International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:4, No:7, 2010

An iterative updating method for damped gyroscopic systems

Yongxin Yuan

Abstract—The problem of updating damped gyroscopic systems using measured modal data can be mathematically formulated as following two problems. **Problem I**: Given $M_a \in \mathbf{R}^{n\times n}, \Lambda = \mathrm{diag}\{\lambda_1, \cdots, \lambda_p\} \in \mathbf{C}^{p\times p}, X = [x_1, \cdots, x_p] \in \mathbf{C}^{n\times p}$, where $p < n$ and both Λ and X are closed under complex conjugation in the sense that $\lambda_{2j} = \bar{\lambda}_{2j-1} \in \mathbf{C}$, $x_{2j} = \bar{x}_{2j-1} \in \mathbf{C}^n$ for $j = 1, \cdots, l$, and $\lambda_k \in \mathbf{R}$, $x_k \in \mathbf{R}^n$ for $k = 2l+1, \cdots, p$, find real-valued symmetric matrices D, K and a real-valued skew-symmetric matrix G (that is, $G^T = -G$) such that $M_a X\Lambda^2 + (D+G)X\Lambda + KX = 0$. **Problem II**: Given real-valued symmetric matrices $D_a, K_a \in \mathbf{R}^{n\times n}$ and a real-valued skew-symmetric matrix G_a, find $(\hat{D}, \hat{G}, \hat{K}) \in \mathbf{S_E}$ such that $\|\hat{D}-D_a\|^2+\|\hat{G}-G_a\|^2+\|\hat{K}-K_a\|^2 = \min_{(D,G,K)\in \mathbf{S_E}}(\|D-D_a\|^2 + \|G-G_a\|^2 + \|K-K_a\|^2)$, where $\mathbf{S_E}$ is the solution set of Problem I and $\|\cdot\|$ is the Frobenius norm. This paper presents an iterative algorithm to solve Problem I and Problem II. By using the proposed iterative method, a solution of Problem I can be obtained within finite iteration steps in the absence of roundoff errors, and the minimum Frobenius norm solution of Problem I can be obtained by choosing a special kind of initial matrices. Moreover, the optimal approximation solution $(\hat{D}, \hat{G}, \hat{K})$ of Problem II can be obtained by finding the minimum Frobenius norm solution of a changed Problem I. A numerical example shows that the introduced iterative algorithm is quite efficient.

Keywords—model updating, iterative algorithm, gyroscopic system, partially prescribed spectral data, optimal approximation.

I. Introduction

THROUGHOUT this paper, we shall adopt the following notation. $\mathbf{C}^{m\times n}$ and $\mathbf{R}^{m\times n}$ denote the set of all $m\times n$ complex and real matrices, $\mathbf{SR}^{n\times n}$ and $\mathbf{SSR}^{n\times n}$ denote the set of all $n\times n$ symmetric and skew-symmetric matrices in $\mathbf{R}^{n\times n}$. A^T, $\mathrm{tr}(A)$ and $R(A)$ stand for the transpose, the trace and the column space of the matrix A, respectively. I_n represents the identity matrix of order n. For $A, B \in \mathbf{R}^{m\times n}$, an inner product in $\mathbf{R}^{m\times n}$ is defined by $(A, B) = \mathrm{tr}(B^T A)$, then $\mathbf{R}^{m\times n}$ is a Hilbert space. The matrix norm $\|\cdot\|$ induced by the inner product is the Frobenius norm. Given two matrices $A = [a_{ij}] \in \mathbf{R}^{m\times n}$ and $B \in \mathbf{R}^{p\times q}$, the Kronecker product of A and B is defined by $A \otimes B = [a_{ij}B] \in \mathbf{R}^{mp\times nq}$. Also, for an $m\times n$ matrix $A = [a_1, a_2, \cdots, a_n]$, where $a_i, i = 1, \cdots, n$, is the i-th column vector of A, the stretching function $\mathrm{vec}(A)$ is defined as $\mathrm{vec}(A) = [a_1^T, a_2^T, \cdots, a_n^T]^T$. Let A, B and X be some matrices with appropriate dimensions, then we have the following well-known identity [1].

$$\mathrm{vec}(AXB) = (B^T \otimes A)\mathrm{vec}(X).$$

Vibrating structures such as bridges, highways, buildings, automobiles and rotating machinery are often modeled using finite element techniques. These techniques generate structured matrix second-order differential equations

$$M_a\ddot{q}(t) + (D_a + G_a)\dot{q}(t) + K_a q(t) = 0. \tag{1}$$

The vector $q(t)$ represents the generalized coordinates of the system. M_a, K_a, D_a and G_a are, respectively, called the analytical mass, stiffness, damping, and gyroscopic matrices. In many practical applications, M_a is symmetric and positive definite ($M_a > 0$), K_a and D_a are real-valued symmetric, and G_a is always real-valued skew-symmetric (that is, $G_a^T = -G_a$). If a fundamental solution to (1) is represented by

$$q(t) = xe^{\lambda t},$$

then the scalar λ and the vector x must solve the quadratic eigenvalue problem (QEP)

$$(\lambda^2 M_a + \lambda(D_a + G_a) + K_a)x = 0. \tag{2}$$

Complex numbers λ and nonzero complex vectors x for which this relation holds are, respectively, the eigenvalues and eigenvectors of the system. It is known that the equation of (2) has $2n$ finite eigenvalues over the complex field, provided that the leading matrix coefficient M_a is nonsingular. Note that the signification of the system (1) usually can be interpreted via the eigenvalues and eigenvectors of Eq.(2). Because of this connection, a lot of efforts have been devoted to the QEP in the literature. Many applications, properties and numerical methods for the QEP are surveyed in the thesis by Tisseur and Meerbergen [2].

Accurate models are essential in analyzing systems under various excitations, boundary conditions and parameter changes. Analytical models, obtained by finite element techniques, inevitably deviate from the true model due to uncertainties in geometry, boundary conditions, discretization error, modeling error of joints, variation of material properties, ignorance of nonlinear effect, and other simplifications, etc. Consequently, eigenvalues and eigenvectors that are extracted

Yongxin Yuan: School of Mathematics and Physics, Jiangsu University of Science and Technology, Zhenjiang 212003, PR China. e-mail: yuanyx_703@163.com

from the test results do not agree with the predicted values from the analytical model. Model updating is the process of using the test results to correct the model so that it agrees, either completely or approximately, with the experimental data.

In the past decades, various techniques for updating mass and stiffness matrices for undamped systems (i.e., $D_a = 0, G_a = 0$) using measured response data have been discussed by Baruch [3], Baruch and Bar-Itzhack [4], Berman [5], Berman and Nagy [6], Wei [7, 8, 9], Yang et al. [10], Yang and Chen [11], and Yuan [12]. For an account of the earlier methods, see the authoritative book by Friswell and Mottershead [13], an integral introduction of the basic theory of finite element model updating is given. For damped structured systems, the theory and computation have been considered by Friswell et al. [14], Pilkey [15], Kuo et al. [16], Chu et al. [17] and Yuan [18]. It is well known that the damped gyroscopic systems are another important class of nonproportionally damped systems. They correspond to spinning structures where the Coriolis inertia forces are taken into account. Examples of such systems include helicopter rotor blades and spin-stabilized satellites with flexible elastic appendages such as solar panels or antennas. The numerical methods for quadratic eigenvalue problems of gyroscopic systems can see [2, 19-24]. In view of in analytical model (1) for structure dynamics, the effect of damping and Coriolis forces on structural dynamic systems is not well understood because it is purely dynamics property that can not be measured statically. Therefore, the correction of damped gyroscopic systems is very important. However, we observe that the iterative methods for model updating have received little attention in these years. In this paper we will develop an iterative method for the finite element model updating of damped gyroscopic systems which can incorporate the measured model data into the finite element model to produce an adjusted finite element model on the damping, gyroscopic and stiffness matrices that closely match the experimental modal data. The problem of updating damping, gyroscopic and stiffness matrices simultaneously can be mathematically formulated as follows.

Problem I. Let $\Lambda = \operatorname{diag}\{\lambda_1, \cdots, \lambda_p\} \in \mathbf{C}^{p\times p}$ and $X = [x_1, \cdots, x_p] \in \mathbf{C}^{n\times p}$ be the measured eigenvalue and eigenvector matrices, where $p < n$ and both Λ and X are closed under complex conjugation in the sense that $\lambda_{2j} = \bar{\lambda}_{2j-1} \in \mathbf{C}$, $x_{2j} = \bar{x}_{2j-1} \in \mathbf{C}^n$ for $j = 1, \cdots, l$, and $\lambda_k \in \mathbf{R}$, $x_k \in \mathbf{R}^n$ for $k = 2l+1, \cdots, p$, find real-valued symmetric matrices D, K and a real-valued skew-symmetric matrix G such that

$$M_a X\Lambda^2 + (D+G)X\Lambda + KX = 0. \tag{3}$$

It is well known that D_a, G_a and K_a are good approximations of D, G and K. The strategy for obtaining an improved model is to find D, G and K that satisfy (3) and deviate as little as possible from D_a, G_a and K_a. Thus, we should further solve the following optimal approximation problem.

Problem II. Let $\mathbf{S_E}$ be the solution set of Problem I. Find $(\hat{D}, \hat{G}, \hat{K}) \in \mathbf{S_E}$ such that

$$\begin{aligned}&\|\hat{D} - D_a\|^2 + \|\hat{G} - G_a\|^2 + \|\hat{K} - K_a\|^2 = \\ &\min_{(D,G,K)\in\mathbf{S_E}}(\|D - D_a\|^2 + \|G - G_a\|^2 + \|K - K_a\|^2).\end{aligned} \tag{4}$$

The paper is organized as follows. In Section 2, an efficient iterative method is presented to solve Problem I and Problem II. Then several properties of Algorithm 1 are proved. By using the proposed iterative method, a solution of Problem I can be obtained within finite iteration steps in the absence of roundoff errors, and the minimum Frobenius norm solution of Problem I can be obtained by choosing a special kind of initial matrices. In addition, the optimal approximation solution of Problem II is provided by finding the minimum Frobenius norm solution of a new matrix equation. In Section 3, a numerical example is used to test the effectiveness of the proposed algorithm.

II. The solution of Problem I and Problem II

Let $\alpha_i = \mathrm{Re}(\lambda_i)$ (the real part of the complex number λ_i), $\beta_i = \mathrm{Im}(\lambda_i)$ (the imaginary part of the complex number λ_i), $y_i = \mathrm{Re}(x_i)$, $z_i = \mathrm{Im}(x_i)$ for $i = 1, 3, \cdots, 2l-1$, and

$$\tilde{\Lambda} = \operatorname{diag}\left\{ \begin{bmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{bmatrix}, \cdots, \begin{bmatrix} \alpha_{2l-1} & \beta_{2l-1} \\ -\beta_{2l-1} & \alpha_{2l-1} \end{bmatrix}, \lambda_{2l+1}, \cdots, \lambda_p \right\} \in \mathbf{R}^{p\times p}, \tag{5}$$

$$\tilde{X} = [y_1, z_1, \cdots, y_{2l-1}, z_{2l-1}, x_{2l+1}, \cdots, x_p] \in \mathbf{R}^{n\times p}. \tag{6}$$

Then, the equation of (3) can be equivalently written as

$$D\tilde{X}\tilde{\Lambda} + G\tilde{X}\tilde{\Lambda} + K\tilde{X} = F, \tag{7}$$
$$\text{s. t. } G \in \mathbf{SSR}^{n\times n},\ D, K \in \mathbf{SR}^{n\times n},$$

where $F = -M_a\tilde{X}\tilde{\Lambda}^2$.

Now, we can describe an iterative algorithm for solving Problem I as follows.

Algorithm 1

S 1. Input matrices $\tilde{X} \in \mathbf{R}^{n\times p}$, $\tilde{\Lambda} \in \mathbf{R}^{p\times p}$ and $M_a \in \mathbf{SR}^{n\times n}$, and choose arbitrary $n\times n$ symmetric matrices D_1, K_1 and a skew-symmetric matrix G_1.

S 2. Calculate
$R_1 = F - D_1\tilde{X}\tilde{\Lambda} - G_1\tilde{X}\tilde{\Lambda} - K_1\tilde{X}$;
$P_1 = \frac{1}{2}(R_1\tilde{\Lambda}^T\tilde{X}^T + \tilde{X}\tilde{\Lambda}R_1^T)$;
$Q_1 = \frac{1}{2}(R_1\tilde{\Lambda}^T\tilde{X}^T - \tilde{X}\tilde{\Lambda}R_1^T)$;
$W_1 = \frac{1}{2}(R_1\tilde{X}^T + \tilde{X}R_1^T)$;
$s := 1$.

S 3. If $R_s = 0$, then stop and (D_s, G_s, K_s) is a solution to the equation of (7), that is, a solution of Problem I; elseif $R_s \neq 0$ but $P_s = 0$, $Q_s = 0$ and $W_s = 0$, then stop and the equation of (7) has no solution; else $s := s + 1$.

International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:4, No:7, 2010

S 4. Calculate
$D_s = D_{s-1} + \frac{\|R_{s-1}\|^2}{\|P_{s-1}\|^2+\|Q_{s-1}\|^2+\|W_{s-1}\|^2} P_{s-1};$
$G_s = G_{s-1} + \frac{\|R_{s-1}\|^2}{\|P_{s-1}\|^2+\|Q_{s-1}\|^2+\|W_{s-1}\|^2} Q_{s-1};$
$K_s = K_{s-1} + \frac{\|R_{s-1}\|^2}{\|P_{s-1}\|^2+\|Q_{s-1}\|^2+\|W_{s-1}\|^2} W_{s-1};$
$R_s = F - D_s\tilde{X}\tilde{\Lambda} - G_s\tilde{X}\tilde{\Lambda} - K_s\tilde{X}$
$= R_{s-1} - \frac{\|R_{s-1}\|^2}{\|P_{s-1}\|^2+\|Q_{s-1}\|^2+\|W_{s-1}\|^2}$
$(P_{s-1}\tilde{X}\tilde{\Lambda} + Q_{s-1}\tilde{X}\tilde{\Lambda} + W_{s-1}\tilde{X});$
$P_s = \frac{1}{2}(R_s\tilde{\Lambda}^T\tilde{X}^T + \tilde{X}\tilde{\Lambda}R_s^T) + \frac{\|R_s\|^2}{\|R_{s-1}\|^2} P_{s-1};$
$Q_s = \frac{1}{2}(R_s\tilde{\Lambda}^T\tilde{X}^T - \tilde{X}\tilde{\Lambda}R_s^T) + \frac{\|R_s\|^2}{\|R_{s-1}\|^2} Q_{s-1};$
$W_s = \frac{1}{2}(R_s\tilde{X}^T + \tilde{X}R_s^T) + \frac{\|R_s\|^2}{\|R_{s-1}\|^2} W_{s-1}.$

S 5. Go to S 3.

From Algorithm 1, we can easily see that $Q_s, G_s \in \mathbf{SSR}^{n\times n}$ and $P_s, D_s, W_s, K_s \in \mathbf{SR}^{n\times n}$ for $s = 1, 2, \cdots$.

Definition 1 Assume that $Y, Z \in \mathbf{R}^{m\times n}$. The matrices Y, Z are called orthogonal each other if $\mathrm{tr}(Y^T Z) = 0$.

About Algorithm 1, we present the following basic properties.

Lemma 1: The sequences $\{R_i\}, \{P_i\}, \{Q_i\}$ and $\{W_i\}$ generated by Algorithm 1 satisfy

$$\begin{aligned} &\mathrm{tr}(R_j^T R_i) = 0, \quad and \quad \mathrm{tr}(P_j^T P_i) + \mathrm{tr}(Q_j^T Q_i) \\ &+\mathrm{tr}(W_j^T W_i) = 0 \quad for \quad i, j = 1, 2, \cdots, s, \ i \neq j. \end{aligned} \tag{8}$$

Proof. Since $\mathrm{tr}(R_j^T R_i) = \mathrm{tr}(R_i^T R_j)$, $\mathrm{tr}(P_j^T P_i) = \mathrm{tr}(P_i^T P_j)$, $\mathrm{tr}(Q_j^T Q_i) = \mathrm{tr}(Q_i^T Q_j)$ and $\mathrm{tr}(W_j^T W_i) = \mathrm{tr}(W_i^T W_j)$, then we only need to show that

$$\begin{aligned} &\mathrm{tr}(R_j^T R_i) = 0, \quad \text{and} \quad \mathrm{tr}(P_j^T P_i) + \mathrm{tr}(Q_j^T Q_i) \\ &+\mathrm{tr}(W_j^T W_i) = 0 \quad \text{for} \quad 1 \leq i < j \leq s. \end{aligned}$$

We use the mathematical induction to prove this conclusion, and we do it in two steps.
We first show that

$$\begin{aligned} &\mathrm{tr}(R_{i+1}^T R_i) = 0, \quad \text{and} \quad \mathrm{tr}(P_{i+1}^T P_i) + \mathrm{tr}(Q_{i+1}^T Q_i) \\ &+\mathrm{tr}(W_{i+1}^T W_i) = 0 \quad \text{for} \quad i = 1, 2, \cdots, s. \end{aligned} \tag{9}$$

For $i = 1$, by Algorithm 1 and noting that $P_1, W_1 \in \mathbf{SR}^{n\times n}$ and $Q_1 \in \mathbf{SSR}^{n\times n}$, we have

$$\begin{aligned} & \mathrm{tr}(R_2^T R_1) \\ = \ & \mathrm{tr}((R_1 - \delta_1(P_1\tilde{X}\tilde{\Lambda} + Q_1\tilde{X}\tilde{\Lambda} + W_1\tilde{X}))^T R_1) \\ = \ & \mathrm{tr}(R_1^T R_1) - \delta_1 \mathrm{tr}(\tilde{\Lambda}^T\tilde{X}^T P_1^T R_1 \\ + \ & \tilde{\Lambda}^T\tilde{X}^T Q_1^T R_1 + \tilde{X}^T W_1^T R_1) \\ = \ & \|R_1\|^2 - \frac{1}{2}\delta_1 \mathrm{tr}(\tilde{\Lambda}^T\tilde{X}^T P_1^T R_1 + R_1^T P_1\tilde{X}\tilde{\Lambda} \\ + \ & \tilde{\Lambda}^T\tilde{X}^T Q_1^T R_1 + R_1^T Q_1\tilde{X}\tilde{\Lambda} + \tilde{X}^T W_1^T R_1 + R_1^T W_1\tilde{X}) \\ = \ & \|R_1\|^2 - \frac{1}{2}\delta_1 \mathrm{tr}(P_1^T R_1\tilde{\Lambda}^T\tilde{X}^T + P_1^T\tilde{X}\tilde{\Lambda}R_1^T \\ + \ & Q_1^T R_1\tilde{\Lambda}^T\tilde{X}^T - Q_1^T\tilde{X}\tilde{\Lambda}R_1^T + W_1^T R_1\tilde{X}^T + W_1^T\tilde{X}R_1^T) \\ = \ & \|R_1\|^2 - \delta_1 \mathrm{tr}(P_1^T P_1 + Q_1^T Q_1 + W_1^T W_1) \\ = \ & 0, \end{aligned}$$

where $\delta_1 = \frac{\|R_1\|^2}{\|P_1\|^2+\|Q_1\|^2+\|W_1\|^2}$.
Applying the proved result $\mathrm{tr}(R_2^T R_1) = 0$, we get

$$\begin{aligned} & \mathrm{tr}(P_2^T P_1) + \mathrm{tr}(Q_2^T Q_1) + \mathrm{tr}(W_2^T W_1) \\ = \ & \frac{1}{2}\mathrm{tr}((R_2\tilde{\Lambda}^T\tilde{X}^T + \tilde{X}\tilde{\Lambda}R_2^T)P_1) + \frac{\|R_2\|^2}{\|R_1\|^2}\|P_1\|^2 \\ + \ & \frac{1}{2}\mathrm{tr}((R_2\tilde{\Lambda}^T\tilde{X}^T - \tilde{X}\tilde{\Lambda}R_2^T)^T Q_1) + \frac{\|R_2\|^2}{\|R_1\|^2}\|Q_1\|^2 \\ + \ & \frac{1}{2}\mathrm{tr}((R_2\tilde{X}^T + \tilde{X}R_2^T)W_1) + \frac{\|R_2\|^2}{\|R_1\|^2}\|W_1\|^2 \\ = \ & \frac{1}{2}\mathrm{tr}(R_2(P_1\tilde{X}\tilde{\Lambda} + Q_1\tilde{X}\tilde{\Lambda} + W_1\tilde{X})^T \\ + \ & (P_1\tilde{X}\tilde{\Lambda} + Q_1\tilde{X}\tilde{\Lambda} + W_1\tilde{X})R_2^T) \\ + \ & \frac{\|R_2\|^2}{\|R_1\|^2}\|P_1\|^2 + \frac{\|R_2\|^2}{\|R_1\|^2}\|Q_1\|^2 + \frac{\|R_2\|^2}{\|R_1\|^2}\|W_1\|^2 \\ = \ & \frac{1}{2}\frac{\|P_1\|^2 + \|Q_1\|^2 + \|W_1\|^2}{\|R_1\|^2} \\ & \mathrm{tr}((R_2(R_1 - R_2)^T + (R_1 - R_2)R_2^T) \\ + \ & \frac{\|R_2\|^2}{\|R_1\|^2}\|P_1\|^2 + \frac{\|R_2\|^2}{\|R_1\|^2}\|Q_1\|^2 + \frac{\|R_2\|^2}{\|R_1\|^2}\|W_1\|^2 \\ = \ & 0. \end{aligned}$$

Suppose that (9) holds for $i = t - 1$. For $i = t$, we have

$$\begin{aligned} & \mathrm{tr}(R_{t+1}^T R_t) \\ = \ & \mathrm{tr}((R_t - \delta_t(P_t\tilde{X}\tilde{\Lambda} + Q_t\tilde{X}\tilde{\Lambda} + W_t\tilde{X}))^T R_t) \\ = \ & \mathrm{tr}(R_t^T R_t) - \delta_t \mathrm{tr}(\tilde{\Lambda}^T\tilde{X}^T P_t^T R_t \\ + \ & \tilde{\Lambda}^T\tilde{X}^T Q_t^T R_t + \tilde{X}^T W_t^T R_t) \\ = \ & \|R_t\|^2 - \frac{1}{2}\delta_t \mathrm{tr}(\tilde{\Lambda}^T\tilde{X}^T P_t^T R_t + R_t^T P_t\tilde{X}\tilde{\Lambda} \\ + \ & \tilde{\Lambda}^T\tilde{X}^T Q_t^T R_t + R_t^T Q_t\tilde{X}\tilde{\Lambda} + \tilde{X}^T W_t^T R_t + R_t^T W_t\tilde{X}) \\ = \ & \|R_t\|^2 - \frac{1}{2}\delta_t \mathrm{tr}(P_t^T R_t\tilde{\Lambda}^T\tilde{X}^T + P_t^T\tilde{X}\tilde{\Lambda}R_t^T \\ + \ & Q_t^T R_t\tilde{\Lambda}^T\tilde{X}^T - Q_t^T\tilde{X}\tilde{\Lambda}R_t^T + W_t^T R_t\tilde{X}^T + W_t^T\tilde{X}R_t^T) \\ = \ & \|R_t\|^2 - \delta_t \mathrm{tr}(P_t^T(P_t - \frac{\|R_t\|^2}{\|R_{t-1}\|^2}P_{t-1}) \\ + \ & Q_t^T(Q_t - \frac{\|R_t\|^2}{\|R_{t-1}\|^2}Q_{t-1}) \\ + \ & W_t^T(W_t - \frac{\|R_t\|^2}{\|R_{t-1}\|^2}W_{t-1})) \\ = \ & \|R_t\|^2 - \delta_t \mathrm{tr}(P_t^T P_t + Q_t^T Q_t + W_t^T W_t) \\ = \ & 0, \end{aligned}$$

where $\delta_t = \frac{\|R_t\|^2}{\|P_t\|^2+\|Q_t\|^2+\|W_t\|^2}$.

$$\begin{aligned} & \mathrm{tr}(P_{t+1}^T P_t) + \mathrm{tr}(Q_{t+1}^T Q_t + \mathrm{tr}(W_{t+1}^T W_t) \\ = \ & \frac{1}{2}\mathrm{tr}((R_{t+1}\tilde{\Lambda}^T\tilde{X}^T + \tilde{X}\tilde{\Lambda}R_{t+1}^T)P_t) + \frac{\|R_{t+1}\|^2}{\|R_t\|^2}\|P_t\|^2 \end{aligned}$$

International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:4, No:7, 2010

$$\begin{aligned}
&+ \quad \frac{1}{2}\mathrm{tr}((R_{t+1}\tilde{\Lambda}^T\tilde{X}^T - \tilde{X}\tilde{\Lambda}R_{t+1}^T)^TQ_t) + \frac{\|R_{t+1}\|^2}{\|R_t\|^2}\|Q_t\|^2 \\
&+ \quad \frac{1}{2}\mathrm{tr}((R_{t+1}\tilde{X}^T + \tilde{X}R_{t+1}^T)W_t) + \frac{\|R_{t+1}\|^2}{\|R_t\|^2}\|W_t\|^2 \\
&= \quad \frac{1}{2}\mathrm{tr}(R_{t+1}(P_t\tilde{X}\tilde{\Lambda} + Q_t\tilde{X}\tilde{\Lambda} + W_t\tilde{X})^T \\
&+ \quad (P_t\tilde{X}\tilde{\Lambda} + Q_t\tilde{X}\tilde{\Lambda} + W_t\tilde{X})R_{t+1}^T) \\
&+ \quad \frac{\|R_{t+1}\|^2}{\|R_t\|^2}\|P_t\|^2 + \frac{\|R_{t+1}\|^2}{\|R_t\|^2}\|Q_t\|^2 + \frac{\|R_{t+1}\|^2}{\|R_t\|^2}\|W_t\|^2 \\
&= \quad \frac{1}{2}\frac{\|P_t\|^2 + \|Q_t\|^2 + \|W_t\|^2}{\|R_t\|^2} \\
&\quad\quad \mathrm{tr}((R_{t+1}(R_t - R_{t+1})^T + (R_t - R_{t+1})R_{t+1}^T) \\
&+ \quad \frac{\|R_{t+1}\|^2}{\|R_t\|^2}\|P_t\|^2 + \frac{\|R_{t+1}\|^2}{\|R_t\|^2}\|Q_t\|^2 + \frac{\|R_{t+1}\|^2}{\|R_t\|^2}\|W_t\|^2 \\
&= \quad 0.
\end{aligned}$$

Therefore, (9) holds for $i = t$. By the principle of induction, we know (9) holds for all i.

Next, assume that $\mathrm{tr}(R_{i+d}^TR_i) = 0$, and $\mathrm{tr}(P_{i+d}^TP_i) + \mathrm{tr}(Q_{i+d}^TQ_i) + \mathrm{tr}(W_{i+d}^TW_i) = 0$ for $1 \leq i \leq s$ and $1 < d < s$. We will prove
$\mathrm{tr}(R_{i+d+1}^TR_i) = 0$, and $\mathrm{tr}(P_{i+d+1}^TP_i) + \mathrm{tr}(Q_{i+d+1}^TQ_i) + \mathrm{tr}(W_{i+d+1}^TW_i) = 0$.

$$\begin{aligned}
&\quad\quad \mathrm{tr}(R_{i+d+1}^TR_i) \\
&= \quad \mathrm{tr}((R_{i+d} - \delta_{i+d}(P_{i+d}\tilde{X}\tilde{\Lambda} \\
&+ \quad Q_{i+d}\tilde{X}\tilde{\Lambda} + W_{i+d}\tilde{X}))^TR_i) \\
&= \quad -\delta_{i+d}\mathrm{tr}(\tilde{\Lambda}^T\tilde{X}^TP_{i+d}^TR_i \\
&+ \quad \tilde{\Lambda}^T\tilde{X}^TQ_{i+d}^TR_i + \tilde{X}^TW_{i+d}^TR_i) \\
&= \quad -\frac{1}{2}\delta_{i+d}\mathrm{tr}(\tilde{\Lambda}^T\tilde{X}^TP_{i+d}^TR_i + R_i^TP_{i+d}\tilde{X}\tilde{\Lambda} \\
&+ \quad \tilde{\Lambda}^T\tilde{X}^TQ_{i+d}^TR_i + R_i^TQ_{i+d}\tilde{X}\tilde{\Lambda} \\
&+ \quad \tilde{X}^TW_{i+d}^TR_i + R_i^TW_{i+d}\tilde{X}) \\
&= \quad -\frac{1}{2}\delta_{i+d}\mathrm{tr}(P_{i+d}^TR_i\tilde{\Lambda}^T\tilde{X}^T + P_{i+d}^T\tilde{X}\tilde{\Lambda}R_i^T \\
&+ \quad Q_{i+d}^TR_i\tilde{\Lambda}^T\tilde{X}^T - Q_{i+d}^T\tilde{X}\tilde{\Lambda}R_i^T \\
&+ \quad W_{i+d}^TR_i\tilde{X}^T + W_{i+d}^T\tilde{X}R_i^T) \\
&= \quad -\delta_{i+d}\mathrm{tr}(P_{i+d}^T(P_i - \frac{\|R_i\|^2}{\|R_{i-1}\|^2}P_{i-1}) \\
&+ \quad Q_{i+d}^T(Q_i - \frac{\|R_i\|^2}{\|R_{i-1}\|^2}Q_{i-1}) \\
&+ \quad W_{i+d}^T(W_i - \frac{\|R_i\|^2}{\|R_{i-1}\|^2}W_{i-1})) \\
&= \quad 0,
\end{aligned}$$

where $\delta_{i+d} = \frac{\|R_{i+d}\|^2}{\|P_{i+d}\|^2 + \|Q_{i+d}\|^2 + \|W_{i+d}\|^2}$.
From the above results, we have $\mathrm{tr}(R_{i+d+1}^TR_i) = 0$ and $\mathrm{tr}(R_{i+d+1}^TR_{i+1}) = 0$. Hence we can get

$$\begin{aligned}
&\quad\quad \mathrm{tr}(P_{i+d+1}^TP_i) + \mathrm{tr}(Q_{i+d+1}^TQ_i) + \mathrm{tr}(W_{i+d+1}^TW_i) \\
&= \quad \frac{1}{2}\mathrm{tr}((R_{i+d+1}\tilde{\Lambda}^T\tilde{X}^T + \tilde{X}\tilde{\Lambda}R_{i+d+1}^T)P_i) \\
&+ \quad \frac{1}{2}\mathrm{tr}((R_{i+d+1}\tilde{\Lambda}^T\tilde{X}^T - \tilde{X}\tilde{\Lambda}R_{i+d+1}^T)^TQ_i) \\
&+ \quad \frac{1}{2}\mathrm{tr}((R_{i+d+1}\tilde{X}^T + \tilde{X}R_{i+d+1}^T)W_i) \\
&= \quad \frac{1}{2}\mathrm{tr}(R_{i+d+1}(P_i\tilde{X}\tilde{\Lambda} + Q_i\tilde{X}\tilde{\Lambda} + W_i\tilde{X})^T \\
&+ \quad (P_i\tilde{X}\tilde{\Lambda} + Q_i\tilde{X}\tilde{\Lambda} + W_i\tilde{X})R_{i+d+1}^T) \\
&= \quad \frac{1}{2}\xi\ \mathrm{tr}((R_{i+d+1}(R_i - R_{i+1})^T + (R_i - R_{i+1})R_{i+d+1}^T) \\
&= \quad 0,
\end{aligned}$$

where $\xi = \frac{\|P_i\|^2 + \|Q_i\|^2 + \|W_i\|^2}{\|R_i\|^2}$.
Thus the conclusion (8) holds by the principle of induction. The proof is completed.

Lemma 2: Let Problem I be consistent, and (D^*, G^*, K^*) be an arbitrary solution of Problem I. Then, for any initial matrix triplet (D_1, G_1, K_1) with $G_1 \in \mathbf{SSR}^{n\times n}$ and $D_1, K_1 \in \mathbf{SR}^{n\times n}$, we have

$$\begin{aligned}
&\mathrm{tr}((D^* - D_i)^TP_i) + \mathrm{tr}((G^* - G_i)^TQ_i) \\
&+\mathrm{tr}((K^* - K_i)^TW_i) = \|R_i\|^2 \quad for \quad i = 1, 2, \cdots,
\end{aligned} \tag{10}$$

where the sequences $\{D_i\}, \{P_i\}, \{G_i\}, \{Q_i\}, \{K_i\}, \{W_i\}$ and $\{R_i\}$ are generated by Algorithm 1.

Proof. We prove the conclusion by induction. For $i = 1$, we have

$$\begin{aligned}
&\quad\quad \mathrm{tr}((D^* - D_1)^TP_1) + \mathrm{tr}((G^* - G_1)^TQ_1) \\
&+ \quad \mathrm{tr}((K^* - K_1)^TW_1) \\
&= \quad \frac{1}{2}\mathrm{tr}((D^* - D_1)^T(R_1\tilde{\Lambda}^T\tilde{X}^T + \tilde{X}\tilde{\Lambda}R_1^T)) \\
&+ \quad \frac{1}{2}\mathrm{tr}((G^* - G_1)^T(R_1\tilde{\Lambda}^T\tilde{X}^T - \tilde{X}\tilde{\Lambda}R_1^T)) \\
&+ \quad \frac{1}{2}\mathrm{tr}((K^* - K_1)^T(R_1\tilde{X}^T + \tilde{X}R_1^T)) \\
&= \quad \frac{1}{2}\mathrm{tr}(FR_1^T - D_1\tilde{X}\tilde{\Lambda}R_1^T - G_1\tilde{X}\tilde{\Lambda}R_1^T - K_1\tilde{X}R_1^T) \\
&+ \quad \frac{1}{2}\mathrm{tr}(F^TR_1 - \tilde{\Lambda}^T\tilde{X}^TD_1^TR_1 \\
&- \quad \tilde{\Lambda}^T\tilde{X}^TG_1^TR_1 - \tilde{X}^TK_1^TR_1) \\
&= \quad \frac{1}{2}\mathrm{tr}(R_1R_1^T) + \frac{1}{2}\mathrm{tr}(R_1^TR_1) \\
&= \quad \|R_1\|^2.
\end{aligned}$$

Now assume the conclusion (10) holds for $1 \leq i \leq t-1$. Then we can get

$$\begin{aligned}
&\quad\quad \mathrm{tr}((D^* - D_t)^TP_t) + \mathrm{tr}((G^* - G_t)^TQ_t) \\
&+ \quad \mathrm{tr}((K^* - K_t)^TW_t) \\
&= \quad \mathrm{tr}((D^* - D_{t-1}
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\|R_{t-1}\|^2}{\|P_{t-1}\|^2+\|Q_{t-1}\|^2+\|W_{t-1}\|^2}P_{t-1})^T P_t) \\
&+ \operatorname{tr}((G^*-G_{t-1} \\
&- \frac{\|R_{t-1}\|^2}{\|P_{t-1}\|^2+\|Q_{t-1}\|^2+\|W_{t-1}\|^2}Q_{t-1})^T Q_t) \\
&+ \operatorname{tr}((K^*-K_{t-1} \\
&- \frac{\|R_{t-1}\|^2}{\|P_{t-1}\|^2+\|W_{t-1}\|^2+\|W_{t-1}\|^2}W_{t-1})^T W_t) \\
&= \operatorname{tr}((D^*-D_{t-1})^T P_t)+\operatorname{tr}((G^*-G_{t-1})^T Q_t) \\
&+ \operatorname{tr}((K^*-K_{t-1})^T W_t) \\
&= \frac{1}{2}\operatorname{tr}((D^*-D_{t-1})^T(R_t\tilde{\Lambda}^T\tilde{X}^T+\tilde{X}\tilde{\Lambda}R_t^T)) \\
&+ \frac{\|R_t\|^2}{\|R_{t-1}\|^2}\operatorname{tr}((D^*-D_{t-1})^T P_{t-1}) \\
&+ \frac{1}{2}\operatorname{tr}((G^*-G_{t-1})^T(R_t\tilde{\Lambda}^T\tilde{X}^T-\tilde{X}\tilde{\Lambda}R_t^T)) \\
&+ \frac{\|R_t\|^2}{\|R_{t-1}\|^2}\operatorname{tr}((G^*-G_{t-1})^T Q_{t-1}) \\
&+ \frac{1}{2}\operatorname{tr}((K^*-K_{t-1})^T(R_t\tilde{X}^T+\tilde{X}R_t^T)) \\
&+ \frac{\|R_t\|^2}{\|R_{t-1}\|^2}\operatorname{tr}((K^*-K_{t-1})^T W_{t-1}) \\
&= \frac{1}{2}\operatorname{tr}((D^*-D_{t-1})^T(R_t\tilde{\Lambda}^T\tilde{X}^T+\tilde{X}\tilde{\Lambda}R_t^T)) \\
&+ \frac{1}{2}\operatorname{tr}((G^*-G_{t-1})^T(R_t\tilde{\Lambda}^T\tilde{X}^T-\tilde{X}\tilde{\Lambda}R_t^T)) \\
&+ \frac{1}{2}\operatorname{tr}((K^*-K_{t-1})^T(R_t\tilde{X}^T+\tilde{X}R_t^T))+\|R_t\|^2 \\
&= \frac{1}{2}\operatorname{tr}(F^T R_t-\tilde{\Lambda}^T\tilde{X}^T D_{t-1}^T R_t \\
&- \tilde{\Lambda}^T\tilde{X}^T G_{t-1}^T R_t-\tilde{X}^T K_{t-1}^T R_t) \\
&+ \frac{1}{2}\operatorname{tr}(F R_t^T-D_{t-1}\tilde{X}\tilde{\Lambda}R_t^T \\
&- G_{t-1}\tilde{X}\tilde{\Lambda}R_t^T-K_{t-1}\tilde{X}R_t^T)+\|R_t\|^2 \\
&= \frac{1}{2}\operatorname{tr}(R_{t-1}^T R_t)+\frac{1}{2}\operatorname{tr}(R_{t-1}R_t^T)+\|R_t\|^2 \\
&= \|R_t\|^2.
\end{aligned}
$$

Thus we complete the proof of Lemma 2 by the principle of induction.

From Lemma 2, we can easily see that if there exists a positive number l such that $P_l=0$, $Q_l=0$ and $W_l=0$ but $R_l\neq 0$, then the equation of (7) has no solution. Hence, the solvability of Eq.(7) can be determined automatically by Algorithm 1.

Theorem 1: Assume that Problem I is consistent. Then for any arbitrary initial matrix triplet (D_1,G_1,K_1) with $G_1\in \mathbf{SSR}^{n\times n}$ and $D_1,K_1\in \mathbf{SR}^{n\times n}$, a solution of Problem I can be obtained with finite iteration steps in the absence of roundoff errors.

Proof. Assume that $R_l\neq 0,\ l=1,2,\cdots,np$. From Lemma 2, we know $\|P_l\|^2+\|Q_l\|^2+\|W_l\|^2\neq 0$. Then we can calculate R_{np+1} and $(D_{np+1},G_{np+1},K_{np+1})$ by Algorithm 1. From Lemma 1, we have

$$\operatorname{tr}(R_{np+1}^T R_t)=0,\ t=1,2,\cdots,np,$$

and

$$\operatorname{tr}(R_j^T R_i)=0,\quad i,j=1,2,\cdots,np,\ i\neq j.$$

Therefore, $\{R_1,R_2,\cdots,R_{np}\}$ forms an orthogonal basis of the real-valued matrix space $\mathbf{R}^{n\times p}$, which implies that $R_{np+1}=0$, that is, $(D_{np+1},G_{np+1},K_{np+1})$ is a solution of Problem I.

Lemma 3: The equation of (7) has a solution (D,G,K) with $G\in\mathbf{SSR}^{n\times n}$ and $D,K\in\mathbf{SR}^{n\times n}$ if and only if the matrix equations

$$
\begin{aligned}
D\tilde{X}\tilde{\Lambda}+G\tilde{X}\tilde{\Lambda}+K\tilde{X}=F,\\
\tilde{\Lambda}^T\tilde{X}^T D-\tilde{\Lambda}^T\tilde{X}^T G+\tilde{X}^T K=F^T,
\end{aligned}
\tag{11}
$$

are consistent.

Proof. If the equation of (7) has a solution (D^*,G^*,K^*) with $G^*\in\mathbf{SSR}^{n\times n}$ and $D^*,K^*\in\mathbf{SR}^{n\times n}$, then $D^*\tilde{X}\tilde{\Lambda}+G^*\tilde{X}\tilde{\Lambda}+K^*\tilde{X}=F$, and $(D^*\tilde{X}\tilde{\Lambda}+G^*\tilde{X}\tilde{\Lambda}+K^*\tilde{X})^T=\tilde{\Lambda}^T\tilde{X}^T D^*-\tilde{\Lambda}^T\tilde{X}^T G^*+\tilde{X}^T K^*=F^T$. That is to say, (D^*,G^*,K^*) is a solution of (11).
Conversely, if the matrix equations of (11) has a solution, say, $D=U,\ G=V,\ K=Z$. Let $D^*=\frac{1}{2}(U+U^T),\ G^*=\frac{1}{2}(V-V^T),\ K^*=\frac{1}{2}(Z+Z^T)$, then G^* is a skew-symmetric matrix and D^*,K^* are symmetric matrices, and

$$
\begin{aligned}
&D^*\tilde{X}\tilde{\Lambda}+G^*\tilde{X}\tilde{\Lambda}+K^*\tilde{X}\\
&=\tfrac{1}{2}(U\tilde{X}\tilde{\Lambda}+V\tilde{X}\tilde{\Lambda}+Z\tilde{X})+\tfrac{1}{2}(U^T\tilde{X}\tilde{\Lambda}-V^T\tilde{X}\tilde{\Lambda}+Z^T\tilde{X})\\
&=\tfrac{1}{2}F+\tfrac{1}{2}(F^T)^T=F.
\end{aligned}
$$

Hence, (D^*,G^*,K^*) is a solution of (7).

The following lemma comes from [25].

Lemma 4: Suppose that the consistent system of linear equations $Ax=b$ has a solution $x\in R(A^T)$, then x is the unique minimum Frobenius norm solution of the system of linear equations.
Using the Kronecker product and the stretching function, we know that the equations of (11) are equivalent to

$$
\begin{bmatrix}
\tilde{\Lambda}^T\tilde{X}^T\otimes I_n & \tilde{\Lambda}^T\tilde{X}^T\otimes I_n & \tilde{X}^T\otimes I_n\\
I_n\otimes\tilde{\Lambda}^T\tilde{X}^T & I_n\otimes(-\tilde{\Lambda}^T\tilde{X}^T) & I_n\otimes\tilde{X}^T
\end{bmatrix}
\begin{bmatrix}\operatorname{vec}(D)\\ \operatorname{vec}(G)\\ \operatorname{vec}(K)\end{bmatrix}
=\begin{bmatrix}\operatorname{vec}(F)\\ \operatorname{vec}(F^T)\end{bmatrix}.
$$

Assume that $H \in \mathbf{R}^{n\times n}$ is an arbitrary matrix, then we have

$$\begin{aligned}
&\begin{bmatrix} \mathrm{vec}(H\tilde{\Lambda}^T\tilde{X}^T + \tilde{X}\tilde{\Lambda}H^T) \\ \mathrm{vec}(H\tilde{\Lambda}^T\tilde{X}^T - \tilde{X}\tilde{\Lambda}H^T) \\ \mathrm{vec}(H\tilde{X}^T + \tilde{X}H^T) \end{bmatrix} \\
= &\begin{bmatrix} \tilde{X}\tilde{\Lambda}\otimes I_n & I_n \otimes \tilde{X}\tilde{\Lambda} \\ \tilde{X}\tilde{\Lambda}\otimes I_n & I_n \otimes (-\tilde{X}\tilde{\Lambda}) \\ \tilde{X}\otimes I_n & I_n \otimes \tilde{X} \end{bmatrix} \begin{bmatrix} \mathrm{vec}(H) \\ \mathrm{vec}(H^T) \end{bmatrix} \\
= &\begin{bmatrix} \tilde{\Lambda}^T\tilde{X}^T\otimes I_n & \tilde{\Lambda}^T\tilde{X}^T\otimes I_n & \tilde{X}^T\otimes I_n \\ I_n\otimes\tilde{\Lambda}^T\tilde{X}^T & I_n\otimes(-\tilde{\Lambda}^T\tilde{X}^T) & I_n\otimes\tilde{X}^T \end{bmatrix}^T \\
&\begin{bmatrix} \mathrm{vec}(H) \\ \mathrm{vec}(H^T) \end{bmatrix} \in \\
R &\left(\begin{bmatrix} \tilde{\Lambda}^T\tilde{X}^T\otimes I_n & \tilde{\Lambda}^T\tilde{X}^T\otimes I_n & \tilde{X}^T\otimes I_n \\ I_n\otimes\tilde{\Lambda}^T\tilde{X}^T & I_n\otimes(-\tilde{\Lambda}^T\tilde{X}^T) & I_n\otimes\tilde{X}^T \end{bmatrix}^T \right).
\end{aligned}$$

It is obvious that if we choose

$$\begin{aligned}
&D_1 = H\tilde{\Lambda}^T\tilde{X}^T + \tilde{X}\tilde{\Lambda}H^T, \quad G_1 = H\tilde{\Lambda}^T\tilde{X}^T - \tilde{X}\tilde{\Lambda}H^T, \\
&K_1 = H\tilde{X}^T + \tilde{X}H^T,
\end{aligned} \tag{12}$$

then all D_s, G_s and K_s generated by Algorithm 1 satisfy

$$\begin{aligned}
&\begin{bmatrix} \mathrm{vec}(D_s) \\ \mathrm{vec}(G_s) \\ \mathrm{vec}(K_s) \end{bmatrix} \in \\
R &\left(\begin{bmatrix} \tilde{\Lambda}^T\tilde{X}^T\otimes I_n & \tilde{\Lambda}^T\tilde{X}^T\otimes I_n & \tilde{X}^T\otimes I_n \\ I_n\otimes\tilde{\Lambda}^T\tilde{X}^T & I_n\otimes(-\tilde{\Lambda}^T\tilde{X}^T) & I_n\otimes\tilde{X}^T \end{bmatrix}^T \right).
\end{aligned}$$

It follows from Lemma 4 that if we choose a initial matrix triplet by (12), where H is an arbitrary matrix, then a solution (D^*, G^*, K^*) obtained by Algorithm 1 is the minimum Frobenius norm solution of Problem I. In summary of above discussion, we have proved the following result.

Theorem 2: Suppose that Problem I is consistent. If we choose the initial matrices by (12), where H is an arbitrary matrix, or especially, $D_1 = 0$, $G_1 = 0$ and $K_1 = 0$, then we can obtain the unique minimum Frobenius norm solution of Problem I within finite iterative steps.

Now we show that the solution of Problem II can be derived by finding the minimum norm solution of a new matrix equation. Assume that Problem I is consistent. Obviously the solution set $\mathbf{S_E}$ of Problem I is nonempty, then for a given matrix triplet (D_a, G_a, K_a), we have

$$\begin{aligned}
& D\tilde{X}\tilde{\Lambda} + G\tilde{X}\tilde{\Lambda} + K\tilde{X} = -M_a\tilde{X}\tilde{\Lambda}^2 \\
\Leftrightarrow \quad & (D - D_a)\tilde{X}\tilde{\Lambda} + (G - G_a)\tilde{X}\tilde{\Lambda} + (K - K_a)\tilde{X} \\
& = -M_a\tilde{X}\tilde{\Lambda}^2 - D_a\tilde{X}\tilde{\Lambda} - G_a\tilde{X}\tilde{\Lambda} - K_a\tilde{X}.
\end{aligned}$$

Let

$$\begin{aligned}
&\tilde{D} = D - D_a, \quad \tilde{G} = G - G_a, \quad \tilde{K} = K - K_a, \\
&\tilde{F} = -M_a\tilde{X}\tilde{\Lambda}^2 - D_a\tilde{X}\tilde{\Lambda} - G_a\tilde{X}\tilde{\Lambda} - K_a\tilde{X},
\end{aligned}$$

then the matrix approximation Problem II is equivalent to finding the minimum Frobenius norm solution of the matrix equation

$$\tilde{D}\tilde{X}\tilde{\Lambda} + \tilde{G}\tilde{X}\tilde{\Lambda} + \tilde{K}\tilde{X} = \tilde{F}, \tag{13}$$

$$\text{s. t.} \quad \tilde{G} \in \mathbf{SSR}^{n\times n}, \ \tilde{D}, \tilde{K} \in \mathbf{SR}^{n\times n}.$$

Applying Algorithm 1, and taking the initial matrices by (12), where H is an arbitrary matrix, or especially, $\tilde{D}_1 = 0$, $\tilde{G}_1 = 0$ and $\tilde{K}_1 = 0$, we can obtain the minimum Frobenius norm solution $(\tilde{D}^*, \tilde{G}^*, \tilde{K}^*)$ of (13). Once $(\tilde{D}^*, \tilde{G}^*, \tilde{K}^*)$ is obtained, the solution of the matrix optimal approximation Problem II can be computed. In this case, can be expressed as

$$\hat{D} = D_a + \tilde{D}^*, \quad \hat{G} = G_a + \tilde{G}^*, \quad \hat{K} = K_a + \tilde{K}^*. \tag{14}$$

III. A NUMERICAL EXAMPLE

In this section, we will give a numerical example to illustrate our results. All the tests are performed using MATLAB 6.5. Because of the influence of the error of calculation, the iteration will not stop within finite steps. Hence, we regard (D_s, G_s, K_s) as a solution of the considered problem if the corresponding residue satisfies $\|R_s\| \le 1.0e-010$.

Example 1. Consider a 7-DOF system modelled analytically with mass, gyroscopic, and stiffness matrices given by

$$M_a = 0.03\times \begin{bmatrix} 52 & 22 & 18 & -13 & 0 & 0 & 0 \\ 22 & 12 & 13 & -9 & 0 & 0 & 0 \\ 18 & 13 & 104 & 0 & 18 & -13 & 0 \\ -13 & -9 & 0 & 24 & 13 & -9 & 0 \\ 0 & 0 & 18 & 13 & 104 & 0 & 18 \\ 0 & 0 & -13 & -9 & 0 & 24 & 13 \\ 0 & 0 & 0 & 0 & 18 & 13 & 104 \end{bmatrix},$$

$$D_a = \begin{bmatrix} 25.8258 & 15.9831 & -6.8154 & 13.3326 & 0.1395 & -0.1053 & 0 \\ 15.9831 & 59.1804 & -12.6864 & 13.8510 & 0.1053 & -0.0792 & 0 \\ -6.8154 & -12.6864 & 59.6436 & 0 & -6.2304 & 12.9024 & 0.1395 \\ 13.3326 & 13.8510 & 0 & 117.1368 & -12.9024 & 14.0112 & 0.1053 \\ 0.1395 & 0.1053 & -6.2304 & -12.9024 & 59.6436 & 0 & -6.2304 \\ -0.1053 & -0.0792 & 12.9024 & 14.0112 & 0 & 117.1368 & -12.9024 \\ 0 & 0 & 0.1395 & 0.1053 & -6.2304 & -12.902 & 59.6436 \end{bmatrix},$$

$$G_a = \begin{bmatrix} 0 & 15.9831 & -6.8154 & 13.3326 & 0.1395 & -0.1053 & 0 \\ -15.9831 & 0 & -12.6864 & 13.8510 & 0.1053 & -0.0792 & 0 \\ 6.8154 & 12.6864 & 0 & 0 & -6.2304 & 12.9024 & 0.1395 \\ -13.3326 & -13.8510 & 0 & 0 & -12.9024 & 14.0112 & 0.1053 \\ -0.1395 & -0.1053 & 6.2304 & 12.9024 & 0 & 0 & -6.2304 \\ 0.1053 & 0.0792 & -12.9024 & -14.0112 & 0 & 0 & -12.9024 \\ 0 & 0 & -0.1395 & -0.1053 & 6.2304 & 12.9024 & 0 \end{bmatrix},$$

and

$$K_a = 600 \times \begin{bmatrix} 2 & 3 & -2 & 3 & 0 & 0 & 0 \\ 3 & 6 & -3 & 3 & 0 & 0 & 0 \\ -2 & -3 & 4 & 0 & -2 & 3 & 0 \\ 3 & 3 & 0 & 12 & -3 & 3 & 0 \\ 0 & 0 & -2 & -3 & 4 & 0 & -2 \\ 0 & 0 & 3 & 3 & 0 & 12 & -3 \\ 0 & 0 & 0 & 0 & -2 & -3 & 4 \end{bmatrix}.$$

The measured eigenvalue and eigenvector matrices Λ and X are given by

$$\begin{aligned} \Lambda &= \operatorname{diag}\{-71.087 + 55.495i, -71.087 - 55.495i, \\ &\quad -19.507 + 39.177i, -19.507 - 39.177i\} \end{aligned}$$

and

$$X = \begin{bmatrix} 0.1714 + 0.3902i & 0.1714 - 0.3902i & 0.4628 + 0.2187i & 0.4628 - 0.2187i \\ -0.3786 - 0.3146i & -0.3786 + 0.3146i & -0.5110 + 0.0243i & -0.5110 - 0.0243i \\ -0.0063 - 0.1350i & -0.0063 + 0.1350i & -0.3022 - 0.2606i & -0.3022 + 0.2606i \\ 0.0855 + 0.5370i & 0.0855 - 0.5370i & 0.1293 - 0.0539i & 0.1293 + 0.0539i \\ -0.0657 - 0.0451i & -0.0657 + 0.0451i & 0.4070 + 0.1619i & 0.4070 - 0.1619i \\ -0.1868 - 0.4171i & -0.1868 + 0.4171i & -0.0065 + 0.0671i & -0.0065 - 0.0671i \\ 0.0738 + 0.0431i & 0.0738 - 0.0431i & -0.3158 - 0.0366i & -0.3158 + 0.0366i \end{bmatrix},$$

Test 1. Choosing initial iterative matrices $D_1 = 0,\ G_1 = 0$ and $K_1 = 0$. By Algorithm 1, after 13 iteration steps, we get the minimum Frobenius norm solution of Problem I as follows.

$$D_{14} = \begin{bmatrix} -1.0350 & -7.3983 & 16.0777 & 10.4515 & -32.2720 & -10.5444 & 23.0742 \\ -7.3983 & 13.4950 & 28.2103 & -23.8379 & -10.7902 & 25.4311 & 4.9442 \\ 16.0777 & 28.2103 & -47.6215 & -7.9163 & 35.6826 & 0.3282 & -11.3227 \\ 10.4515 & -23.8379 & -7.9163 & -21.6123 & 31.6409 & 3.7644 & -21.6979 \\ -32.2720 & -10.7902 & 35.6826 & 31.6409 & 57.1259 & 26.1558 & -85.4500 \\ -10.5444 & 25.4311 & 0.3282 & 3.7644 & 26.1558 & 25.3634 & -29.2406 \\ 23.0742 & 4.9442 & -11.3227 & -21.6979 & -85.4500 & -29.2406 & 99.6122 \end{bmatrix},$$

$$G_{14} = \begin{bmatrix} -0.0000 & -6.7377 & -8.0588 & 1.2158 & 38.6231 & 7.8206 & -36.4310 \\ 6.7377 & 0.0000 & -30.9780 & 34.0188 & 19.7042 & -21.6911 & -15.7616 \\ 8.0588 & 30.9780 & -0.0000 & -18.5950 & -61.9164 & -3.8357 & 70.7559 \\ -1.2158 & -34.0188 & 18.5950 & -0.0000 & -67.2390 & -29.2425 & 61.9985 \\ -38.6231 & -19.7042 & 61.9164 & 67.2390 & 0.0000 & -15.1970 & -49.8023 \\ -7.8206 & 21.6911 & 3.8357 & 29.2425 & 15.1970 & 0.0000 & -27.4853 \\ 36.4310 & 15.7616 & -70.7559 & -61.9985 & 49.8023 & 27.4853 & 0.0000 \end{bmatrix},$$

$$K_{14} = \begin{bmatrix} 0.9201 & -0.2805 & -2.7237 & -0.1034 & 4.9203 & 0.8169 & -3.7965 \\ -0.2805 & -0.0944 & 1.5355 & -0.0387 & -3.3070 & -0.4975 & 2.6824 \\ -2.7237 & 1.5355 & 3.9447 & 0.1669 & -5.9154 & -1.0632 & 4.2561 \\ -0.1034 & -0.0387 & 0.1669 & 1.2746 & -0.9910 & -1.0895 & 0.6674 \\ 4.9203 & -3.3070 & -5.9154 & -0.9910 & 8.3247 & 1.9996 & -5.7547 \\ 0.8169 & -0.4975 & -1.0632 & -1.0895 & 1.9996 & 1.0734 & -1.3660 \\ -3.7965 & 2.6824 & 4.2561 & 0.6674 & -5.7547 & -1.3660 & 3.9314 \end{bmatrix},$$

with corresponding residual

$$\|R_{14}\| = \|F - D_{14}\tilde{X}\tilde{\Lambda} - G_{14}\tilde{X}\tilde{\Lambda} - K_{14}\tilde{X}\| = 2.5677e-012.$$

International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:4, No:7, 2010

Test 2. Choosing initial iterative matrices $\tilde{D}_1 = 0, \tilde{G}_1 = 0$ and $\tilde{K}_1 = 0$. By Algorithm 1, we get the minimum Frobenius norm solution $(\tilde{D}^*, \tilde{G}^*, \tilde{K}^*)$ of Eq.(13) as follows.

$$\tilde{D}^* = \tilde{D}_{14} = \begin{bmatrix} -0.8613 & -1.5193 & 2.7099 & 0.2863 & -3.2564 & -0.3986 & 2.0024 \\ -1.5193 & -1.6506 & 3.2150 & -3.2481 & -1.0203 & 2.0852 & 0.3289 \\ 2.7099 & 3.2150 & -5.9458 & -0.9176 & 3.7313 & -0.0354 & -0.7599 \\ 0.2863 & -3.2481 & -0.9176 & -5.9295 & 2.9333 & 1.9747 & -1.3325 \\ -3.2564 & -1.0203 & 3.7313 & 2.9333 & 3.9074 & 1.2042 & -6.6729 \\ -0.3986 & 2.0852 & -0.0354 & 1.9747 & 1.2042 & 0.2575 & -1.6034 \\ 2.0024 & 0.3289 & -0.7599 & -1.3325 & -6.6729 & -1.6034 & 7.7265 \end{bmatrix},$$

$$\tilde{G}^* = \tilde{G}_{14} = \begin{bmatrix} -0.0000 & -1.2427 & -0.9730 & -0.4217 & 3.6149 & 0.5062 & -3.2792 \\ 1.2427 & 0.0000 & -2.5346 & 2.9375 & 1.7040 & -1.6916 & -1.4012 \\ 0.9730 & 2.5346 & 0.0000 & -1.3438 & -5.7363 & -0.4763 & 6.5728 \\ 0.4217 & -2.9375 & 1.3438 & 0.0000 & -6.1061 & -2.6883 & 5.9064 \\ -3.6149 & -1.7040 & 5.7363 & 6.1061 & 0.0000 & -0.9247 & -4.4698 \\ -0.5062 & 1.6916 & 0.4763 & 2.6883 & 0.9247 & -0.0000 & -1.9688 \\ 3.2792 & 1.4012 & -6.5728 & -5.9064 & 4.4698 & 1.9688 & 0.0000 \end{bmatrix},$$

$$\tilde{K}^* = \tilde{K}_{14} = \begin{bmatrix} 0.1477 & -0.0097 & -0.3159 & -0.0129 & 0.4606 & 0.0448 & -0.3314 \\ -0.0097 & -0.0503 & 0.1516 & 0.0445 & -0.2844 & -0.0463 & 0.2180 \\ -0.3159 & 0.1516 & 0.4305 & 0.0358 & -0.5687 & -0.0820 & 0.3875 \\ -0.0129 & 0.0445 & 0.0358 & 0.1819 & -0.1254 & -0.1320 & 0.0809 \\ 0.4606 & -0.2844 & -0.5687 & -0.1254 & 0.7523 & 0.1703 & -0.5036 \\ 0.0448 & -0.0463 & -0.0820 & -0.1320 & 0.1703 & 0.1043 & -0.1162 \\ -0.3314 & 0.2180 & 0.3875 & 0.0809 & -0.5036 & -0.1162 & 0.3362 \end{bmatrix},$$

with corresponding residual

$$\|R_{14}\| = \|\tilde{F} - \tilde{D}_{14}\tilde{X}\tilde{\Lambda} - \tilde{G}_{14}\tilde{X}\tilde{\Lambda} - \tilde{K}_{14}\tilde{X}\| = 2.2388e-013.$$

Therefore, by (14), the optimal approximation solution of Problem II is

$$\hat{D} = \begin{bmatrix} 24.9645 & 14.4638 & -4.1055 & 13.6189 & -3.1169 & -0.5039 & 2.0024 \\ 14.4638 & 57.5298 & -9.4714 & 10.6029 & -0.9150 & 2.0060 & 0.3289 \\ -4.1055 & -9.4714 & 53.6978 & -0.9176 & -2.4991 & 12.8670 & -0.6204 \\ 13.6189 & 10.6029 & -0.9176 & 111.2073 & -9.9691 & 15.9859 & -1.2272 \\ -3.1169 & -0.9150 & -2.4991 & -9.9691 & 63.5510 & 1.2042 & -12.9033 \\ -0.5039 & 2.0060 & 12.8670 & 15.9859 & 1.2042 & 117.3943 & -14.5058 \\ 2.0024 & 0.3289 & -0.6204 & -1.2272 & -12.9033 & -14.5058 & 67.3701 \end{bmatrix},$$

$$\hat{G} = \begin{bmatrix} -0.0000 & 14.7404 & -7.7884 & 12.9109 & 3.7544 & 0.4009 & -3.2792 \\ -14.7404 & 0.0000 & -15.2210 & 16.7885 & 1.8093 & -1.7708 & -1.4012 \\ 7.7884 & 15.2210 & 0.0000 & -1.3438 & -11.9667 & 12.4261 & 6.7123 \\ -12.9109 & -16.7885 & 1.3438 & 0.0000 & -19.0085 & 11.3229 & 6.0117 \\ -3.7544 & -1.8093 & 11.9667 & 19.0085 & 0.0000 & -0.9247 & -10.7002 \\ -0.4009 & 1.7708 & -12.4261 & -11.3229 & 0.9247 & -0.0000 & -14.8712 \\ 3.2792 & 1.4012 & -6.7123 & -6.0117 & 10.7002 & 14.8712 & 0.0000 \end{bmatrix},$$

$$\hat{K} = 1000 \times \begin{bmatrix} 1.2001 & 1.8000 & -1.2003 & 1.8000 & 0.0005 & 0.0000 & -0.0003 \\ 1.8000 & 3.5999 & -1.7998 & 1.8000 & -0.0003 & -0.0000 & 0.0002 \\ -1.2003 & -1.7998 & 2.4004 & 0.0000 & -1.2006 & 1.7999 & 0.0004 \\ 1.8000 & 1.8000 & 0.0000 & 7.2002 & -1.8001 & 1.7999 & 0.0001 \\ 0.0005 & -0.0003 & -1.2006 & -1.8001 & 2.4008 & 0.0002 & -1.2005 \\ 0.0000 & -0.0000 & 1.7999 & 1.7999 & 0.0002 & 7.2001 & -1.8001 \\ -0.0003 & 0.0002 & 0.0004 & 0.0001 & -1.2005 & -1.8001 & 2.4003 \end{bmatrix}.$$

References

[1] P. Lancaster, M. Tismenetsky. The Theory of Matrices. 2rd Edition, Academic Press, London, 1985.

[2] F. Tisseur, K. Meerbergen. The quadratic eigenvalue problem. SIAM Review, 43 (2001) 235-286.
[3] M. Baruch. Optimization procedure to correct stiffness and flexibility matrices using vibration tests. AIAA Journal, 16 (1978) 1208-1210.
[4] M. Baruch, I. Y. Bar-Itzhack. Optimal weighted orthogonalization of measured modes. AIAA Journal, 16 (1978) 346-351.
[5] A. Berman. Mass matrix correction using an incomplete set of measured modes. AIAA Journal, 17 (1979) 1147-1148.
[6] A. Berman, E. J. Nagy. Improvement of a large analytical model using test data. AIAA Journal, 21 (1983) 1168-1173.
[7] F. S. Wei. Stiffness matrix correction from incomplete test data. AIAA Journal, 18 (1980) 1274-1275.
[8] F. S. Wei. Mass and stiffness interaction effects in analytical model modification. AIAA Journal, 28 (1990) 1686-1688.
[9] F. S. Wei. Analytical dynamic model improvement using vibration test data. AIAA Journal, 28 (1990) 174-176.
[10] Y.B. Yang, Y. J. Chen, T. W. Hsu. Direct updating method for structural models based on orthogonality constraints. Mechanics of Advanced Materials and Structures, 16 (2009) 390-401.
[11] Y.B. Yang, Y. J. Chen. A new direct method for updating structural models based on measured modal data. Engineering Structures, 31 (2009) 32-42.
[12] Y. Yuan. A model updating method for undamped structural systems. Journal of Computational and Applied Mathematics, 219 (2008) 294-301.
[13] M. I. Friswell, J. E. Mottershead. Finite Element Moodel Updating in Structural Dynamics. Dordrecht: Klumer Academic Publishers, 1995.
[14] M. I. Friswell, D. J. Inman, D. F. Pilkey. The direct updating of damping and stiffness matrices. AIAA Journal, 36 (1998) 491-493.
[15] D. F. Pilkey. Computation of a damping matrix for finite element model updating. Ph. D. Thesis, Dept. of Engineering Mechanics, Virginia Polytechnical Institute and State University, 1998.
[16] Y. C. Kuo, W. W. Lin, S. F. Xu. New methods for finite element moodel updating problems. AIAA Journal, 44 (2006) 1310-1316.
[17] D. L. Chu, M. Chu, W. W. Lin. Quadratic model updating with symmetry, positive definiteness, and no spill-over. SIAM Journal on Matrix Analysis And Applications, 31 (2009) 546-564.
[18] Y. Yuan. An inverse quadratic eigenvalue problem for damped structural systems. Mathematical Problems in Engineering, (2008), Article ID 730358, 9 pages.
[19] B. N. Datta, D. R. Sarkissian. Feedback control in distributed parameter gyroscopic systems: a solution of the partial eigenvalue assignment problem. Mechanical Systems and signal Processing, 16 (2002) 3-17.
[20] C. H. Guo. Numerical solution of a quadratic eigenvalue problem. Linear Algebra and its Applications, 385 (2004) 391-406.
[21] L. Meirovitch. A new method of solution of the eigenvalue problem for gyroscopic systems. AIAA Journal, 12 (1974) 1337-1342.
[22] R. H. Plaut. Alternative formulations for discrete gyroscopic eigenvalue problems. AIAA Journal, 14 (1976) 431-435.
[23] J. Qian, W. W. Lin. A numerical method for quadratic eigenvalue problems of gyroscopic systems. Journal of Sound and Vibration, 306 (2007) 284-296.
[24] D. R. Sarkissian. Theory and computations of partial eigenvalue and eigenstructure assignment problems in matrix second-order and distributed-parameter systems. Ph. D. thesis: Department of Mathematical Science, Northern Illinois University, 2001.
[25] A. Ben-Israel, T. N. E. Greville. Generalized Inverses. Theory and Applications (second ed). New York: Springer, 2003.

Yongxin Yuan received his Ph.D from Nanjing University of Aeronautics and Astronautics in China. His research interests are mainly on numerical algebra and matrix theory, especially in the theory and computation of the inverse problems in Structural Dynamics.