=[$b_1$,$b_2$,……$b_N$] to minimize the cost function (18) in accordance with the constraints of equations(15), (16) and (17).

## IV. Simulation and Results Analysis

The constraints are dealt with nonlinear programming multiplier method, and the combined DFP method is used to solve the optimization problem after the conversion. The simulation parameters are a coplanar orbital rendezvous problem with given initial point and end point. The initial and final orbit parameters are given below:

$$\begin{cases} a_0 = 7078145m \\ i_0 = 98.18^\circ \\ \Omega_0 = 30^\circ \end{cases} \begin{cases} a_f = 7206904m \\ i_f = 98.75^\circ \\ \Omega_f = 30^\circ \end{cases}$$

The constraints of thrust direction angle are set as follows：

$$\begin{aligned} &-\pi/2 \leq \Psi \leq \pi/2 \\ &-\pi \leq \theta \leq \pi \end{aligned}$$

The values of the other parameters used in this scenario are summarized here:

$$\begin{cases} v_e = 3000\,\mathrm{m/s} \\ m_0 = 2850kg \\ T = 980N \\ \mu = 398600.44k\,m^3/s^2 \\ R_e = 6378145m \end{cases}$$

Orbital boundary conditions: The values of state variables and initial and terminal constraints are shown in Table I.

The optimized performance of spacecraft orbit transfer is selected as the minimum fuel consumption during the entire process: $J = -m(t_f)$

LGL points taken are N=25. Orbit transfer time and the loss of mass recorded is 1766.5 sec & 576.9 Kg respectively.

Following Table I demonstrate the terminal constraint satisfaction

TABLE I
THE VALUE OF STATE VARIABLES, INITIAL & TERMINAL RESTRICTION

| | *x(km)* | *y (km)* | *z(km)* | *$V_X$(m/s)* | *Vy(m/s)* | *Vz(m/s)* | *m(kg)* |
|---|---|---|---|---|---|---|---|
| $x(t_0)$ | 6123.7 | 3535.5 | 0.0 | 534.1 | -925.1 | 7431.7 | 2850 |
| $x(t_f)$ | -1238.7 | -1929.5 | 6832.5 | -6337.7 | -3289.8 | -2078.0 | free |
| $x(t=t_f)$ | -1238.7 | -1929.5 | 6832.5 | -6337.7 | -3289.8 | -2078.0 | 2273.1 |

Results depicted in Table I and Fig. 1 & 2 shows that the optimized orbit changes smoothly with considerably fast convergence rate compared with the value of terminal constraints with high accuracy. Moreover, the results show that the spacecraft can be accurately transferred to the target from the initial orbit. The control curve of the Orbit transfer process is shown in Fig. 4. The azimuth angle and the angle of site changes smoothly, and appropriately meeting the constraints. Therefore, the entire flight transfer orbit is under control. In addition, the simulation of the initial parameters selected is relatively free which results that the pseudospectral method is not sensitive to the initial guess and having good robustness.

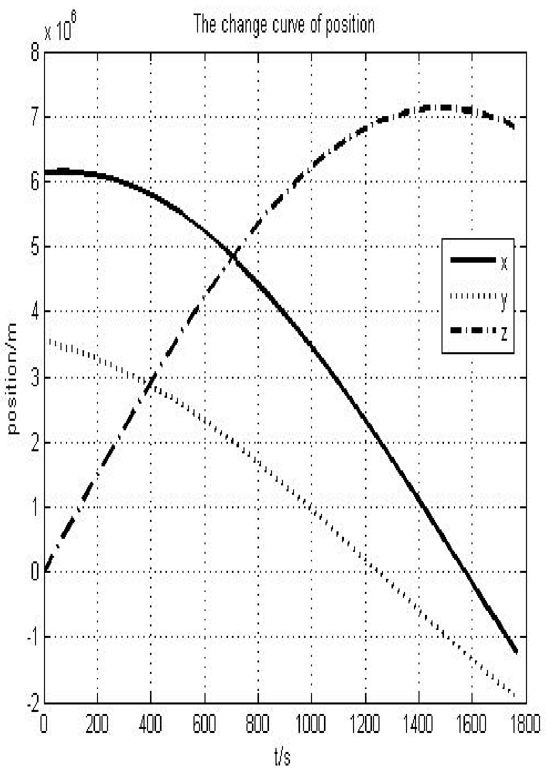


Fig. 1 The change curve of position

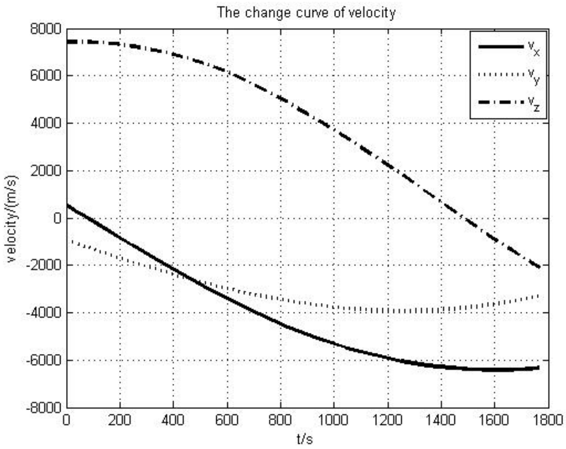


Fig. 2 The change curve of velocity

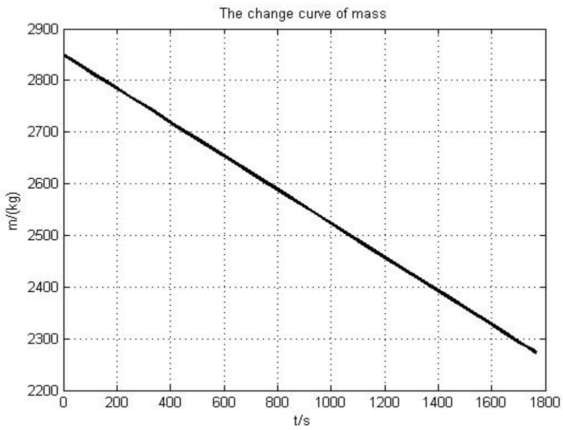


Fig. 3 The change curve of mass

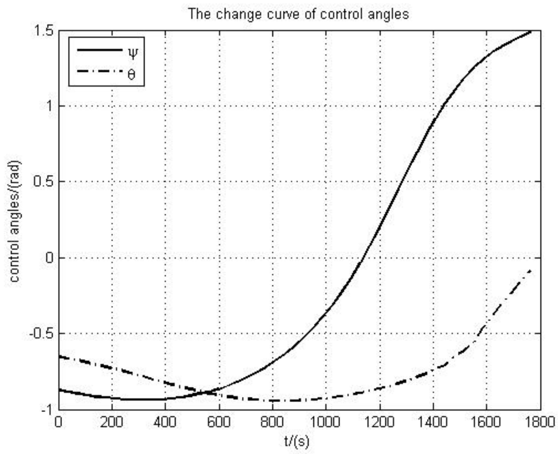


Fig. 4 The change curve of control angles

## V. CONCLUSION

Initially, the finite-thrust orbit transfer optimal control problem is transformed into nonlinear programming problem by using the Legendre pseudospectral method. Subsequently, the dynamic optimization problem is conversed to a static state parameter optimization problem.

The simulation results demonstrate that Legendre pseudospectral method is not sensitive to the initial conditions of orbital transfer. The results of a numerical simulation verified the validity of the proposed optimization method. The results indicate that the method can provide good performance on accuracy and fast convergence. It is expected that this novel optimization algorithm can be used to solve the similar optimization problems.

## REFERENCES

[1] Qibo Peng "Finite-Thrust Trajectory Optimization Using a Combinationof Gauss Pseudospectral Method and Genetic Algorithm". (Chapter 4 of Genetic Algorithms in Applications Edited by Rustem Popa, ISBN 978-953-51-0400-1, hard cover, 328 pages, Publisher: In Tech, Published: March 21, 2012 under CC BY 3.0 license, in subject Artificial Intelligence DOI: 10.5772/2675.
[2] http://en.wikipedia.org/wiki/Optimal_control
[3] http://en.wikipedia.org/wiki/Pseudospectral_optimal_control
[4] Q. Gong, W. Kang and I. M. Ross, A Pseudospectral Method for The Optimal Control of Constrained Feedback Linearizable Systems, IEEE Trans. Auto. Cont., Vol.~51, No.~7, July 2006, pp.~1115–1129.
[5] J. S. Hesthaven, S. Gottlieb and D. Gottlieb, Spectral methods for time-dependent problems, Cambridge University Press, 2007. ISBN 978-0-521-79211-0
[6] Q. Gong, I. M. Ross, W. Kang and Fahroo, F., Connections between the covector mapping theorem and convergence of pseudospectral methods for optimal control, Computational Optimization and Applications, Springer Netherlands, published online: 31 October 2007, to appear in Journal, 2008.