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An H^1-Galerkin mixed method for the coupled Burgers equation

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Abstract—In this paper, an H^1-Galerkin mixed finite element method is discussed for the coupled Burgers equations. The optimal error estimates of the semi-discrete and fully discrete schemes of the coupled Burgers equation are derived.

Keywords—The coupled Burgers equation; H^1-Galerkin mixed finite element method; Backward Euler's method; Optimal error estimates.

I. Introduction

With the research and development of the mixed finite element methods and H^1-Galerkin method, Pani [2] (in 1998) proposed a new mixed finite element method called H^1-Galerkin mixed finite element procedure which is applied to a mixed system in u and its flux q. The approximating finite element spaces V_h and W_h are allowed to be of differing polynomial degrees. Hence, estimations have been obtained which distinguish the better approximation properties of V_h and W_h. Compared to standard mixed methods, the proposed one is not subject to LBB consistency condition. Although we require extra regularity on the solution, a better order of convergence for the flux in L^2-norm is obtained. From then on, the method was applied to the evolution integro-differential equation[3],[4],[5],[6], hyperbolic problems[9],[10],[11],[14],[16], fourth-order parabolic equation[15], Sobolev equation[7],[8], Schrodinger equation[13] and nonlinear evolution equations[17],[18],[12] and so on. In this paper, we propose H^1-Galerkin mixed finite element scheme for the following coupled Burgers equation[1]

$$\begin{cases} u_t - u_{xx} - 2uu_x + (uv)_x = f(x,t), (x,t) \in \Omega \times J, \\ v_t - v_{xx} - 2vv_x + (uv)_x = g(x,t), (x,t) \in \Omega \times J, \\ u(x,t) = 0, v(x,t) = 0, (x,t) \in \partial\Omega \times \bar{J}, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), x \in \Omega, \end{cases} \tag{1}$$

where $\Omega = [0,1] \subset R^1$ with Lipschitz continuous boundary $\partial\Omega$, $J = (0,T]$ is the time interval with $0 < T < \infty$, $f(x,t), g(x,t)$ are two functions.

II. H^1-Galerkin mixed finite element method

Denote the natural inner product on $L^2(I)$ as $(\cdot,\cdot)$. Let $H_0^1 = \{z \in H^1(I) \mid z(0) = z(1) = 0\}$. Further, we call the classical Sobolev spaces $W^{m,p}(I), 1 \le p \le \infty$ as $W^{m,p}$ with norm$||\cdot||_{m,p}$. When $p = 2$, we simply write $W^{m,p}$ as H^m with norm $||\cdot||_m$.

With $q = u_x$,$\sigma = v_x$ we reformulate the formulation (1) as the first-order system:

$$\begin{cases} u_x = q, v_x = \sigma, \\ u_t - q_x - 2uq + qv + u\sigma = f(x,t), \\ v_t - \sigma_x - 2v\sigma + \sigma u + qv = g(x,t), \end{cases} \tag{2}$$

To derive the H^1-Galerkin mixed finite element method, we consider the following weak formulation of (1): find $\{u,v;q,\sigma\}$:$[0,T] \to H_0^1 \times H^1$ satisfying:

$$\begin{cases} (u_x, \chi_x) = (q, \chi_x), \forall \chi \in H_0^1, (a) \\ (v_x, w_x) = (\sigma, w_x), \forall w \in H_0^1, (b) \\ (q_t, \phi) + (q_x, \phi_x) + 2(uq, \phi_x) - (qv, \phi_x) - (u\sigma, \phi_x) \\ = -(f, \phi_x), \forall \phi \in H^1, (c) \\ (\sigma_t, \psi) + (\sigma_x, \psi_x) + 2(v\sigma, \psi_x) - (\sigma u, \psi_x) - (qv, \psi_x) \\ = -(g, \psi_x), \forall \psi \in H^1, (d) \end{cases} \tag{3}$$

For (3c,d), we have used integration by parts, and the *Dirichlet* boundary conditions $u_t(0,t) = u_t(1,t) = 0, v_t(0,t) = v_t(1,t) = 0$.

Let V_h and W_h be finite dimensional subspaces of H_0^1 and H^1, respectively, with the following approximation properties: for $1 \le p \le \infty$ and k, r positive integers

$$\begin{aligned} &\inf_{v_h \in V_h} \{||v - v_h||_{L^p} + h||v - v_h||_{W^{1,p}}\} \\ &\le Ch^{k+1}||v||_{W^{k+1,p}}, v \in \ H_0^1 \cap W^{k+1,p}, \\ &\inf_{w_h \in W_h} \{||w - w_h||_{L^p} + h||w - w_h||_{W^{1,p}}\} \\ &\le Ch^{r+1}||w||_{W^{r+1,p}}, w \in W^{r+1,p}. \end{aligned}$$

The semidiscrete H^1-Galerkin mixed finite element for (3) consists in determining $\{u^h, v^h; q^h, \sigma^h\}$:$[0,T] \to V_h \times W_h$ such that:

$$\begin{cases} (u_x^h, \chi_x^h) = (q^h, \chi_x^h), \forall \chi^h \in V_h, (a) \\ (v_x^h, w_x^h) = (\sigma^h, w_x^h), \forall w^h \in V_h, (b) \\ (q_t^h, \phi^h) + (q_x^h, \phi_x^h) + 2(u^h q^h, \phi_x^h) - (q^h v^h, \phi_x^h) \\ - (u^h \sigma^h, \phi_x^h) = -(f^h, \phi_x^h), \forall \phi^h \in W_h, (c) \\ (\sigma_t^h, \psi^h) + (\sigma_x^h, \psi_x^h) + 2(v^h \sigma^h, \psi_x^h) - (\sigma^h u^h, \psi_x^h) \\ - (q^h v^h, \psi_x^h) = -(g^h, \psi_x^h), \forall \psi^h \in W_h, (d) \end{cases} \tag{4}$$

For use in the error analysis, we define the elliptic projection $\tilde{u}^h, \tilde{v}^h \in V_h$ by

$$(u_x - \tilde{u}_x^h, \chi_x^h) = 0, (v_x - \tilde{v}_x^h, \omega_x^h) = 0, \chi^h, \omega^h \ \in V_h. \tag{5}$$

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Further, we also define a Ritz projection $\tilde{q}^h, \tilde{\sigma}^h \in W_h$ of q, σ as the solution of

$$A(q - \tilde{q}^h, \phi^h) = 0, A(\sigma - \tilde{\sigma}^h, \psi^h) = 0, \phi^h, \psi_h \in W_h. \quad (6)$$

where $A(z, w) = (z_x, w_x) + \lambda(z, w)$. Here λ is chosen appropriately so that A is H^1-coercive, i.e.,

$$A(w, w) \geq \mu_0 ||w||_1^2, w \in H^1$$

where μ_0 is a positive constant. Moreover, it is not hard to check that $A(\cdot, \cdot)$ is bounded.

With $\eta = u - \widetilde{u}^h$,$\tau = v - \widetilde{v}^h$,$\rho = q - \widetilde{q}^h$, $\delta = \sigma - \widetilde{\sigma}^h$, the following estimates are well known [19]: for $j = 0, 1$

$$\|\eta\|_j + \|\eta_t\|_j \leq Ch^{k+1-j}[\|u\|_{k+1} + \|u_t\|_{k+1}], \|\widetilde{u}^h\|_{0,\infty} \leq C(u) \quad (7)$$

$$\|\tau\|_j + \|\tau_t\|_j \leq Ch^{k+1-j}[\|v\|_{k+1} + \|v_t\|_{k+1}], \|\widetilde{v}^h\|_{0,\infty} \leq C(v) \quad (8)$$

$$\|\rho\|_j \leq Ch^{r+1-j}\|q\|_{r+1}, \|\rho_t\|_j \leq Ch^{r+1-j}\|q_t\|_{r+1} \quad (9)$$

$$\|\delta\|_j \leq Ch^{r+1-j}\|\sigma\|_{r+1}, \|\delta_t\|_j \leq Ch^{r+1-j}\|\sigma_t\|_{r+1} \quad (10)$$

III. Error estiamtes for semi-discrete scheme

For a priori error estimates, we decompose the errors as $u - u^h = u - \widetilde{u}^h + \widetilde{u}^h - u^h = \eta + \varsigma$; $v - v^h = v - \widetilde{v}^h + \widetilde{v}^h - v^h = \tau + \theta$; $q - q^h = q - \widetilde{q}^h + \widetilde{q}^h - q^h = \rho + \xi$; $\sigma - \sigma^h = \sigma - \widetilde{\sigma}^h + \widetilde{\sigma}^h - \sigma^h = \delta + \gamma$ From (3)-(6), we then obtain

$$\begin{cases} (\varsigma_x, \chi_x^h) = (\rho, \chi_x^h) + (\xi, \chi_x^h), \forall \chi^h \in V_h, (a) \\ (\theta_x, w_x^h) = (\delta, w_x^h) + (\gamma, w_x^h), \forall w^h \in V_h, (b) \\ (\xi_t, \phi^h) + (\xi_x, \phi_x^h) + 2(uq - u^h q^h, \phi_x^h) \\ -(qv - q^h v^h, \phi_x^h) - (u\sigma - u^h \sigma^h, \phi_x^h) \\ = -(\rho_t, \phi^h) + \lambda(\rho, \phi^h), \forall \phi^h \in W_h, (c) \\ (\gamma_t, \psi^h) + (\gamma_x, \psi_x^h) + 2(v\sigma - v^h \sigma^h, \psi_x^h) \\ -(\sigma u - \sigma^h u^h, \psi_x^h) - (qv - q^h v^h, \psi_x^h) \\ = -(\delta_t, \psi^h) + \lambda(\delta, \psi^h), \forall \psi^h \in W_h, (d) \end{cases} \quad (11)$$

Theorem 3.1: Assuming that $u^h(0) = \widetilde{u}^h(0)$,$v^h(0) = \widetilde{v}^h(0)$,$q^h(0) = \widetilde{q}^h(0)$, $\sigma^h(0) = \widetilde{\sigma}^h(0)$, we have

$$\|u - u^h\|^2 + h^2\|u - u^h\|_1^2 \leq Ch^{2\min(k+1,r+1)},$$

$$\|v - v^h\|^2 + h^2\|v - v^h\|_1^2 \leq Ch^{2\min(k+1,r+1)},$$

$$\|q - q^h\|^2 + h^2\|q - q^h\|_1^2 \leq Ch^{2\min(k+1,r+1)},$$

$$\|\sigma - \sigma^h\|^2 + h^2\|\sigma - \sigma^h\|_1^2 \leq Ch^{2\min(k+1,r+1)}.$$

Proof: Since estimates of ρ,δ,η and τ are given, respectively, it is sufficient to estimate ξ,γ,ς and θ. Choosing $\chi^h = \varsigma$ in (11a) and $w^h = \theta$ in (11b), using the Cauchy-Schwarz's inequality and Young's inequality, we have

$$\|\varsigma_x\| \leq C(\|\rho\| + \|\xi\|), \|\theta_x\| \leq C(\|\delta\| + \|\gamma\|). \quad (12)$$

Using Poincaré inequality, we have

$$\|\varsigma\| \leq C(\|\rho\| + \|\xi\|), \|\theta\| \leq C(\|\delta\| + \|\gamma\|). \quad (13)$$

Take $\phi^h = \xi$ in (11c) and use (13) to obtain

$$\begin{aligned} &(\xi_t, \xi) + A(\xi, \xi) \\ \leq &C(\|\rho\|^2 + \|\rho_t\|^2 + \|\xi\|^2) \\ &+ C\|q\|_{0,\infty}(\|\eta\| + \|\rho\| + \|\xi\| + \|\tau\| + \|\delta\| + \|\gamma\|)\|\xi\|_1 \\ &+ C\|u^h\|_{0,\infty}(\|\rho\| + \|\xi\| + \|\delta\| + \|\gamma\|)\|\xi\|_1 \\ &+ C\|\sigma\|_{0,\infty}(\|\eta\| + \|\rho\| + \|\xi\|)\|\xi\|_1 \\ &+ C\|v^h\|_{0,\infty}(\|\rho\| + \|\xi\|)\|\xi\|_1. \end{aligned} \quad (14)$$

Take $\psi^h = \gamma$ in (11d) and use (13) to have

$$\begin{aligned} &(\gamma_t, \gamma) + A(\gamma, \gamma) \\ \leq &C(\|\delta\|^2 + \|\delta_t\|^2 + \|\gamma\|^2) \\ &+ C\|\sigma\|_{0,\infty}(\|\tau\| + \|\delta\| + \|\gamma\| + \|\eta\| + \|\rho\| + \|\xi\|)\|\gamma\|_1 \\ &+ C\|v^h\|_{0,\infty}(\|\delta\| + \|\gamma\| + \|\rho\| + \|\xi\|)\|\gamma\|_1 \\ &+ C\|q\|_{0,\infty}(\|\tau\| + \|\delta\| + \|\gamma\|)\|\gamma\|_1 \\ &+ C\|u^h\|_{0,\infty}(\|\tau\| + \|\delta\| + \|\gamma\|)\|\gamma\|_1. \end{aligned} \quad (15)$$

Add (14) and (15) to get

$$\begin{aligned} &\frac{d}{dt}[\|\xi\|^2 + \|\gamma\|^2] + A(\xi, \xi) + A(\gamma, \gamma) \\ \leq &C(\|\xi\|^2 + \|\gamma\|^2) + C(\|\rho\|^2 + \|\rho_t\|^2 + \|\delta\|^2 \\ &+ \|\delta_t\|^2 + \|\eta\|^2 + \|\tau\|^2) + \frac{\mu_0}{2}(\|\xi\|_1^2 + \|\gamma\|_1^2). \end{aligned} \quad (16)$$

Integrate (16) with respect to time t to get

$$\begin{aligned} &\|\xi\|^2 + \|\gamma\|^2 + \int_0^t [A(\xi, \xi) + A(\gamma, \gamma)]ds \\ \leq &C\int_0^t (\|\xi\|^2 + \|\gamma\|^2)ds + C\int_0^t (\|\rho\|^2 + \|\rho_t\|^2 + \|\delta\|^2 \\ &+ \|\delta_t\|^2 + \|\eta\|^2 + \|\tau\|^2)ds + \frac{\mu_0}{2}\int_0^t (\|\xi\|_1^2 + \|\gamma\|_1^2)ds. \end{aligned} \quad (17)$$

For (17), we use the Gronwall lemma and $A(w, w) \geq \mu_0\|\xi\|_1^2$ to get

$$\begin{aligned} &\|\xi\|^2 + \|\gamma\|^2 + \int_0^t (\|\xi\|_1^2 + \|\gamma\|_1^2)ds \\ \leq &C\int_0^t (\|\rho\|^2 + \|\rho_t\|^2 + \|\delta\|^2 + \|\delta_t\|^2 + \|\eta\|^2 + \|\tau\|^2)ds. \end{aligned} \quad (18)$$

Choosing $\phi^h = \xi_t$ in (11c) and using (13), we have

$$\begin{aligned} &(\xi_t, \xi_t) + \frac{1}{2}\frac{d}{dt}A(\xi, \xi) \\ \leq &C(\|\rho_t\|^2 + \|\rho\|^2 + \|\xi\|^2) + C\|u^h\|_{0,\infty}(\|\rho\| + \|\xi\| \\ &+ \|\delta\| + \|\gamma\|)\|\xi_t\|_1 + C\|q\|_{0,\infty}(\|\eta\| + \|\rho\| + \|\xi\| + \\ &\|\tau\| + \|\delta\| + \|\gamma\|)\|\xi_t\|_1 + C\|v^h\|_{0,\infty}(\|\rho\| + \|\xi\|)\|\xi_t\|_1 \\ &+ C\|\sigma\|_{0,\infty}(\|\eta\| + \|\rho\| + \|\xi\|)\|\xi_t\|_1 + \mu_1\|\xi_t\|^2 \end{aligned} \quad (19)$$

Take $\psi^h = \gamma_t$ in (11d) to get

$$\begin{aligned}
&(\gamma_t, \gamma_t) + \frac{1}{2}\frac{d}{dt}A(\gamma, \gamma) \\
\leq& C(\|\delta_t\|^2 + \|\delta\|^2 + \|\gamma\|^2) + C\|\sigma\|_{0,\infty}(\|\eta\| + \|\varsigma\| \\
&+ \|\tau\| + \|\theta\|)\|\gamma_t\|_1 + C\|v^h\|_{0,\infty}(\|\rho\| + \|\xi\| \\
&+ \|\delta\| + \|\gamma\|)\|\gamma_t\|_1 + C\|u^h\|_{0,\infty}(\|\delta\| + \|\gamma\|)\|\gamma_t\|_1 \\
&+ C\|q\|_{0,\infty}(\|\tau\| + \|\theta\|)\|\gamma_t\|_1 + \mu_2\|\gamma_t\|^2 \\
\leq& C(\|\delta_t\|^2 + \|\delta\|^2 + \|\gamma\|^2) + C\|\sigma\|_{0,\infty}(\|\eta\| + \|\rho\| \\
&+ \|\xi\| + \|\tau\| + \|\delta\| + \|\gamma\|)\|\gamma_t\|_1 + C\|v^h\|_{0,\infty}(\|\rho\| + \|\xi\| \\
&+ \|\delta\| + \|\gamma\|)\|\gamma_t\|_1 + C\|u^h\|_{0,\infty}(\|\delta\| + \|\gamma\|)\|\gamma_t\|_1 \\
&+ C\|q\|_{0,\infty}(\|\tau\| + \|\delta\| + \|\gamma\|)\|\gamma_t\|_1 + \mu_2\|\gamma_t\|^2.
\end{aligned} \tag{20}$$

Add (19) and (20), integrate with respect to time t and use the Gronwall lemma to get

$$\begin{aligned}
&\mu_3\int_0^t[\|\xi\|^2 + \|\gamma\|^2]ds + \|\xi\|_1^2 + \|\gamma\|_1^2 \\
\leq& C\int_0^t(\|\rho\|^2 + \|\rho_t\|^2 + \|\delta\|^2 + \|\delta_t\|^2 + \|\eta\|^2 + \|\tau\|^2)ds.
\end{aligned} \tag{21}$$

Combining (12), (13), (18) and (21), we have

$$\begin{aligned}
\|\varsigma\|^2 \leq \|\varsigma\|_1^2 \leq& C(\|\rho\|^2 + \|\xi\|^2) \\
\leq& C\|\rho\|^2 + C\int_0^t(\|\rho\|^2 + \|\rho_t\|^2 + \|\delta\|^2 \\
&+ \|\delta_t\|^2 + \|\eta\|^2 + \|\tau\|^2)ds.
\end{aligned} \tag{22}$$

$$\begin{aligned}
\|\theta\|^2 \leq \|\theta\|_1^2 \leq& C(\|\delta\|^2 + \|\gamma\|^2) \\
\leq& C\|\delta\|^2 + C\int_0^t(\|\rho\|^2 + \|\rho_t\|^2 + \|\delta\|^2 \\
&+ \|\delta_t\|^2 + \|\eta\|^2 + \|\tau\|^2)ds.
\end{aligned} \tag{23}$$

Using (18), (21)-(23), (7)-(10) with the triangle inequality, we obtain the conclusion. ■

IV. Fully-discrete error estimates

For the backward Euler procedure, let $0 = t_0 < t_1 < \cdots < t_M = T$ be a given partition of the time interval $[0, T]$ with step length $\Delta t = T/M$, for some positive integer M. For a smooth function ϕ on $[0, T]$, define $\phi^n = \phi(t_n)$ and $\bar{\partial}_t\phi^n = (\phi^n - \phi^{n-1})/\Delta t$.

Let U^n, V^n, Q^n and Z^n, respectively, be the approximations of u, v, q and σ at $t = t_n$ which we shall define through the following scheme. Given $\{U^{n-1}, V^{n-1}; Q^{n-1}, Z^{n-1}\}$ in $V_h \times W_h$, we now determine $\{U^n, V^n; Q^n, Z^n\}$ in $V_h \times W_h$ satisfying

$$\begin{cases}
(U_x^n, \chi_x^h) = (Q^h, \chi_x^h), \forall\chi^h \in V_h, (a) \\
(V_x^n, w_x^h) = (Z^n, w_x^h), \forall w^h \in V_h, (b) \\
(\overline{\partial}_t Q^n, \phi^h) + (Q_x^n, \phi_x^h) + 2(U^n Q^n, \phi_x^h) - (Q^n V^n, \phi_x^h) \\
- (U^n Z^n, \phi_x^h) = -(f^n, \phi_x^h), \forall\phi^h \in W_h, (c) \\
(\overline{\partial}_t Z^n, \psi^h) + (Z_x^n, \psi_x^h) + 2(V^n Z^n, \psi_x^h) - (Z^n U^n, \psi_x^h) \\
- (Q^n V^n, \psi_x^h) = -(g^n, \psi_x^h), \forall\psi^h \in W_h, (d)
\end{cases} \tag{24}$$

we now split the errors

$$\begin{aligned}
u(t_n) - U^n &= u(t_n) - \widetilde{u}^h(t_n) + \widetilde{u}^h(t_n) - U^n = \eta^n + \varsigma^n \\
v(t_n) - V^n &= v(t_n) - \widetilde{v}^h(t_n) + \widetilde{v}^h(t_n) - V^n = \tau^n + \theta^n \\
q(t_n) - Q^n &= q(t_n) - \widetilde{q}^h(t_n) + \widetilde{q}^h(t_n) - Q^n = \rho^n + \xi^n \\
\sigma(t_n) - Z^n &= \sigma(t_n) - \widetilde{\sigma}^h(t_n) + \widetilde{\sigma}^h(t_n) - Z^n = \sigma^n + \gamma^n
\end{aligned}$$

Using (5)-(6) and (24), we obtain the following error equation

$$\begin{cases}
(\varsigma_x^n, \chi_x^h) = (\rho^n + \xi^n, \chi_x^h), \forall\chi^h \in V_h, (a) \\
(\theta_x^n, w_x^h) = (\delta^n + \gamma^n, w_x^h), \forall w^h \in V_h, (b) \\
(\overline{\partial}_t\xi^n, \phi^h) + (\xi_x^n, \phi_x^h) + 2(u^n q^n - U^n Q^n, \phi_x^h) \\
- (q^n v^n - Q^n V^n, \phi_x^h) - (u^n\sigma^n - U^n Z^n, \phi_x^h) \\
= -(\pi^n + \overline{\partial}_t\rho^n, \phi^h) + \lambda(\rho^n, \phi^h), \forall\phi^h \in W_h, (c) \\
(\overline{\partial}_t\gamma^n, \psi^h) + (\gamma_x^n, \psi_x^h) + 2(v^n\sigma^n - V^n Z^n, \psi_x^h) \\
- (\sigma^n u^n - Z^n U^n, \psi_x^h) - (q^n v^n - Q^n V^n, \psi_x^h) \\
= -(\varepsilon^n + \overline{\partial}_t\delta^n, \psi^h) + \lambda(\delta^n, \psi^h), \forall\psi^h \in W_h, (d)
\end{cases} \tag{25}$$

where $\pi^n = q_t(t_n) - \overline{\partial}_t q_(t_n)$, $\varepsilon^n = \sigma_t(t_n) - \overline{\partial}_t\sigma_(t_n)$.

Theorem 4.1: With $Q^0(0) = \widetilde{q}^h(0)$,$Z^0(0) = \widetilde{\sigma}^h(0)$ and $1 \leq J \leq M$, we have

$$\begin{aligned}
&\|q^J - Q^J\| + \|\sigma^J - Z^J\| + \|u^J - U^J\| + \|v^J - V^J\| \\
&+h\|u^J - U^J\|_1 + h\|v^J - V^J\|_1 \leq Ch^{min(r+1,k+1)}
\end{aligned}$$

Proof: Take $\chi^h = \varsigma^n$ and $w^h = \theta^n$ in (25a,b) to get

$$\|\varsigma_x^n\| \leq C(\|\rho^n\| + \|\xi^n\|), \|\theta_x^n\| \leq C(\|\delta^n\| + \|\gamma^n\|). \tag{26}$$

Using the Poincare inequality, we have

$$\|\varsigma^n\| \leq C(\|\rho^n\| + \|\xi^n\|), \|\theta^n\| \leq C(\|\delta^n\| + \|\gamma^n\|). \tag{27}$$

Choose $\phi^h = \xi^n$ and $\psi^h = \gamma^n$ in (25c,d) and add the two equations to get

$$\begin{aligned}
&(\overline{\partial}_t\xi^n, \xi^n) + (\overline{\partial}_t\gamma^n, \gamma^n) + (\xi_x^n, \xi_x^n) + (\gamma_x^n, \gamma_x^n) \\
=& -2(u^n q^n - U^n Q^n, \xi_x^n) + (q^n v^n - Q^n V^n, \xi_x^n) \\
&+ (u^n\sigma^n - U^n Z^n, \xi_x^n) - (\pi^n + \overline{\partial}_t\rho^n, \xi^n) + \lambda(\rho^n, \xi^n) \\
&- 2(v^n\sigma^n - V^n Z^n, \gamma_x^n) + (\sigma^n u^n - Z^n U^n, \gamma_x^n) \\
&+ (q^n v^n - Q^n V^n, \gamma_x^n) - (\varepsilon^n + \overline{\partial}_t\delta^n, \gamma^n) + \lambda(\delta^n, \gamma^n).
\end{aligned} \tag{28}$$

Noting that $(\overline{\partial}_t\xi^n, \xi^n) \geq \frac{1}{2}\overline{\partial}_t\|\xi^n\|^2$, $(\overline{\partial}_t\gamma^n, \gamma^n) \geq \frac{1}{2}\overline{\partial}_t\|\gamma^n\|^2$, and using (27), we have

$$\begin{aligned}
&\frac{1}{2}\overline{\partial}_t\|\xi^n\|^2 + \frac{1}{2}\overline{\partial}_t\|\gamma^n\|^2 + \|\xi_x^n\|^2 + \|\gamma_x^n\|^2 \\
\leq& C(\|\rho^n\|^2 + \|\eta^n\|^2 + \|\delta^n\|^2 + \|\tau^n\|^2) + C(\|\pi^n\|^2 \\
&+ \|\varepsilon^n\|^2) + C(\overline{\partial}_t\|\rho^n\|^2 + \overline{\partial}_t\|\delta^n\|^2) + C(\|\xi^n\|^2 + \|\gamma^n\|^2).
\end{aligned} \tag{29}$$

Noting that $\overline{\partial}_t\|\xi^n\|^2 = (\|\xi^n\|^2 - \|\xi^{n-1}\|^2)/\Delta t$, $\overline{\partial}_t\|\gamma^n\|^2 = (\|\gamma^n\|^2 - \|\gamma^{n-1}\|^2)/\Delta t$, and combining (29), we have

$$\begin{aligned}
&\|\xi^n\|^2 - \|\xi^{n-1}\|^2 + \|\gamma^n\|^2 - \|\gamma^{n-1}\|^2 \\
&+ 2\Delta t(\|\xi_x^n\|^2 + \|\gamma_x^n\|^2). \leq C\Delta t(\|\rho^n\|^2 + \|\eta^n\|^2 \\
&+ \|\delta^n\|^2 + \|\tau^n\|^2) + C\Delta t(\|\pi^n\|^2 + \|\varepsilon^n\|^2) \\
&+ C\Delta t(\overline{\partial}_t\|\rho^n\|^2 + \overline{\partial}_t\|\delta^n\|^2) + C\Delta t(\|\xi^n\|^2 + \|\gamma^n\|^2).
\end{aligned} \tag{30}$$

Sum (30) from $n = 1$ to J $(1 \leq J \leq M)$ and use Gronwall lemma to get

$$\begin{aligned}
&(1 - C\Delta t)(\|\xi^J\|^2 + \|\gamma^J\|^2) + 2\Delta t \sum_{n=1}^{J} (\|\xi_x^n\|^2 + \|\gamma_x^n\|^2) \\
\leq& C(\|\xi^0\|^2 + \|\gamma^0\|^2) + C\Delta t \sum_{n=1}^{J} (\|\rho^n\|^2 + \|\eta^n\|^2 \\
&+ \|\delta^n\|^2 + \|\tau^n\|^2) + C\Delta t \sum_{n=1}^{J} (\|\pi^n\|^2 + \|\varepsilon^n\|^2) \\
&+ C\Delta t \sum_{n=1}^{J} (\overline{\partial}_t \|\rho^n\|^2 + \overline{\partial}_t \|\delta^n\|^2).
\end{aligned} \tag{31}$$

Use

$$\overline{\partial}_t \|\rho^n\|^2 \leq \frac{h^{2(r+1)}}{\Delta t} \int_{t_{n-1}}^{t_n} \|q_t\|_{r+1}^2 ds,$$

$$\overline{\partial}_t \|\delta^n\|^2 \leq \frac{h^{2(r+1)}}{\Delta t} \int_{t_{n-1}}^{t_n} \|\sigma_t\|_{r+1}^2 ds$$

$$\|\pi^n\|^2 \leq C\Delta t \int_{t_{n-1}}^{t_n} \|q_{tt}\|_{r+1}^2 ds,$$

$$\|\varepsilon^n\|^2 \leq C\Delta t \int_{t_{n-1}}^{t_n} \|\sigma_{tt}\|_{r+1}^2 ds,$$

and (7)-(10) to obtain

$$\begin{aligned}
&(\|\xi^J\|^2 + \|\gamma^J\|^2) + 2\Delta t \sum_{n=1}^{J} (\|\xi_x^n\|^2 + \|\gamma_x^n\|^2) \\
\leq& Ch^{2min(r+1,k+1)}(\|u\|_{L^\infty(H^{k+1})}^2 + \|u_t\|_{L^\infty(H^{k+1})}^2 \\
&+ \|v\|_{L^\infty(H^{k+1})}^2 + \|v_t\|_{L^\infty(H^{k+1})}^2 + \|q\|_{L^\infty(H^{r+1})}^2 \\
&+ \|\sigma\|_{L^\infty(H^{r+1})}^2 + \|q_t\|_{L^2(H^{k+1})}^2 + \|\sigma_t\|_{L^2(H^{k+1})}^2) \\
&+ C\Delta t^2(\|q_{tt}\|_{L^2(L^2)}^2 + \|\sigma_{tt}\|_{L^2(L^2)}^2).
\end{aligned} \tag{32}$$

Using (26)-(27), we have

$$\begin{aligned}
&\|\varsigma^J\|_1 + \|\theta^J\|_1 \\
\leq& Ch^{r+1}(\|q\|_{L^\infty H^{r+1}} + \|\sigma\|_{L^\infty H^{r+1}}) \\
&+ Ch^{min(r+1,k+1)}(\|u\|_{L^\infty(H^{k+1})} + \|u_t\|_{L^\infty(H^{k+1})} \\
&+ \|v\|_{L^\infty(H^{k+1})} + \|v_t\|_{L^\infty(H^{k+1})} + \|q\|_{L^\infty(H^{r+1})} \\
&+ \|\sigma\|_{L^\infty(H^{r+1})} + \|q_t\|_{L^2(H^{k+1})} + \|\sigma_t\|_{L^2(H^{k+1})}) \\
&+ C\Delta t(\|q_{tt}\|_{L^2(L^2)} + \|\sigma_{tt}\|_{L^2(L^2)}).
\end{aligned} \tag{33}$$

We apply the triangle inequality to get the conclusion. ■

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