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# An Evaluation of Average Run Length of MaxEWMA and MaxGWMA Control Charts

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***Abstract*—**Exponentially weighted moving average control chart (EWMA) is a popular chart used for detecting shift in the mean of parameter of distributions in quality control. The objective of this paper is to compare the efficiency of control chart to detect an increase in the mean of a process. In particular, we compared the Maximum Exponentially Weighted Moving Average (MaxEWMA) and Maximum Generally Weighted Moving Average (MaxGWMA) control charts when the observations are Exponential distribution. The criteria for evaluate the performance of control chart is called, the Average Run Length (ARL). The result of comparison show that in the case of process is small sample size, the MaxEWMA control chart is more efficiency to detect shift in the process mean than MaxGWMA control chart. For the case of large sample size, the MaxEWMA control chart is more sensitive to detect small shift in the process mean than MaxGWMA control chart, and when the process is a large shift in mean, the MaxGWMA control chart is more sensitive to detect mean shift than MaxEWMA control chart.

***Keywords*—**Maximum Exponentially Weighted Moving Average, Maximum General Weighted Moving Average, Average Run Length.

## I. Introduction

SOME of the most widely used form of control charts such as $\overline{X}$ - R charts and Individuals charts. These are often referred to as Shewhart control charts and it was first introduced by Walter Shewhart. The Shewhart control charts are sensitive to detecting relatively large shifts in the process mean. An alternative control chart is primarily used to detect smaller shifts, namely Exponentially Weighted Moving Average (EWMA) control chart. Roberts, S.W. [10] originally developed the EWMA control chart. It has been used in various industries especially the chemical industry. An abbreviation of EWMA control chart, this technique is used in statistical process control to monitor the output of manufacturing process by tracking the moving average of performance over lifetime of the process. The Cumulative Sum (CUSUM) procedure introduced by Page [2] and the Shiryaev–Roberts procedure introduced by Shiryaev for the Brownian motion case.

Its properties have been thoroughly studied in the literature (see, e.g., Hawkins and Olwell [1]). A numerical comparison of EWMA and CUSUM control charts was given by Lucas and Saccucci [7] and Yashchin [3], [4]. Srivastava and Wu [8], [9] and Wu [12] considred design of the optimal EWMA control chart and compared it with the CUSUM and Shiryaev-Roberts control charts. Xie [6] was introducing the MaxEWMA control chart and Chen et al. [5] extended Xie's research. The MaxEWMA combines two EWMA charts into a single chart such that its can detect the change-point in the process mean and variability.

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A common characteristic used for comparing the performance of control charts is Average Run Length (ARL), the expected number of observations taken from an in-control process until the control chart falsely signals out-of-control is denoted by $ARL_0$. An $ARL_0$ will be regarded as acceptable if it is large enough to keep the level of false alarms at an acceptable level. A second common characteristic is the expected number of observations taken from an out-of-control process until the control chart signals that the process is out-of-control is denoted by $ARL_1$.

The objective of this paper is concerned with the use control charts for detecting change-point in mean of exponential distribution. Our goal is to provide a comparative study of the main competitor control charts: the Maximum Exponentially Weighted Moving Average (MaxEWMA) and Maximum Generally Weighted Moving Average (MaxGWMA) control charts when the observations are exponential distribution.

This paper is divided into five sections: in Section I, we introduce the statistical process control charts. Section II presents the characteristics of the MaxEWMA and MaxGWMA control charts. In Section III, we show the results of comparison when the process are exponential distribution. Finally, we present the conclusions.

## II. The Characteristics of Control Charts

### A. MaxEWMA Control Chart

Xie [6] was first introduced the MaxEWMA control chart. The MaxEWMA control chart can be used to monitor the process mean and variability. In this section, we describe the characteristics of MaxEWMA control chart for exponential distribution. The MaxEWMA control chart is based on a weighted average of current and previous data. In this article, we consider the simplest version of the change-point detection problem where we assume that the observations are exponential distributed before the change-point in the mean with a common density function and after the change-point in the mean with a different density function, both of which are considered known.

Let $X$ be a quality characteristic of a process and it has a exponential distribution with the process mean ( $\beta$ ). In situation the process is in-control with exponential parameter $\beta = \beta_0$ and the process is out-of-control with exponential parameter

$\beta = \beta_1$ where $\beta_1 = \beta_0 + \delta\beta_0$. Let $X_{ij}$ , $i = 1,2,...$ and $j = 1,2,...,n_i$ are the observations of the random variable $X$ arranged in groups of size $n_i$, where $i$ is the index of the subgroup. We defined two statistics as following:

$$U_i = \frac{\bar{X}_i - \mu}{\sigma / \sqrt{n_i}} \text{ and } V_i = \Phi^{-1}\left\{ F\left[ \frac{(n_i - 1)S_i^2}{\sigma^2}, n_i - 1 \right] \right\} \quad (1)$$

where $\bar{X}_i$ and $S_i^2$ denote the sample mean and sample variance of $i^{th}$ sample, respectively. Let $\Phi^{-1}(.)$ denote the inverse standard normal distribution function and $F(h,\upsilon)$ denote the $\chi^2$ distribution function with $\upsilon$ degrees of freedom.

Two EWMA statistics for the mean and variance can be defined from the statistics $U_i$ and $V_i$ as following:

$$A_i = \lambda U_i + (1-\lambda)A_{i-1} \quad ; i = 1,2,.... \quad (2)$$

$$B_i = \lambda V_i + (1-\lambda)B_{i-1} \quad ; i = 1,2,.... \quad (3)$$

where $0 < \lambda \le 1$ is smoothing parameter and $A_0$ = 0 and $B_0$ = 0 are the initial value of EWMA statistics. $A_i \sim N\left(0,\sigma_{A_i}^2\right)$ and $B_i \sim N\left(0,\sigma_{B_i}^2\right)$ are independent.

The MaxEWMA statistic is defined as:

$$ME_i = max\{|A_i|,|B_i|\} \quad ; i = 1,2,.... \quad (4)$$

The Upper Control Limit for MaxEWMA control chart is:

$$UCL_{ME} = E(ME_i) + L_{ME}\sqrt{Var(ME_i)} \quad (5)$$

where $E(ME_i)$ and $Var(ME_i)$ are the mean and variance of the MaxEWMA statistics respectively and $L_{ME}$ is the width of the control limit when the process is in control state.

Therefore, the process will be declared to be in an out-of-control state when the MaxEWMA statistics $ME_i > UCL_{ME}$.

### B. MaxGWMA Control Chart

The Generally Weighted Moving Average (GWMA) is a moving average of a set of past data in which a weight is assigned to each data point. The GWMA control chart is briefly introduced herein for completeness. Sheu et al. [11] combined between the MaxEWMA and GWMA control charts and proposed the new control chart is known as the Maximum Generally Weighted Moving Average (MaxGWMA) control chart.

Among the sequence of independent samples, let M represent the number of samples until the first occurrence of event A since the previous occurrence of event A.

$$\sum_{m=1}^{\infty} P(M=m) = P(M=1) + P(M=2) + \ldots + P(M=t) + P(M>t) = 1 \quad (6)$$

where $P(M=1)$, $P(M=2)$,... $P(M=t)$ be the weights of the current sample, the previous sample, …., and the most out-of-data sample, respectively. Thus, $P(M>t)$ is weighted with the target value of the process. And the weights can be compute; $P(M>i) = q^{i^{\alpha}}$.

$$P(M=i) = P(M>i-1) - P(M>i) = q^{(i-1)^{\alpha}} - q^{i^{\alpha}} \quad (7)$$

where $0 \le q \le 1$ is a constant parameter and $\alpha > 0$ is the adjustment parameter.

The GWMA statistic used in this paper is the same as that used by Roberts [10] when $\alpha = 1$ and $q = 1-\lambda$. The GWMA statistics for the mean and variance can be defined from the statistics $U_i$ and $V_i$ as following:

$$G_i = P(M=1)\,U_i + P(M=2)\,U_{i-1} + \ldots + P(M=i)\,U_1 + P(M>i)G_0 \quad (8)$$

$$H_i = P(M=1)\,V_i + P(M=2)\,V_{i-1} + \ldots + P(M=i)\,V_1 + P(M>i)V_0 \quad (9)$$

where $G_0$ = 0 and $H_0$ = 0 are the initial value of GWMA statistics; $G_i \sim N\left(0,\sigma_{G_i}^2\right)$ and $H_i \sim N\left(0,\sigma_{H_i}^2\right)$ are independent.

The MaxGWMA statistic is defined as:

$$MG_i = max\{|G_i|,|H_i|\} \quad ; i = 1,2,.... \quad (10)$$

The Upper Control Limit for MaxGWMA control chart is:

$$UCL_{MG} = E(MG_i) + L_{MG}\sqrt{Var(MG_i)} \quad (11)$$

where $E(MG_i)$ and $Var(MG_i)$ are the mean and variance of the MaxGWMA statistics respectively and $L_{MG}$ is the width of the control limit when the process is in control state.

Therefore, the process will be declared to be in an out-of-control state when the MaxGWMA statistics $MG_i > UCL_{MG}$.

## III. The Results of Comparison Performance of Control Charts

Generally, the statistical performance of a control chart is evaluated using the ARL. The ARL usually needs to be sufficiently large to avoid false alarms when the process is in-control control, but it needs to be sufficiently small so as to rapidly detect shifts when the process is out-of-control. We calculate the ARL of MaxEWMA and MaxGWMA control charts by using Monte Carlo simulations technique.

The MaxEWMA control chart with different values for the parameters specifications $\lambda$ (smoothing parameter) and $L$ (the width of the control limit) and the MaxGWMA control chart with different values for parameters $q$ (constant parameter), $\alpha$ (adjustment parameter) and $L$.

In this section, we compare the efficiency of control charts between MaxEWMA and MaxGWMA control charts. We first consider the case in which the observations are exponential distributions with parameter $\beta$. In situation the process is in-control state, we let the exponential parameter $\beta = \beta_0$ = 1 and the process is out-of-control state, the exponential parameter $\beta = \beta_1 = \beta_0 + \delta\beta_0$ where $\delta$ is the magnitude of shift size; $\delta$ = 0.0, 0.25, 0.50, 1.00, 2.00 and 3.00 respectively. In Table I gives the results for $ARL_0$ = 370 when $\lambda$ = 0.01, 0.05, 0.10 and 0.50 respectively and sample size n = 5. From the Table I, we find that the MaxEWMA control chart appears as good as the MaxGWMA control chart.

TABLE I
THE CORRESPONDING ARL FOR MAXEWMA AND MAXGWMA CONTROL CHARTS WHEN $ARL_0$ = 370 AND $n$ = 5

| $\lambda$ | shift ($\delta$) | MaxEWMA | MaxGWMA |
|---|---|---|---|
| 0.01 | 0.00 | 369.045 | 369.108 |
| | 0.25 | 96.320* | 245.620 |
| | 0.50 | 51.424* | 153.068 |
| | 1.00 | 32.634* | 48.046 |
| | 2.00 | 26.394 | 24.286* |
| | 3.00 | 23.846 | 20.966* |
| 0.05 | 0.00 | 371.980 | 370.206 |
| | 0.25 | 161.592* | 268.790 |
| | 0.50 | 59.738* | 159.534 |
| | 1.00 | 30.704* | 58.884 |
| | 2.00 | 23.894* | 26.148 |
| | 3.00 | 22.478* | 23.698 |
| 0.10 | 0.00 | 369.590 | 370.844 |
| | 0.25 | 216.528* | 268.818 |
| | 0.50 | 78.434* | 164.18 |
| | 1.00 | 30.642* | 57.392 |
| | 2.00 | 22.980* | 25.774 |
| | 3.00 | 21.114* | 21.342 |
| 0.50 | 0.00 | 370.890 | 370.380 |
| | 0.25 | 240.328* | 270.850 |
| | 0.50 | 125.914* | 167.084 |
| | 1.00 | 33.322* | 55.304 |
| | 2.00 | 21.962* | 25.284 |
| | 3.00 | 20.962* | 22.594 |

*Minimum Average Run Length

Table II gives the results for $ARL_0$ = 370 when $\lambda$ = 0.01, 0.05, 0.10 and 0.50 respectively and sample size n = 30. From the Table II, we find that the MaxEWMA is more sensitive to detect small shift in the process mean than MaxGWMA chart.

TABLE II
THE CORRESPONDING ARL FOR MAXEWMA AND MAXGWMA CONTROL CHARTS WHEN $ARL_0$ = 370 AND AND $n$ = 30

| $\lambda$ | shift ($\delta$) | MaxEWMA | MaxGWMA |
|---|---|---|---|
| 0.01 | 0.00 | 372.068 | 371.378 |
| | 0.25 | 44.746* | 126.44 |
| | 0.50 | 31.144* | 34.712 |
| | 1.00 | 25.244 | 21.466* |
| | 2.00 | 22.230 | 20.008* |
| | 3.00 | 21.342 | 20.000* |
| 0.05 | 0.00 | 372.592 | 372.916 |
| | 0.25 | 43.630* | 136.420 |
| | 0.50 | 26.486* | 33.618 |
| | 1.00 | 22.686 | 20.700* |
| | 2.00 | 21.002 | 20.026* |
| | 3.00 | 20.442 | 20.000* |
| 0.10 | 0.00 | 370.404 | 371.72 |
| | 0.25 | 46.872* | 138.636 |
| | 0.50 | 25.160* | 35.676 |
| | 1.00 | 21.828 | 21.322* |
| | 2.00 | 20.538 | 20.018* |
| | 3.00 | 20.158 | 20.002* |
| 0.50 | 0.00 | 369.326 | 370.428 |
| | 0.25 | 78.328* | 140.278 |
| | 0.50 | 24.556* | 42.212 |
| | 1.00 | 21.632 | 21.056* |
| | 2.00 | 20.034 | 20.022* |
| | 3.00 | 20.003 | 20.002* |

*Minimum Average Run Length

## IV. CONCLUSION

Comparing our results from the MaxEWMA and MaxGWMA control charts shows that for the case of a one-sided shift, it has been shown that the MaxEWMA control chart is the best control chart in the sense that it has minimizes the supremum of the conditional Average Run Length ($ARL_1$) when the process is small sample size and the process has a small shift ($0.25 \le \delta \le 0.50$). In situation of the process is large sample and large shift ($1.00 \le \delta \le$ 3.00) the MaxGWMA is more sensitive to detect shift than MaxEWMA control chart.

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## REFERENCES

[1] D.M. Hawkins and D.H. Olwell, "Cumulative Sum Charts and Charting for Quality Improvement," 1998, Springer, New York.
[2] E.S. Page, "Continuous inspection schemes," Biometrika 41, 1995, pp. 100-115.
[3] E. Yashchin, "Statistical control schemes: Methods, Applications and generalization," International Statistical 61, 1993, pp. 41-66.
[4] E. Yashchin, "Performance of CUSUM control schemes for serially correlated observations," Technometrics 35, 1993, pp. 37-52.
[5] G. Chen, S. W. Cheng, and H. Xie, "Monitoring process mean and variability with one EWMA chart," Journal of Quality Technology 33, 2001, pp. 223-233.
[6] H. Xie, "Contributions to qualimetry," Ph.D. Thesis. Winnipeg, Canada: University of Manitoba, pp. 23-37.

[7] J.M. Lucas and M.S. Saccucci, “Exponentially weighted moving average control schemes: properties and enhancements,” Technometrics 32, 1990, pp. 1-29.
[8] M.S. Srivastava and Y.H. Wu, “Comparison of EWMA, CUSUM and Shiryayev-Roberts procedures for detecting a shift in the mean,” The Annals of Statistics 21, 1993, pp. 645-670.
[9] M.S. Srivastava and Y.H. Wu, “Evaluation of optimum weights and average run lengths in EWMA control schemes,” Communications in Statistics: Theory and Methods 26, 1997, pp.1253-1267.
[10] S.W. Roberts, “Control chart tests based on geometric moving average,” Technometrics 1, 1959, pp. 239-250.
[11] S. Sheu, C. Huang and T. Hsu, “Extended maximum generally weighted moving average control chart for monitoring process mean and variability,” Computers & Industrial Engineering 62, 2012, pp. 216-225.
[12] Y.H. Wu, “Design of control charts for detecting the change point, In Change-Point Problems,” 1994, pp. 330-345.