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An Optimal Control of Water Pollution in a Stream Using a Finite Difference Method

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Abstract—Water pollution assessment problems arise frequently in environmental science. In this research, a finite difference method for solving the one-dimensional steady convection-diffusion equation with variable coefficients is proposed; it is then used to optimize water treatment costs.

Keywords—Finite difference, One-dimensional, Steady state, Water pollution control, Optimization, Convection-diffusion equation.

I. INTRODUCTION

IN [3] and [6], the finite element method for solving the water pollution models in one- and two-dimensional water areas are presented, respectively. In [4] and [5], they used the finite difference method to the hydrodynamic model with constant coefficients in the uniform reservoir and stream, respectively.

This paper describes a mathematical model for solving the dispersion of pollutant in a a stream. A finite difference method for assessment of the chemical oxygen demand(COD) concentration in a stream is considered. This model requires the calculation of the substance dispersion given the water velocity in the stream. A finite difference model is used to compute the concentration of the pollutant for variable inputs. These are then subjected to the optimal control of the water treatment plants to achieve minimum cost. A numerical example given.

II. DISPERSION IN A STREAM

The dispersion of COD is described by the advection-dispersion-reaction equation(ADRE) [2] in the domain $[a, b]$,

$$-D_x \frac{d^2c}{dx^2} + u\frac{dc}{dx} + Rc - Q = 0 \qquad (1)$$

where $c(x)$ is the concentration of COD at the point $x \in [a, b]$ (kg/m^3), u is the flow velocity in the x-direction (m/s) , D_x is the diffusion coefficient (m^2/s), R is the substance decay rate (s^{-1}) and Q is the rate of change of substance concentration due to a source $(kg/m^3 s)$. The boundary conditions are $c = c_0$ at $x = a$ and $\frac{dc}{dx} = t_0$ at $x = b$.

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III. A NUMERICAL TECHNIQUE

Consider Eq.(1) in a form

$$c'' = p(x)c' + q(x)c + r(x). \qquad (2)$$

First, we select an integer $N > 0$ and divide the interval $[a, b]$ into $N + 1$ equal subintervals, whose endpoints are the mesh points $x_i = a + ih$, for all $i = 1, 2, \ldots, N + 1$, where $h = \frac{(b-a)}{N+1}$. At the interior mesh points, x_i for all $i = 1, 2, \ldots, N$, the differential equation to be approximated is

$$c''(x_i) = p(x_i)c'(x_i) + q(x_i)c(x_i) + r(x_i). \qquad (3)$$

Expanding y in a third Taylor polynomial about x_i evaluated at x_{i+1} and x_{i-1}, we have

$$\begin{aligned} c(x_{i+1}) = c(x_i + h) &= c(x_i) + hc'(x_i) + \frac{h^2}{2}c''(x_i) \\ &+ \frac{h^3}{6}c'''(x_i) + \frac{h^4}{24}c^{(4)}(\xi_i^+), \end{aligned} \qquad (4)$$

for some ξ in (x_i, x_{i+1}), and

$$\begin{aligned} c(x_{i-1}) = c(x_i - h) &= c(x_i) - hc'(x_i) + \frac{h^2}{2}c''(x_i) \\ &- \frac{h^3}{6}c'''(x_i) + \frac{h^4}{24}c^{(4)}(\xi_i^-), \end{aligned} \qquad (5)$$

for some ξ in (x_{i-1}, x_i), assuming $c \in C^4[x_{i-1}, x_{i+1}]$. If these equations are added, the terms involving $c'(x_i)$ and $c'''(x_i)$ are eliminated and simple algebraic manipulation gives

$$c''(x_i) = \frac{1}{h^2}[c(x_{i+1}) - 2c(x_i) + c(x_{i-1})] - \frac{h^2}{24}[c^{(4)}(\xi_i^+) + c^{(4)}(\xi_i^-)]. \qquad (6)$$

The intermediate value theorem can be used to simplify this even further

$$c''(x_i) = \frac{1}{h^2}[c(x_{i+1}) - 2c(x_i) + c(x_{i-1})] - \frac{h^2}{12}[c^{(4)}(\xi_i)], \qquad (7)$$

for some ξ_i in (x_{i-1}, x_{i+1}). A centered-difference formula for $y'(x_i)$ is obtained in a similar manner resulting in

$$c'(x_i) = \frac{1}{2h}[c(x_{i+1}) - c(x_{i-1})] - \frac{h^2}{6}[c'''(\eta_i)], \qquad (8)$$

for some η_i in (x_{i-1}, x_{i+1}). The use of these centered-difference formulas in Eqs.(7)-(8) results in the equation

$$\begin{aligned} \frac{c(x_{i+1}) - 2c(x_i) + c(x_{i-1})}{h^2} &= p(x_i)[\frac{c(x_{i+1}) - c(x_{i-1})}{2h}] \\ +q(x_i)c(x_i) + r(x_i) &- \frac{h^2}{12}[2p(x_i)c'''(\eta_i) - c^{(4)}(\xi_i)]. \end{aligned} \qquad (9)$$

A finite difference method with truncation error of order $O(h^2)$ results by using the Eq.(9) together with the boundary conditions $c(a) = c_0$ and a Neumann boundary condition $c'(b) = t_0$ become [1]

$$c_{N+1} = 2ht_0 + c_{N-1}. \tag{10}$$

Using the central difference method, we can obtain

$$\left(\frac{2c_i - c_{i+1} - 2c_{i-1}}{h^2}\right) + p(x_i)\left[\frac{c_{i+1} - c_{i-1}}{2h}\right] + q(x_i)c_i = -r(x_i), \tag{11}$$

$$-(1 + \frac{h}{2}p(x_i)c_{i-1}) + (2 + h^2 q(x_i))c_i - (1 - \frac{h}{2}p(x_i))c_{i+1} = -h^2 c(x_i), \tag{12}$$

and the resulting system of equation is expressed in the tridiagonal $N \times N$-matrix form

$$[K]\{c\} = \{G\}, \tag{13}$$

where

$$\{c\} = \begin{Bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{N-1} \\ c_N \end{Bmatrix}, \tag{15}$$

$$\{G\} = \begin{Bmatrix} -h^2 r(x_1) + ((1 + \frac{h}{2}p(x_1))c_0) \\ 0 \\ \vdots \\ 0 \\ -h^2 r(x_N) + ((-1 + \frac{h}{2}p(x_N))(2ht_0)) \end{Bmatrix} \tag{16}$$

The entries of $[K]$ are k_{ij} for each $1 \le i, j \le N$ where N is the number of nodes. We apply the boundary condition to Eq.(13), to give

$$k_{\alpha\alpha} = 1, \tag{17}$$

$$g_\alpha = c_\alpha \tag{18}$$

where α indicates any inflow node and c_α is the corresponding given value. We denote the corresponding matrices by K^0 and G^0, respectively. Let B be the inverse matrix of K^0. Then

$$\{\widetilde{c}\} = [B]\{G^0\}, \tag{19}$$

and we obtain

$$\widetilde{c}_i = \sum_{j=1}^{N} b_{ij} f_j, \tag{20}$$

where $1 \le i \le N$.

IV. Optimal control of cost

Let x_β be the observation nodes and r_α be the COD concentration that is removed at inflow points. It follows that $c_\alpha - r_\alpha$ is the concentration of the pollutant after partial purification. Then

$$\widetilde{c}_\beta = b_{\beta 1} g_1 + \ldots + b_{\beta\alpha}(g_\alpha - r_\alpha) + \ldots + b_{\beta N} g_N. \tag{21}$$

Let C_{ST} be the standard COD concentration. The water quality $\widetilde{c}_\beta$ must be at or below the standard water quality. That is,

$$\widetilde{c}_\beta = \sum_{i=1}^{m} b_{\beta i} g_i + \sum_{j=1}^{n} b_{\beta\alpha_j}(g_{\alpha_j} - r_{\alpha_j}) \le C_{ST} \tag{22}$$

where m is the number of observation points and n is the number of inflow points $(N = m+n)$. The objective function J is the cost of wastewater treatment in the system, so

$$J(x) = \sum_{j=1}^{m} \omega_j r_{\alpha_j}, \tag{23}$$

where ω_j is the cost of wastewater treatment for the required reduction of the COD concentration. The constraints are

$$\widetilde{c}_\beta = \sum_{i=1}^{m} b_{\beta i} g_i + \sum_{j=1}^{n} b_{\beta\alpha_j}(g_{\alpha_j} - r_{\alpha_j}) \le C_{ST}, \tag{24}$$

the upper bound of the control (treatment plant) is

$$r_{\alpha_j} \le u_{\alpha_j}, \tag{25}$$

the lower bound of the control (treatment plant) is

$$r_{\alpha_j} \ge l_{\alpha_j} \tag{26}$$

and the controls are non-negative, that is

$$r_{\alpha_j} \ge 0, \tag{27}$$

where l_{α_j} and u_{α_j} are the lower and upper bounds respectively of the points control variables. The optimal control problem is solved by the simplex method.

V. Numerical example

Assume that there are plants A,B and C which discharge wastewater into the stream at 0.0 km, 0.4 km and 0.8 km. and that the COD concentrations of the wastewater are 1.2500, 1.4918 and 1.9312 mg/l, respectively. The physical parameters are: diffusion coefficient 2 m^2/s, flow velocity $u(x) = 5 - x\ m/s$, substance decay rate 3 day^{-1} and rate of change of substance concentration due to a source 1 $mg/l{\cdot}day$. The legal requirement is that the plant has to decrease the COD concentration in the wastewater to less than 0.1 mg/l in the stretch from the plant A to a point 2.0 km. downstream from A. At the observation points the COD concentration must be less than 1.2000 mg/l. Plants A,B and C are capable of treating the wastewater, so that the COD concentration is not greater than 1.0, 1.0 and 1.0 mg/l, respectively. The costs of wastewater treatment for the reduction by 1 mg/l of COD concentration are 400, 600 and 720 Euro for plants A, B and C, respectively. It turns out that the least cost of wastewater treatment is 813 Euro in this case.

VI. Conclusion

We have established a simulation process by means of which water pollution levels can be reduced to an agreed standard at least cost.

$$[K] = \begin{bmatrix} 2+h^2q(x_1) & -(1-\frac{h}{2}p(x_1)) & & & & \\ -(1+\frac{h}{2}p(x_2)) & 2+h^2q(x_2) & -(1-\frac{h}{2}p(x_2)) & & & \\ & -(1+\frac{h}{2}p(x_3)) & 2+h^2q(x_3) & -(1-\frac{h}{2}p(x_3)) & & \\ & & \ddots & \ddots & \ddots & \\ & & & & -2p(x_N) & 2+h^2q(x_N) \end{bmatrix} \quad (14)$$

TABLE I
COD CONCENTRATION ASSESSMENT

Node	1	2	3	4	5	6	7	8	9	10	11
Unpurified Inflow	1.2500	-	1.4918	-	1.9312	-	-	-	-	-	-
Observations	1.2500	0.6978	1.4918	1.6083	1.9312	1.7516	1.5923	1.4542	1.3405	1.2581	1.2214
Purified Inflow	1.1500	-	1.1838	-	1.3231	-	-	-	-	-	-
Observations	1.1500	0.6978	1.1838	1.2000	1.3231	1.2000	1.0909	0.9963	0.9184	0.8619	0.8368

TABLE II
OPTIMAL COST OF WASTEWATER TREATMENT

Plant	Optimal Reduction of COD Concentration (mg/l)	Optimal Cost of Reduction (Euro)
A	0.1000	40
B	0.5580	335
C	0.6081	438
Minimum cost (Euro)		813

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