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An approximate solution of the classical Van der Pol oscillator coupled gyroscopically to a linear oscillator using parameter-expansion method

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Abstract—In this article, we are dealing with a model consisting of a classical Van der Pol oscillator coupled gyroscopically to a linear oscillator. The major problem is analyzed. The regular dynamics of the system is considered using analytical methods. In this case, we provide an approximate solution for this system using parameter-expansion method. Also, we find approximate values for frequencies of the system. In parameter-expansion method the solution and unknown frequency of oscillation are expanded in a series by a bookkeeping parameter. By imposing the non-secularity condition at each order in the expansion the method provides different approximations to both the solution and the frequency of oscillation. One iteration step provides an approximate solution which is valid for the whole solution domain.

Keywords—Parameter-expansion method, classical Van der Pol oscillator.

I. INTRODUCTION

CONSIDER the following model of a classical Van der Pol oscillator coupled gyroscopically to a linear oscillator

$$\begin{cases} y'' + \varepsilon(y^2-1)y' + y + fx'' = E\cos(nt), \\ x'' + \lambda x' + x - dy = 0 \end{cases} \tag{1}$$

where a prime denotes time derivative. The Van der Pol oscillator is represented by the variable y while x stands for the linear oscillator, ε and λ are respectively the Van der Pol parameters and the damping coefficient of the linear oscillator. The quantities f and d are the coupling coefficients. E and n are the amplitude and frequency of the external excitation while t is the non-dimensional time. We have restricted our analysis to the case where the natural frequencies of both oscillators are identical (internal resonance). To solve non-linear evolution equations many effective methods have been introduced, such as the variational iteration method [1], [2], [3], the Adomian decomposition method [4], [5], the homotopy perturbation method [6], [7], [8], parameter expansion method [9], [10], [11], [12], spectral collocation method [13], [14], [15], [16], [17], homotopy analysis method [18], [19], [20], [21], three-wave method [22], [23], [24], extended homoclinic test approach [25], [26], [27], the $(\frac{G'}{G})-$expansion method [28] and the Exp-function method [29], [30], [31], [32], [33], [34].

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In this article we apply the parameter-expansion method to obtain approximate solution of system (1), also we provide numerical approximations for frequencies of x and y.

II. PARAMETER EXPANSION METHOD

To solve (1) by parameter-expansion method we rewrite the system as

$$\begin{cases} y'' + 1\cdot y = E\cos(nt) - fx'' + \varepsilon(1-y^2)y', \\ x'' + 1\cdot x = dy - \lambda x'. \end{cases} \tag{2}$$

According to the parameter-expansion method, all variables x and y can be expanded into a series of an artificial parameter p such as

$$\begin{aligned} x &= x_0 + px_1 + p^2x_2 + \cdots \\ y &= y_0 + py_1 + p^2y_2 + \cdots \end{aligned} \tag{3}$$

where p is called a bookkeeping parameter [35]. We also expand all coefficients of the system (1) into a series of p in a similar way

$$\begin{aligned} 1 &= \alpha^2 + p\alpha_1 + p^2\alpha_2 + \cdots \\ E &= pE_1 + p^2E_2 + \cdots \\ \varepsilon &= p\varepsilon_1 + p^2\varepsilon_2 + \cdots \\ f &= pf_1 + p^2f_2 + \cdots \\ 1 &= \beta^2 + p\beta_1 + p^2\beta_2 + \cdots \\ d &= pd_1 + p^2d_2 + \cdots \\ \lambda &= p\lambda_1 + p^2\lambda_2 + \cdots . \end{aligned} \tag{4}$$

By substituting the above expansions (3) and (4) into the system (2), we have

$$\begin{aligned} &(y_0'' + py_1'' + p^2y_2'' + \cdots) + (\alpha^2 + p\alpha_1 + p^2\alpha_2 + \cdots)\times \\ &\quad (y_0 + py_1 + p^2y_2 + \cdots) = \\ &\quad (pE_1 + p^2E_2 + \cdots)\cos(nt) - (pf_1 + p^2f_2 + \cdots)\times \\ &\quad (x_0'' + px_1'' + p^2x_2'' + \cdots) + (p\varepsilon_1 + p^2\varepsilon_2 + \cdots)\times \\ &\quad [1 - (y_0'' + py_1'' + p^2y_2'' + \cdots)^2](y_0' + py_1' + p^2y_2' + \cdots), \\ \\ &(x_0'' + px_1'' + p^2x_2'' + \cdots) + (\beta^2 + p\beta_1 + p^2\beta_2 + \cdots)\times \\ &\quad (x_0 + px_1 + p^2x_2 + \cdots) = (pd_1 + p^2d_2 + \cdots)\times \\ &\quad (y_0 + py_1 + p^2y_2 + \cdots) - (p\lambda_1 + p^2\lambda_2 + \cdots)\times \\ &\quad (x_0' + px_1' + p^2x_2' + \cdots). \end{aligned} \tag{5}$$

Equating in the powers of p, we have

$$p^0 : \begin{cases} y_0'' + \alpha^2 y_0 = 0 \\ \\ x_0'' + \beta^2 x_0 = 0 \end{cases} \tag{6}$$

and

$$p^1 : \begin{cases} y_1'' + \alpha^2 y_1 = \\ \qquad E_1 \cos(nt) + \varepsilon_1 y_0' - \varepsilon_1 y_0^2 y_0' - \alpha_1 y_0 - f_1 x_0'', \\ \\ x_1'' + \beta^2 x_1 = \\ \qquad d_1 y_0 - \beta_1 x_0 - \lambda_1 x_0'. \end{cases} \tag{7}$$

Solving the equation (6), we obtain

$$\begin{cases} y_0 = A_1 \cos(\alpha t) + A_2 \sin(\alpha t) \\ \\ x_0 = B_1 \cos(\beta t) + B_2 \sin(\beta t). \end{cases} \tag{8}$$

where A_1,A_2,B_1 and B_2 are arbitrary constants. Substituting (8) into (7), we obtain

$$\begin{cases} y_1'' + \alpha^2 y_1 = \\ \qquad E_1 \cos(nt) + f_1[B_1\beta^2 \cos(\beta t) - B_2 \beta^2 \sin(\beta t)] \\ \qquad + \sin(\alpha t)[\frac{1}{4}A_1 A_2^2 \alpha\varepsilon_1 + \frac{1}{4}A_1^3 \alpha\varepsilon_1 - A_1\alpha\varepsilon_1 - \alpha_1 A_2] \\ \qquad + \cos(\alpha t)[-\frac{1}{4}A_2^3 \alpha\varepsilon_1 - \frac{1}{4}A_1^2 A_2 \alpha\varepsilon_1 + A_2\alpha\varepsilon_1 - \alpha_1 A_1] \\ \qquad + \sin(3\alpha t)[\frac{1}{4}A_1^3 \alpha\epsilon_1 - \frac{3}{4}A_1 A_2^2 \alpha\varepsilon_1] + \\ \qquad \cos(3\alpha t)[\frac{3}{4}A_1^2 A_2 \alpha\epsilon_1 - \frac{1}{4}A_2^3\alpha\varepsilon_1], \\ \\ x_1'' + \beta^2 x_1 = \\ \qquad d_1 A_1 \cos(\alpha t) + d_1 A_2 \sin(\alpha t) \\ \quad + \sin(\beta t)[\lambda_1 B_1 \beta - \beta_1 B_2] + \cos(\beta t)[-\lambda_1 B_2 \beta - \beta_1 B_1]. \end{cases} \tag{9}$$

If the first-order approximation is enough, then, setting $p = 1$ in both equations (3) and (4), we have

$$\begin{aligned} &y = y_0 + y_1, \quad \alpha^2 + \alpha_1 = 1, \quad E = E_1, \quad \varepsilon = \varepsilon_1, \quad f = f_1 \\ &x = x_0 + x_1, \quad \beta^2 + \beta_1 = 1, \quad d = d_1, \quad \lambda = \lambda_1. \end{aligned} \tag{10}$$

Now substituting (10) into (9) yields:

$$\begin{cases} y_1'' + \alpha^2 y_1 = \\ \quad E \cos(nt) + f[B_1\beta^2 \cos(\beta t) - B_2 \beta^2 \sin(\beta t)] \\ \quad + \sin(\alpha t)[\frac{1}{4}A_1 A_2^2 \alpha\varepsilon + \frac{1}{4}A_1^3 \alpha\varepsilon - A_1\alpha\varepsilon - \alpha_1 A_2] \\ \quad + \cos(\alpha t)[-\frac{1}{4}A_2^3 \alpha\varepsilon - \frac{1}{4}A_1^2 A_2 \alpha\varepsilon + A_2\alpha\varepsilon - \alpha_1 A_1] \\ \quad + \sin(3\alpha t)[\frac{1}{4}A_1^3 \alpha\epsilon - \frac{3}{4}A_1 A_2^2 \alpha\varepsilon] \\ \quad + \cos(3\alpha t)[\frac{3}{4}A_1^2 A_2 \alpha\varepsilon - \frac{1}{4}A_2^3\alpha\varepsilon], \\ \\ x_1'' + \beta^2 x_1 = \\ \quad dA_1 \cos(\alpha t) + dA_2 \sin(\alpha t) \\ + \sin(\beta t)[\lambda B_1 \beta - \beta_1 B_2] + \cos(\beta t)[-\lambda B_2 \beta - \beta_1 B_1] \end{cases} \tag{11}$$

No secular term in y_1 and x_1 requires that

$$\begin{cases} \frac{1}{4}A_1 A_2^2 \alpha\varepsilon + \frac{1}{4}A_1^3 \alpha\varepsilon - A_1\alpha\varepsilon - \alpha_1 A_2 = 0, \\ \\ -\frac{1}{4}A_2^3 \alpha\varepsilon - \frac{1}{4}A_1^2 A_2 \alpha\varepsilon + A_2\alpha\varepsilon - \alpha_1 A_1 = 0 \end{cases} \tag{12}$$

and

$$\begin{cases} \lambda B_1 \beta - \beta_1 B_2 = 0, \\ \\ -\lambda B_2 \beta - \beta_1 B_1 = 0 \end{cases} \tag{13}$$

and using (12), (13) and (10), we have

$$\alpha = \beta = 1. \tag{14}$$

and the frequencies of equations are

$$\begin{cases} T_y = \frac{2\pi}{\alpha} = 2\pi, \\ \\ T_x = \frac{2\pi}{\beta} = 2\pi. \end{cases} \tag{15}$$

Furthermore, equation (11) can be simplified as

$$\begin{cases} y_1'' + y_1 = \\ \quad E \cos(nt) + f[B_1 \cos t + B_2 \sin t] \\ \quad + \sin 3t[\frac{1}{4}A_1^3 \epsilon - \frac{3}{4}A_1 A_2^2 \varepsilon] + \cos 3t[\frac{3}{4}A_1^2 A_2 \varepsilon - \frac{1}{4}A_2^3 \varepsilon], \\ \\ x_1'' + x_1 = \\ \quad dA_1 \cos t + dA_2 \sin t \end{cases} \tag{16}$$

Solving equation (16) yields

$$\begin{cases} y_1 = \\ \quad \frac{E}{1-n^2} \cos(nt) + \frac{t}{2}(B_1 f \sin t - B_2 f \cos t) \\ \\ + \sin 3t[\frac{3}{32}A_1 A_2^2 \varepsilon - \frac{1}{32}A_1^3 \varepsilon] + \cos 3t[\frac{1}{32}A_2^3 \varepsilon - \frac{3}{32}A_1^2 A_2 \varepsilon], \\ \\ x_1 = \\ \quad \frac{t}{2}(dA_1 \sin t - dA_2 f \cos t). \end{cases} \tag{17}$$

Now using (17), (8) and (10) we obtain the first order solution for x and y as follows

$$\begin{cases} y = y_0 + y_1 = A_1 \cos t + A_2 \sin t \\ \\ + \frac{E}{1-n^2} \cos(nt) + \frac{t}{2}(B_1 f \sin t - B_2 f \cos t) \\ \\ + \sin 3t[\frac{3}{32}A_1 A_2^2 \varepsilon - \frac{1}{32}A_1^3 \varepsilon] + \cos 3t[\frac{1}{32}A_2^3 \varepsilon - \frac{3}{32}A_1^2 A_2 \varepsilon], \\ \\ x = x_0 + x_1 = \\ \qquad B_1 \cos t + B_2 \sin t + \frac{t}{2}(dA_1 \sin t - dA_2 f \cos t). \end{cases} \tag{18}$$

III. Conclusions

In this study, we have applied the parameter-expansion method to solve the classical Van der Pol oscillator coupled gyroscopically to a linear oscillator equation. The method more efficient than perturbation method [36] for this problem because the method is independent of perturbation parameter assumption and one iteration step provide an approximate solution which is valid for the whole solution domain. One can apply the parameter-expansion method on another nonlinear oscillators, easily. Because, the method can be easily comprehended with only a basic knowledge of advanced calculus. The method by help of Maple, is utter simplicity, and can be easily extended to all kinds of non-linear equations.

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