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An adaptive least-squares mixed finite element method for pseudo-parabolic integro-differential equations

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Abstract—In this article, an adaptive least-squares mixed finite element method is studied for pseudo-parabolic integro-differential equations. The solutions of least-squares mixed weak formulation and mixed finite element are proved. A posteriori error estimator is constructed based on the least-squares functional and the posteriori errors are obtained.

Keywords—Pseudo-parabolic integro-differential equation; least-squares mixed finite element method; Adaptive method; A posteriori error estimates.

I. Introduction

In this article, we consider the following pseudo-parabolic integro-differential equation

$$\begin{cases} u_t - \nabla\cdot(a(x)\nabla u_t + b(x)\nabla u \\ \quad + d(x)\displaystyle\int_0^t \nabla u(x,\tau)d\tau) + qu = f(x,t), & x\in\Omega, \quad t\in J, \\ u(x,t) = 0, & x\in\partial\Omega, \ t\in J, \\ u(x,0) = u_0(x), & x\in\Omega, \end{cases} \tag{1}$$

where Ω is a bounded convex polygonal domain in $R^d(d=1,2,3)$ with $Lipschitz$ continuous boundary $\partial\Omega$, $J=(0,T]$ is the time interval with $0<T<\infty$. $q=q(x)\geq 0$ and $f=f(x,t)$ in (1) are given functions. We shall make the following assumptions on the coefficients c,a,b and d: there exist some positive constants c_*, c^*,a_*, a^*,b_*, b^* and d_*, d^* such that $0<c_*\leq c(x)\leq c^*, 0<a_*\leq a(x)\leq a^*, 0<b_*\leq b(x)\leq b^*, 0<d_*\leq d(x)\leq d^*$.

Evolution integro-differential equations are a class of important evolution partial differential equations, and have a lot of applications in many physical problems, such as the transport of reactive and passive contaminates in aquifers. Ewing et al. [1] studied finite volume element approximations for two-dimensional parabolic integro-differential equations. Lin et al. [2] proposed the Ritz-Volterra projection onto finite element spaces for integro-differential and related equations. H^1-Galerkin mixed element methods were studied for parabolic, pseudo-parabolic, and pseudo-hyperbolic integro-differential equations in [3], [4], [5], [6]. Refs.[7], [8] analysed the space-time finite element methods for second-order and fourth-order parabolic partial integro-differential equations. Refs.[9], [10] studied the least-squares mixed finite element method for parabolic partial integro-differential equations. Cui [11] studied pseudo-parabolic, and pseudo-hyperbolic integro-differential equations, proposed a new Sobolev-Volterra projection. Zhang [12] investigated the superconvergence properties of finite element approximations to parabolic and hyperbolic integro-differential equations.

In recent years, a lot of researchers have studied the least-squares mixed finite element methods for partial differential equations. Luo et al. [13] proposed a nonlinear Galerkin/Petrov-least squares mixed element method for the stationary Navier-Stokes equations. Yang [14] studied the least-square mixed finite element methods for nonlinear parabolic problems. Chen et al. [15] studied least-squares mixed finite element method for degenerate elliptic problems. Duan and Lin [17] studied least-squares mixed finite elements for elasticity. Cai et al. [16] proposed and analysed an adaptive least-squares mixed finite element method for the stress-displacement formulation of linear elasticity Gu and Li [18] studied an adaptive least-squares mixed finite element method for nonlinear parabolic problems. In this article, we study an adaptive least-squares mixed finite element method for the pseudo-parabolic integro-differential equations, and obtain a posteriori error estimates.

II. A Least-Squares Formulation

Introduce the auxiliary variable $\sigma = -(a(x)\nabla u_t + b(x)\nabla u + d(x)\int_0^t \nabla u(x,\tau)d\tau)$ to have

$$\begin{cases} u_t + div\sigma + qu = f, & x\in\Omega, \quad t\in J, \\ \sigma + a(x)\nabla u_t + b(x)\nabla u \\ \quad + d(x)\displaystyle\int_0^t \nabla u(x,\tau)d\tau = 0, & x\in\Omega, \quad t\in J, \\ u = 0, & x\in\partial\Omega, \ t\in J, \\ u(x,0) = u_0(x), & x\in\Omega, \end{cases} \tag{2}$$

We differentiate the second equation of system (2) with respect to time t to obtain

$$\sigma_t = -a(x)\nabla u_{tt} - b(x)\nabla u_t - d(x)\nabla u. \tag{3}$$

Taking $\omega = u_t$, we can derive the following equivalent

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system of (2)

$$\begin{cases} \omega + div\sigma + qu = f, & x \in \Omega, \quad t \in J, \\ \sigma_t + a(x)\nabla\omega_t + b(x)\nabla\omega + d(x)\nabla u = 0, & x \in \Omega, \quad t \in J, \\ \omega - u_t = 0, & x \in \Omega, \quad t \in J, \\ u = 0, & x \in \partial\Omega, \ t \in J, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{4}$$

Using the finite difference to discretize (4) in t. Let $\tau > 0, \tau^n = n\tau, n = 0, 1 \cdots, N = T/\tau$. Let $\omega^n = (u^n - u^{n-1})/\tau, n \geq 1$. We can rewrite the system (4) :

$$\begin{cases} \omega^n + div\sigma^n + qu^n = f^n, & x \in \Omega, \quad t \in J, \\ \dfrac{\sigma^n - \sigma^{n-1}}{\tau} + a(x)\dfrac{\nabla\omega^n - \nabla\omega^{n-1}}{\tau} & \\ + b(x)\nabla\omega^n + d(x)\nabla u^n = 0, & x \in \Omega, \quad t \in J, \\ \omega^n = \dfrac{u^n - u^{n-1}}{\tau}, & x \in \Omega, \ t \in J, \\ u^n = 0, & x \in \partial\Omega, \ t \in J, \\ u^n(x,0) = u_0^n(x), & x \in \Omega, \end{cases} \tag{5}$$

Introduce Sobolev spaces:

$$L \doteq H^1(\Omega) = \{p \in L^2(\Omega) : \nabla p \in L^2(\Omega)^2\},$$
$$H \doteq H(div, \Omega) = \{\mathbf{w} \in L^2(\Omega)^2 : div\mathbf{w} \in L^2(\Omega)\}.$$

We can use the least-squares mixed finite element method for (5). The least-squares minimization problem is: find $(u^n, \sigma^n, \omega^n) \in X := L \times H \times L$ such that

$$I(u^n, \sigma^n, \omega^n) = \inf_{(\nu,\xi,\eta)\in X} I(\nu, \xi, \eta), \tag{6}$$

where I is the quadratic functional defined by

$$\begin{aligned} I(\nu^n, \xi^n, \eta^n) =& \|\eta^n + div\xi^n + q\nu^n - f^n\|_{0,\Omega}^2 \\ &+ \|\frac{\xi^n - \xi^{n-1}}{\tau} + a(x)\frac{\nabla\eta^n - \nabla\eta^{n-1}}{\tau} \\ &+ b(x)\nabla\eta^n + d(x)\nabla\nu^n\|_{0,\Omega}^2. \end{aligned}$$

Set $J^n = [t^{n-1}, t^n]$. The corresponding variational statement is: find $(u^n, \sigma^n, \omega^n) : J^n \to X$

$$B(u^n, \sigma^n, \omega^n; \nu, \xi, \eta) = 0, \ \forall (\nu, \xi, \eta) \in X, \tag{7}$$

where

$$\begin{aligned} &B(u^n, \sigma^n, \omega^n; \nu, \xi, \eta) \\ &= \tau(\omega^n + div\sigma^n + qu^n, \eta + div\xi + q\nu)_{0,\Omega} \\ &+ (\sigma^n - \sigma^{n-1} + a(x)(\nabla\omega^n - \nabla\omega^{n-1}) + \tau b(x)\nabla\omega^n \\ &+ \tau d(x)\nabla u^n, \xi + a(x)\nabla\eta + \tau b(x)\nabla\eta + \tau d(x)\nabla\nu)_{0,\Omega}. \end{aligned} \tag{8}$$

Theorem 2.1: The bilinear form $B(\cdot,\cdot,\cdot;\cdot,\cdot,\cdot)$ is continuous and coercive. That is that there exist positive constants α and β such that

$$\begin{aligned} &B(u^n, \sigma^n, \omega^n; \nu, \xi, \eta) \\ \leq& \alpha(\|\omega^n\|_{0,\Omega}^2 + \|div\sigma^n\|_{0,\Omega}^2 + \|u^n\|_{0,\Omega}^2 + \|\sigma^n\|_{0,\Omega}^2 \\ &+ \|\nabla u^n\|_{0,\Omega}^2 + \|\nabla\omega^n\|_{0,\Omega}^2)^{\frac{1}{2}}(\|\eta\|_{0,\Omega}^2 + \|div\xi\|_{0,\Omega}^2 \\ &+ \|v\|_{0,\Omega}^2 + \|\xi\|_{0,\Omega}^2 + \|\nabla\eta\|_{0,\Omega}^2 + \|\nabla\nu\|_{0,\Omega}^2)^{\frac{1}{2}}, \end{aligned} \tag{9}$$

and

$$\begin{aligned} B(\nu, \xi, \eta; \nu, \xi, \eta) \geq& \beta(\|\eta\|_{0,\Omega}^2 + \|div\xi\|_{0,\Omega}^2 + \|v\|_{0,\Omega}^2 \\ &+ \|\xi\|_{0,\Omega}^2 + \|\nabla\eta\|_{0,\Omega}^2 + \|\nabla\nu\|_{0,\Omega}^2). \end{aligned} \tag{10}$$

Proof: **(I). For the upper bound**

Noting that the functions $a(x), b(x), d(x)$ and $q(x)$ are bounded, we have

$$\begin{aligned} &B(\nu, \xi, \eta; \nu, \xi, \eta) \\ =& \tau(\eta + div\xi + q\nu, \eta + div\xi + q\nu)_{0,\Omega} \\ &+ (\xi + a(x)\nabla\eta + \tau b(x)\nabla\eta + \tau d(x)\nabla\nu, \\ &\xi + a(x)\nabla\eta + \tau b(x)\nabla\eta + \tau d(x)\nabla\nu)_{0,\Omega} \\ =& \tau\|\eta + div\xi + q\nu\|_{0,\Omega}^2 + \|\xi + a(x)\nabla\eta \\ &+ \tau b(x)\nabla\eta + \tau d(x)\nabla\nu\|_{0,\Omega}^2 \\ \leq& c(\|\eta\|_{0,\Omega}^2 + \|div\xi\|_{0,\Omega}^2 + \|v\|_{0,\Omega}^2 + \|\xi\|_{0,\Omega}^2 + \|\nabla\eta\|_{0,\Omega}^2 \\ &+ \|\tau b(x)\nabla\eta\|_{0,\Omega}^2 + \|\tau d(x)\nabla\nu\|_{0,\Omega}^2) \\ \leq& c(\|\eta\|_{0,\Omega}^2 + \|div\xi\|_{0,\Omega}^2 + \|v\|_{0,\Omega}^2 \\ &+ \|\xi\|_{0,\Omega}^2 + \|\nabla\eta\|_{0,\Omega}^2 + \|\nabla\nu\|_{0,\Omega}^2). \end{aligned}$$

Since the bilinear form is symmetric, this is sufficient for the upper bound in Theorem 2.1.

(II).For the lower bound

Note that the bounded functions $a(x), b(x), d(x)$ and $q(x)$ and use the Holder inequality, the Cauchy inequality to get

$$\begin{aligned} &B(\nu, \xi, \eta; \nu, \xi, \eta) \\ =& \tau(\eta + div\xi + q\nu, \eta + div\xi + q\nu)_{0,\Omega} \\ &+ (\xi + a(x)\nabla\eta + \tau b(x)\nabla\eta + \tau d(x)\nabla\nu, \\ &\xi + a(x)\nabla\eta + \tau b(x)\nabla\eta + \tau d(x)\nabla\nu)_{0,\Omega} \\ =& \tau(\eta, \eta)_{0,\Omega} + \tau(div\xi, div\xi)_{0,\Omega} + \tau(q\nu, q\nu)_{0,\Omega} \\ &+ 2\tau(\eta, div\xi)_{0,\Omega} + 2\tau(\eta, q\nu)_{0,\Omega} + 2\tau(div\xi, q\nu)_{0,\Omega} \\ &+ (\xi, \xi)_{0,\Omega} + a^2(x)(\nabla\eta, \nabla\eta)_{0,\Omega} + \tau^2 b^2(x)(\nabla\eta, \nabla\eta)_{0,\Omega} \\ &+ \tau^2 d^2(x)(\nabla\nu, \nabla\nu)_{0,\Omega} + 2(\xi, a(x)\nabla\eta)_{0,\Omega} + 2\tau(\xi, b(x)\nabla\eta)_{0,\Omega} \\ &+ 2\tau(\xi, d(x)\nabla\nu)_{0,\Omega} + 2\tau(a(x)\nabla\eta, b(x)\nabla\eta)_{0,\Omega} \\ &+ 2\tau(a(x)\nabla\eta, d(x)\nabla\nu)_{0,\Omega} + 2\tau^2(b(x)\nabla\eta, d(x)\nabla\nu)_{0,\Omega}. \end{aligned}$$

$$\begin{aligned} \geq& \tau\|\eta\|_{0,\Omega}^2 + \tau\|div\xi\|_{0,\Omega}^2 + c_1\|v\|_{0,\Omega}^2 + \|\xi\|_{0,\Omega}^2 + c_2\|\nabla\eta\|_{0,\Omega}^2 \\ &+ c_3\|\nabla\nu\|_{0,\Omega}^2 - \epsilon_1(\|\eta\|_{0,\Omega}^2 + \|div\xi\|_{0,\Omega}^2) - \epsilon_2(\|\eta\|_{0,\Omega}^2 \\ &+ \|\nu\|_{0,\Omega}^2) - \epsilon_3(\|div\xi\|_{0,\Omega}^2 + \|\nu\|_{0,\Omega}^2) - \epsilon_4(\|\xi\|_{0,\Omega}^2 + \|\nabla\eta\|_{0,\Omega}^2) \\ &- \epsilon_5(\|\xi\|_{0,\Omega}^2 + \|\nabla\eta\|_{0,\Omega}^2) - \epsilon_6(\|\xi\|_{0,\Omega}^2 + \|\nabla\nu\|_{0,\Omega}^2) \\ &- \epsilon_7(\|\nabla\eta\|_{0,\Omega}^2 + \|\nabla\nu\|_{0,\Omega}^2) - \epsilon_8(\|\nabla\eta\|_{0,\Omega}^2 + \|\nabla\nu\|_{0,\Omega}^2) \\ =& (\tau - \epsilon_1 - \epsilon_2)\|\eta\|_{0,\Omega}^2 + (\tau - \epsilon_1 - \epsilon_3)\|div\xi\|_{0,\Omega}^2 \\ &+ (c_1 - \epsilon_2 - \epsilon_3)\|\nu\|_{0,\Omega}^2 + (1 - \epsilon_4 - \epsilon_5 - \epsilon_6)\|\xi\|_{0,\Omega}^2 \\ &+ (c_2 - \epsilon_5 - \epsilon_7 - \epsilon_8)\|\nabla\eta\|_{0,\Omega}^2 + (c_3 - \epsilon_6 - \epsilon_7 - \epsilon_8)\|\nabla\nu\|_{0,\Omega}^2. \end{aligned}$$

So we can select constants $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7, \epsilon_8, c_1, c_2, c_3$ such that

$$B(\nu, \xi, \eta; \nu, \xi, \eta) \geq c\|\eta\|_{0,\Omega}^2,$$
$$B(\nu, \xi, \eta; \nu, \xi, \eta) \geq c\|div\xi\|_{0,\Omega}^2,$$
$$B(\nu, \xi, \eta; \nu, \xi, \eta) \geq c\|\nu\|_{0,\Omega}^2,$$

$$B(\nu,\xi,\eta;\nu,\xi,\eta) \geq c\|\xi\|_{0,\Omega}^2,$$
$$B(\nu,\xi,\eta;\nu,\xi,\eta) \geq c\|\nabla\eta\|_{0,\Omega}^2,$$
$$B(\nu,\xi,\eta;\nu,\xi,\eta) \geq c\|\nabla\nu\|_{0,\Omega}^2,$$

So we obtain

$$B(\nu,\xi,\eta;\nu,\xi,\eta) \geq \beta(\|\eta\|_{0,\Omega}^2 + \|div\xi\|_{0,\Omega}^2 + \|\nu\|_{0,\Omega}^2 + \|\xi\|_{0,\Omega}^2 + \|\nabla\eta\|_{0,\Omega}^2 + \|\nabla\nu\|_{0,\Omega}^2).$$

From the above inequality, we can get the proof of Theorem 2.1. ■

Theorem 2.2: We obtain that Eq.(7) have a unique solution $(u^n,\sigma^n,\omega^n) \in X := L \times H \times L$.

Proof: Use the similar proof to Ref [18] to obtain the conclusion of Theorem 2.2.

III. Finite Element Approximation

Assume the spaces $L_h \subset L, H_h \subset H$. The completely discrete least-squares mixed finite method approximation is: find $(u_h^n,\sigma_h^n,\omega_h^n) \in X_h := L_h \times H_h \times L_h$ such that

$$I(u_h^n,\sigma_h^n,\omega_h^n) = \inf_{(\nu_h,\xi_h,\eta_h)\in X} I(\nu_h,\xi_h,\eta_h). \quad (11)$$

where

$$\begin{aligned} & I(\nu_h^n,\xi_h^n,\eta_h^n) \\ &= \sum_{T\in T_h} (\|\eta_h^n + div\xi_h^n + q\nu_h^n - f^n\|_{0,T}^2 \\ &+ \|\frac{\xi_h^n - \xi_h^{n-1}}{\tau} + a(x)\frac{\nabla\eta_h^n - \nabla\eta_h^{n-1}}{\tau} \\ &+ b(x)\nabla\eta_h^n + d(x)\nabla\nu_h^n\|_{0,T}^2). \end{aligned}$$

We defined the bilinear form $B(\cdot,\cdot,\cdot;\cdot,\cdot,\cdot)$ as follows

$$\begin{aligned} & B(u_h^n,\sigma_h^n,\omega_h^n;\nu_h,\xi_h,\eta_h) \\ &= \sum_{T\in T_h} (\tau(\omega_h^n + div\sigma_h^n + qu_h^n, \eta_h + div\xi_h + q\nu_h)_{0,T} \\ &+ (\sigma_h^n - \sigma_h^{n-1} + a(x)(\nabla\omega_h^n - \nabla\omega_h^{n-1}) \\ &+ \tau b(x)\nabla\omega_h^n + \tau d(x)\nabla u_h^n, \\ & \xi_h + a(x)\nabla\eta_h + \tau b(x)\nabla\eta_h + \tau d(x)\nabla\nu_h)_{0,T}). \end{aligned} \quad (12)$$

■

Theorem 3.1: The bilinear $B(\cdot,\cdot,\cdot;\cdot,\cdot,\cdot)$ is continuous and coercive, that is that there exist positive constants α_h and β_h such that

$$\begin{aligned} & B(u_h^n,\sigma_h^n,\omega_h^n;\nu_h,\xi_h,\eta_h) \\ &\leq \alpha_h(\sum_{T\in T_h} \|\omega_h^n\|_{0,T}^2 + \|div\sigma_h^n\|_{0,T}^2 + \|u_h^n\|_{0,T}^2 + \|\sigma_h^n\|_{0,T}^2 \\ &+ \|\nabla u_h^n\|_{0,T}^2 + \|\nabla\omega_h^n\|_{0,T}^2)^{\frac{1}{2}}(\sum_{T\in T_h} \|\eta_h\|_{0,T}^2 + \|div\xi_h\|_{0,T}^2 \\ &+ \|v_h\|_{0,T}^2 + \|\xi_h\|_{0,T}^2 + \|\nabla\eta_h\|_{0,T}^2 + \|\nabla\nu_h\|_{0,T}^2)^{\frac{1}{2}}, \end{aligned} \quad (13)$$

and

$$\begin{aligned} & B(\nu_h,\xi_h,\eta_h;\nu_h,\xi_h,\eta_h) \\ &\geq \beta_h \sum_{T\in T_h} (\|\eta_h\|_{0,\Omega}^2 + \|div\xi_h\|_{0,\Omega}^2 + \|v_h\|_{0,\Omega}^2 \\ &+ \|\xi_h\|_{0,\Omega}^2 + \|\nabla\eta_h\|_{0,\Omega}^2 + \|\nabla\nu_h\|_{0,\Omega}^2). \end{aligned} \quad (14)$$

which holds for all $(u_h^n,\sigma_h^n,\omega_h^n) \in X_h := L_h \times H_h \times L_h, (\nu_h,\xi_h,\eta_h) \in X_h := L_h \times H_h \times L_h$.

Proof: **(I). For the upper bound**

Since the functions $a(x), b(x), d(x)$ and $q(x)$ are bounded, we have

$$\begin{aligned} & B(\nu_h,\xi_h,\eta_h;\nu_h,\xi_h,\eta_h) \\ &= \sum_{T\in T_h} (\tau(\eta_h + div\xi_h + q\nu_h, \eta_h + div\xi_h + q\nu_h)_{0,T} \\ &+ (\xi_h + a(x)\nabla\eta_h + \tau b(x)\nabla\eta_h + \tau d(x)\nabla\nu_h, \\ & \xi_h + a(x)\nabla\eta_h + \tau b(x)\nabla\eta_h + \tau d(x)\nabla\nu_h)_{0,T}) \\ &= \sum_{T\in T_h} (\tau\|\eta_h + div\xi_h + q\nu_h\|_{0,T}^2 + \|\xi_h + a(x)\nabla\eta_h \\ &+ \tau b(x)\nabla\eta_h + \tau d(x)\nabla\nu_h\|_{0,T}^2) \\ &\leq c\sum_{T\in T_h} (\|\eta_h\|_{0,T}^2 + \|div\xi_h\|_{0,T}^2 + \|v_h\|_{0,T}^2 + \|\xi_h\|_{0,T}^2 \\ &+ \|\nabla\eta_h\|_{0,T}^2 + \|\tau b(x)\nabla\eta_h\|_{0,T}^2 + \|\tau d(x)\nabla\nu_h\|_{0,T}^2) \\ &\leq c\sum_{T\in T_h} (\|\eta_h\|_{0,T}^2 + \|div\xi_h\|_{0,T}^2 + \|v_h\|_{0,T}^2 \\ &+ \|\xi_h\|_{0,T}^2 + \|\nabla\eta_h\|_{0,T}^2 + \|\nabla\nu_h\|_{0,T}^2). \end{aligned}$$

Since the bilinear form is symmetric, this is sufficient for the upper bound in Theorem 3.1.

(II). For the lower bound

Note that the bounded functions $a(x), b(x), d(x)$ and $q(x)$ and use the Holder inequality, the Cauchy inequality to get

$$\begin{aligned} & B(\nu_h,\xi_h,\eta_h;\nu_h,\xi_h,\eta_h) \\ &= \sum_{T\in T_h} (\tau(\eta_h + div\xi_h + q\nu_h, \eta_h + div\xi_h + q\nu_h)_{0,T} \\ &+ (\xi_h + a(x)\nabla\eta_h + \tau b(x)\nabla\eta_h + \tau d(x)\nabla\nu_h, \\ & \xi_h + a(x)\nabla\eta_h + \tau b(x)\nabla\eta_h + \tau d(x)\nabla\nu_h)_{0,T}) \\ &= \sum_{T\in T_h} (\tau(\eta_h,\eta_h)_{0,T} + \tau(div\xi_h, div\xi_h)_{0,T} + \tau(q\nu_h, q\nu_h)_{0,T} \\ &+ 2\tau(\eta_h, div\xi_h)_{0,T} + 2\tau(\eta_h, q\nu_h)_{0,T} + 2\tau(div\xi_h, q\nu_h)_{0,T} \\ &+ (\xi_h,\xi_h)_{0,T} + a^2(x)(\nabla\eta_h,\nabla\eta_h)_{0,T} \\ &+ \tau^2 b^2(x)(\nabla\eta_h,\nabla\eta_h)_{0,T} + \tau^2 d^2(x)(\nabla\nu_h,\nabla\nu_h)_{0,T} \\ &+ 2(\xi_h, a(x)\nabla\eta_h)_{0,T} + 2\tau(\xi_h, b(x)\nabla\eta_h)_{0,T} \\ &+ 2\tau(\xi_h, d(x)\nabla\nu_h)_{0,T} + 2\tau(a(x)\nabla\eta_h, b(x)\nabla\eta_h)_{0,T} \\ &+ 2\tau(a(x)\nabla\eta_h, d(x)\nabla\nu_h)_{0,T} + 2\tau^2(b(x)\nabla\eta_h, d(x)\nabla\nu_h)_{0,T}) \\ &\geq \sum_{T\in T_h} (\tau\|\eta_h\|_{0,T}^2 + \tau\|div\xi_h\|_{0,T}^2 + \tilde{c_1}\|v_h\|_{0,T}^2 + \|\xi_h\|_{0,T}^2 \\ &+ \tilde{c_2}\|\nabla\eta_h\|_{0,T}^2 + \tilde{c_3}\|\nabla\nu_h\|_{0,T}^2 - \tilde{\epsilon_1}(\|\eta_h\|_{0,T}^2 + \|div\xi_h\|_{0,T}^2) \\ &- \tilde{\epsilon_2}(\|\eta_h\|_{0,T}^2 + \|\nu_h\|_{0,T}^2) - \tilde{\epsilon_3}(\|div\xi_h\|_{0,T}^2 + \|\nu_h\|_{0,T}^2) \end{aligned}$$

$$
\begin{aligned}
&- \tilde{\epsilon_4}(\|\xi_h\|^2_{0,T} + \|\nabla\eta_h\|^2_{0,T}) - \tilde{\epsilon_5}(\|\xi_h\|^2_{0,T} + \|\nabla\eta_h\|^2_{0,T})\\
&- \tilde{\epsilon_6}(\|\xi_h\|^2_{0,T} + \|\nabla\nu_h\|^2_{0,T}) - \tilde{\epsilon_7}(\|\nabla\eta_h\|^2_{0,T} + \|\nabla\nu_h)\|^2_{0,T}\\
&- \tilde{\epsilon_8}(\|\nabla\eta_h\|^2_{0,T} + \|\nabla\nu_h\|^2_{0,T}))\\
= &\sum_{T\in T_h}((\tau - \tilde{\epsilon_1} - \tilde{\epsilon_2})\|\eta_h\|^2_{0,T} + (\tau - \tilde{\epsilon_1} - \tilde{\epsilon_3})\|div\xi_h\|^2_{0,T}\\
&+ (\tilde{c_1} - \tilde{\epsilon_2} - \tilde{\epsilon_3})\|\nu_h\|^2_{0,T} + (1 - \tilde{\epsilon_4} - \tilde{\epsilon_5} - \tilde{\epsilon_6})\|\xi_h\|^2_{0,T}\\
&+ (\tilde{c_2} - \tilde{\epsilon_5} - \tilde{\epsilon_7} - \tilde{\epsilon_8})\|\nabla\eta_h\|^2_{0,T}\\
&+ (\tilde{c_3} - \tilde{\epsilon_6} - \tilde{\epsilon_7} - \tilde{\epsilon_8})\|\nabla\nu_h\|^2_{0,T}).
\end{aligned}
$$

So we can select constants $\tilde{\epsilon_1}, \tilde{\epsilon_2}, \tilde{\epsilon_3}, \tilde{\epsilon_4}, \tilde{\epsilon_5}, \tilde{\epsilon_6}, \tilde{\epsilon_7}, \tilde{\epsilon_8}, \tilde{c_1}, \tilde{c_2}, \tilde{c_3}$ such that

$$
\begin{aligned}
B(\nu,\xi,\eta;\nu,\xi,\eta) &\ge c\sum_{T\in T_h}\|\eta_h\|^2_{0,T},\\
B(\nu,\xi,\eta;\nu,\xi,\eta) &\ge c\sum_{T\in T_h}\|div\xi_h\|^2_{0,T},\\
B(\nu,\xi,\eta;\nu,\xi,\eta) &\ge c\sum_{T\in T_h}\|\nu_h\|^2_{0,T},\\
B(\nu,\xi,\eta;\nu,\xi,\eta) &\ge c\sum_{T\in T_h}\|\xi_h\|^2_{0,T},\\
B(\nu,\xi,\eta;\nu,\xi,\eta) &\ge c\sum_{T\in T_h}\|\nabla\eta_h\|^2_{0,T},\\
B(\nu,\xi,\eta;\nu,\xi,\eta) &\ge c\sum_{T\in T_h}\|\nabla\nu_h\|^2_{0,T},
\end{aligned}
$$

So we can obtain

$$
\begin{aligned}
&B(\nu_h,\xi_h,\eta_h;\nu_h,\xi_h,\eta_h)\\
\ge&\beta_h\sum_{T\in T_h}(\|\eta_h\|^2_{0,T} + \|div\xi_h\|^2_{0,T} + \|\nu_h\|^2_{0,T}\\
&+ \|\xi_h\|^2_{0,T} + \|\nabla\eta_h\|^2_{0,T} + \|\nabla\nu_h\|^2_{0,T}).
\end{aligned}
$$

We can obtain the proof for Theorem 3.1. ■

Theorem 3.2: It is easy to show that Eq.(12) have a unique solution $(u_h^n, \sigma_h^n, \omega_h^n) \in X_h := L_h \times H_h \times L_h$.

IV. The Least-Squares Functional of an A Posteriori Error Estimator

In this section, we use the similar method to Ref [18]. From the least-squares functional

$$
\begin{aligned}
I(u_h^n,\sigma_h^n,\omega_h^n) = &\sum_{T\in T_h}(\|\omega_h^n + div\sigma_h^n + qu_h^n - f^n\|^2_{0,T}\\
&+ \|\frac{\sigma_h^n - \sigma_h^{n-1}}{\tau} + a(x)\frac{\nabla\omega_h^n - \nabla\omega_h^{n-1}}{\tau}\\
&+ b(x)\nabla\omega_h^n + d(x)\nabla u_h^n\|^2_{0,T}).
\end{aligned}
$$

So we obtain

$$
\begin{aligned}
&B(u_h^n - u^n, \sigma_h^n - \sigma^n, \omega_h^n - \omega^n; u_h^n - u^n, \sigma_h^n - \sigma^n, \omega_h^n - \omega^n)\\
=&\sum_{T\in T_h}(\|\omega_h^n - \omega^n + div(\sigma_h^n - \sigma^n) + q(u_h^n - u^n)\|^2_{0,T}\\
&+ \|\frac{\sigma_h^n - \sigma^n - (\sigma_h^{n-1} - \sigma^{n-1})}{\tau}\\
&+ a(x)\frac{\nabla(\omega_h^n - \omega^n) - \nabla(\omega_h^{n-1} - \omega^{n-1})}{\tau}\\
&+ b(x)\nabla(\omega_h^n - \omega^n) + d(x)\nabla(u_h^n - u^n)\|^2_{0,T}).
\end{aligned}
$$

For obtaining the definition of the a posteriori estimator, we can see that the least-squares functional is the sum of its element-wise contributions

$$
\begin{aligned}
I(u_h^n,\sigma_h^n,\omega_h^n) = &\sum_{T\in T_h}(\|\omega_h^n + div\sigma_h^n + qu_h^n - f^n\|^2_{0,T}\\
&+ \|\frac{\sigma_h^n - \sigma_h^{n-1}}{\tau} + a(x)\frac{\nabla\omega_h^n - \nabla\omega_h^{n-1}}{\tau}\\
&+ b(x)\nabla\omega_h^n + d(x)\nabla u_h^n\|^2_{0,T}) =: \sum_{T\in T_h}\theta_T^2.
\end{aligned}
$$

So we can obtain the following theorem.

Theorem 4.1: The element-wise computation of the least-squares functional constitutes an a posteriori error estimator. That is, for

$$
\begin{aligned}
&\|\omega_h^n + div\sigma_h^n + qu_h^n - f^n\|^2_{0,T} + \|\frac{\sigma_h^n - \sigma_h^{n-1}}{\tau}\\
&+ a(x)\frac{\nabla\omega_h^n - \nabla\omega_h^{n-1}}{\tau} + b(x)\nabla\omega_h^n + d(x)\nabla u_h^n\|^2_{0,T} = \theta_T^2.
\end{aligned}
$$

there exist positive constants α_T and β_T such that

$$
\begin{aligned}
\sum_{T\in T_h}\theta_T^2 \ge&\beta_T\sum_{T\in T_h}(\|\omega_h^n - \omega^n\|^2_{0,T} + \|div(\sigma_h^n - \sigma^n)\|^2_{0,T}\\
&+ \|u_h^n - u^n\|^2_{0,T} + \|\sigma_h^n - \sigma^n\|^2_{0,T}\\
&+ \|\nabla(\omega_h^n - \omega^n)\|^2_{0,T} + \|\nabla(u_h^n - u^n)\|^2_{0,T}),
\end{aligned}
$$

$$
\begin{aligned}
\sum_{T\in T_h}\theta_T^2 \le&\alpha_T\sum_{T\in T_h}(\|\omega_h^n - \omega^n\|^2_{0,T} + \|div(\sigma_h^n - \sigma^n)\|^2_{0,T}\\
&+ \|u_h^n - u^n\|^2_{0,T} + \|\sigma_h^n - \sigma^n\|^2_{0,T}\\
&+ \|\nabla(\omega_h^n - \omega^n)\|^2_{0,T} + \|\nabla(u_h^n - u^n)\|^2_{0,T}).
\end{aligned}
$$

where α_T and β_T are the same constants as in Theorem 3.1.

V. Error Estimates

In this section, we assume that the spaces L_h, H_h can be the spaces of Raviart-Thomas for $r \ge 0$. The completely discrete least-squares mixed finite element approximation is: find $(u_h^n, \sigma_h^n, \omega_h^n) \in X_h := L_h \times H_h \times L_h$ such that

$$
\begin{aligned}
&B(u_h^n,\sigma_h^n,\omega_h^n;\nu_h,\xi_h,\eta_h)\\
=&\tau(\omega_h^n + div\sigma_h^n + qu_h^n, \eta_h + div\xi_h + q\nu_h)_{0,\Omega}\\
&+ (\sigma_h^n - \sigma_h^{n-1} + a(x)(\nabla\omega_h^n - \nabla\omega_h^{n-1}) + \tau b(x)\nabla\omega_h^n\\
&+ \tau d(x)\nabla u_h^n, \xi_h + a(x)\nabla\eta_h + \tau b(x)\nabla\eta_h + \tau d(x)\nabla\nu_h)_{0,\Omega}.
\end{aligned}
$$

We shall split the error as a sum of two terms,

$$u - u_h = (u - \tilde{u}) + (\tilde{u} - u_h).$$

where $\tilde{u}$ is an elliptic projection in L_h of the exact solution u. The projection $\tilde{u} = \tilde{u}(t)$ is defined by

$$(d(x)\nabla(u - \tilde{u}), \nabla\chi) = 0, \forall\chi \in L_h. \qquad (15)$$

It is known that the projection $(\tilde{u}, \tilde{\sigma}, \tilde{\omega}) \in X_h := L_h \times H_h \times L_h$ exits and has the following properties [16], [18], for $1 \le r \le k+1$

$$\|u - \tilde{u}\| \le ch^r\|u\|_r,$$

$$\|(u-\tilde{u})_t\| \le ch^r(\|u\|_r + \|u_t\|_r),$$

$$\|\sigma - \tilde{\sigma}\| \le ch^r\|\sigma\|_r,$$

$$\|div(\sigma - \tilde{\sigma})\| \le ch^r\|\sigma\|_{r+1},$$

$$\|(\sigma-\tilde{\sigma})_t\| \le ch^r(\|\sigma\|_r + \|\sigma_t\|_r),$$

From the bilinear form $B(\cdot,\cdot,\cdot;\cdot,\cdot,\cdot)$ in section 2, we have

$$\begin{aligned}
&B(u^n,\sigma^n,\omega^n;\nu,\xi,\eta)\\
=&\tau(\omega^n + div\sigma^n + qu^n, \eta + div\xi + q\nu)_{0,\Omega}\\
&+(\sigma^n - \sigma^{n-1} + a(x)(\nabla\omega^n - \nabla\omega^{n-1})\\
&+\tau b(x)\nabla\omega^n + \tau d(x)\nabla u^n, \xi + a(x)\nabla\eta\\
&+\tau b(x)\nabla\eta + \tau d(x)\nabla\nu)_{0,\Omega}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&B(u_h^n - u^n, \sigma_h^n - \sigma^n, \omega_h^n - \omega^n; \nu_h, \xi_h, \eta_h)\\
=&\tau(\omega^n - \omega_h^n + div(\sigma^n - \sigma_h^n) + q(u^n - u_h^n),\\
&\eta_h + div\xi_h + q\nu_h)_{0,\Omega} + (\sigma^n - \sigma_h^n - (\sigma^{n-1} - \sigma_h^{n-1})\\
&+a(x)(\nabla(\omega^n - \omega_h^n) - \nabla(\omega^{n-1} - \omega_h^{n-1}))\\
&+\tau b(x)\nabla(\omega^n - \omega_h^n) + \tau d(x)\nabla(u^n - u_h^n),\\
&\xi_h + a(x)\nabla\eta_h + \tau b(x)\nabla\eta_h + \tau d(x)\nabla\nu_h)_{0,\Omega} = 0.
\end{aligned}$$

Let $\theta = \sigma - \tilde{\sigma}, \pi = \sigma_h - \tilde{\sigma}, \rho = u - \tilde{u}, \varepsilon = u_h - \tilde{u}, \rho_t = \omega - \tilde{\omega}, \varepsilon_t = \omega_h - \tilde{\omega}$. We shall use the following notation for the difference operator:

$$\bar{\partial}_t f^n = \frac{f^n - f^{n-1}}{\triangle t},$$

We then have the following estimate:

$$\|\bar{\partial}_t\rho^n\|^2 \le \frac{1}{\triangle t}\int_{t_{n-1}}^{t_n}\|\rho_t\|^2 dt,$$

Lemma 5.1: With $\tilde{u}$ difined in (15), we have

$$\|\nabla\tilde{u}\|_{L\infty} \le c, \forall t \in J$$

$$\|\nabla\tilde{u}_{tt}\| \le c, \forall t \in J$$

Please see [16], [18].

From the above equation $B(u^n - u_h^n, \sigma^n - \sigma_h^n, \omega^n - \omega_h^n; \nu_h, \xi_h, \eta_h)$ and the boundedness of $a(x), b(x), d(x), q(x)$, we can get

$$\begin{aligned}
&B(u^n - u_h^n, \sigma^n - \sigma_h^n, \omega^n - \omega_h^n; \nu_h, \xi_h, \eta_h)\\
=&\tau(\rho_t^n - \varepsilon_t^n + div(\theta^n - \pi^n) + q(\rho^n - \varepsilon^n),\\
&\eta_h + div\xi_h + q\nu_h)_{0,\Omega} + (\theta^n - \pi^n - (\theta^{n-1} - \pi^{n-1})\\
&+a(x)(\nabla(\rho_t^n - \varepsilon_t^n) - \nabla(\rho_t^{n-1} - \varepsilon_t^{n-1}))\\
&+\tau b(x)\nabla(\rho_t^n - \varepsilon_t^n) + \tau d(x)\nabla(\rho^n - \varepsilon^n),\\
&\xi_h + a(x)\nabla\eta_h + \tau b(x)\nabla\eta_h + \tau d(x)\nabla\nu_h)_{0,\Omega}\\
&=0.
\end{aligned}$$

Take $\xi_h = \pi^n, \nu_h = \varepsilon^n, \eta_h = \varepsilon_t^n$ to get

$$\begin{aligned}
&B(u^n - u_h^n, \sigma^n - \sigma_h^n, \omega^n - \omega_h^n; \varepsilon^n, \pi^n, \varepsilon_t^n)\\
=&\tau(\rho_t^n - \varepsilon_t^n + div(\theta^n - \pi^n) + q(\rho^n - \varepsilon^n),\\
&\varepsilon_t^n + div\pi^n + q\varepsilon^n)_{0,\Omega} + (\theta^n - \pi^n - (\theta^{n-1} - \pi^{n-1})\\
&+a(x)(\nabla(\rho_t^n - \varepsilon_t^n) - \nabla(\rho_t^{n-1} - \varepsilon_t^{n-1}))\\
&+\tau b(x)\nabla(\rho_t^n - \varepsilon_t^n) + \tau d(x)\nabla(\rho^n - \varepsilon^n),\\
&\pi^n + a(x)\nabla\varepsilon_t^n + \tau b(x)\nabla\varepsilon_t^n + \tau d(x)\nabla\varepsilon^n)_{0,\Omega}\\
=&0.
\end{aligned}$$

The left side of the above equation is

$$\begin{aligned}
I_3 =&\tau(\rho_t^n + div\theta^n + q\rho^n, \varepsilon_t^n + div\pi^n + q\varepsilon^n)_{0,\Omega}\\
&+(\theta^n - \theta^{n-1} + a(x)\nabla(\rho_t^n - \rho_t^{n-1})\\
&+\tau b(x)\nabla\rho_t^n + \tau d(x)\nabla\rho^n, \pi^n + a(x)\nabla\varepsilon_t^n\\
&+\tau b(x)\nabla\varepsilon_t^n + \tau d(x)\nabla\varepsilon^n)_{0,\Omega}.
\end{aligned}\tag{16}$$

The right side of the above equation is

$$\begin{aligned}
&I_1 + I_2\\
&= \tau(\varepsilon_t^n + div\pi^n + q\varepsilon^n, \varepsilon_t^n + div\pi^n + q\varepsilon^n)_{0,\Omega}\\
&+(\pi^n - \pi^{n-1} + a(x)\nabla(\varepsilon_t^n - \varepsilon_t^{n-1}) + \tau b(x)\nabla\varepsilon_t^n\\
&+\tau d(x)\nabla\varepsilon^n, \pi^n + a(x)\nabla\varepsilon_t^n + \tau b(x)\nabla\varepsilon_t^n + \tau d(x)\nabla\varepsilon^n)_{0,\Omega}.
\end{aligned}\tag{17}$$

From the boundedness of $a(x), b(x), d(x)$ and $q(x)$, we obtain

$$\begin{aligned}
I_2 =&c(\pi^n + \nabla\varepsilon_t^n + \tau\nabla\varepsilon_t^n + \tau\nabla\varepsilon^n, \pi^n + \nabla\varepsilon_t^n\\
&+\tau\nabla\varepsilon_t^n + \tau\nabla\varepsilon^n)_{0,\Omega} - c(\pi^{n-1} + \nabla\varepsilon_t^{n-1}, \pi^n\\
&+\nabla\varepsilon_t^n + \tau\nabla\varepsilon_t^n + \tau\nabla\varepsilon^n)_{0,\Omega}\\
\ge&\frac{c}{2}[(\pi^n + \nabla\varepsilon_t^n + \tau\nabla\varepsilon_t^n + \tau\nabla\varepsilon^n, \pi^n\\
&+\nabla\varepsilon_t^n + \tau\nabla\varepsilon_t^n + \tau\nabla\varepsilon^n)_{0,\Omega}\\
&-(\pi^{n-1}, \pi^{n-1})_{0,\Omega} - (\nabla\varepsilon_t^{n-1}, \nabla\varepsilon_t^{n-1})_{0,\Omega}]\\
=&\frac{c}{2}[(\pi^n, \pi^n)_{0,\Omega} - (\pi^{n-1}, \pi^{n-1})_{0,\Omega} + (\nabla\varepsilon_t^n, \nabla\varepsilon_t^n)_{0,\Omega}\\
&-(\nabla\varepsilon_t^{n-1}, \nabla\varepsilon_t^{n-1})_{0,\Omega} + 2(\pi^n, \nabla\varepsilon_t^n + \tau\nabla\varepsilon_t^n\\
&+\tau\nabla\varepsilon^n)_{0,\Omega} + 2(\nabla\varepsilon_t^n, \pi^n + \tau\nabla\varepsilon_t^n + \tau\nabla\varepsilon^n)_{0,\Omega}\\
&+(\nabla\varepsilon_t^n + \tau\nabla\varepsilon_t^n + \tau\nabla\varepsilon^n, \nabla\varepsilon_t^n + \tau\nabla\varepsilon_t^n + \tau\nabla\varepsilon^n)_{0,\Omega}\\
&+(\pi^n + \tau\nabla\varepsilon_t^n + \tau\nabla\varepsilon^n, \pi^n + \tau\nabla\varepsilon_t^n + \tau\nabla\varepsilon^n)_{0,\Omega}].
\end{aligned}$$

Use $I_1 \ge \frac{c}{2}I_1, (c \le 2)$ to get

$$\begin{aligned}
&I_1 + I_2\\
\ge&\frac{c}{2}[(\pi^n, \pi^n)_{0,\Omega} - (\pi^{n-1}, \pi^{n-1})_{0,\Omega} + (\nabla\varepsilon_t^n, \nabla\varepsilon_t^n)_{0,\Omega}\\
&-(\nabla\varepsilon_t^{n-1}, \nabla\varepsilon_t^{n-1})_{0,\Omega} + 2(\pi^n, \nabla\varepsilon_t^n + \tau\nabla\varepsilon_t^n + \tau\nabla\varepsilon^n)_{0,\Omega}\\
&+2(\nabla\varepsilon_t^n, \pi^n + \tau\nabla\varepsilon_t^n + \tau\nabla\varepsilon^n)_{0,\Omega} + (\nabla\varepsilon_t^n + \tau\nabla\varepsilon_t^n\\
&+\tau\nabla\varepsilon^n, \nabla\varepsilon_t^n + \tau\nabla\varepsilon_t^n + \tau\nabla\varepsilon^n)_{0,\Omega}\\
&+(\pi^n + \tau\nabla\varepsilon_t^n + \tau\nabla\varepsilon^n, \pi^n + \tau\nabla\varepsilon_t^n + \tau\nabla\varepsilon^n)_{0,\Omega}\\
&+\tau(\varepsilon_t^n + div\pi^n + q\varepsilon^n, \varepsilon_t^n + div\pi^n + q\varepsilon^n)_{0,\Omega}]\\
=&I_4.
\end{aligned}$$

Summing from $n=0$ to N, we obtain

$$\sum_{n=0}^{N} I_4 = c(\|\pi^N\|_{0,\Omega}^2 - \|\pi^0\|_{0,\Omega}^2) + c(\|\nabla\varepsilon_t^N\|_{0,\Omega}^2 - \|\nabla\varepsilon_t^0\|_{0,\Omega}^2) + c\sum_{n=0}^{N}(\|\pi^n\|_{0,\Omega}^2 + \|\nabla\varepsilon_t^n\|_{0,\Omega}^2 + \tau^2\|\nabla\varepsilon^n\|_{0,\Omega}^2 + \tau^2\|\nabla\varepsilon_t^n\|_{0,\Omega}^2) \le \sum_{n=0}^{N} I_3$$

Let $\|\pi^{N^*}\|_{0,\Omega}^2$ be a maximum of the $\|\pi^n\|_{0,\Omega}^2$ and $\|\nabla\varepsilon_t^{N^*}\|_{0,\Omega}^2$ be a maximum of the $\|\nabla\varepsilon_t^n\|_{0,\Omega}^2$, we obtain

$$\sum_{n=0}^{N} I_4 = c(\|\pi^N\|_{0,\Omega}^2 + \|\nabla\varepsilon_t^N\|_{0,\Omega}^2) + c\sum_{n=0}^{N}(\|\pi^n\|_{0,\Omega}^2 + \|\nabla\varepsilon_t^n\|_{0,\Omega}^2 + \tau^2\|\nabla\varepsilon^n\|_{0,\Omega}^2 + \tau^2\|\nabla\varepsilon_t^n\|_{0,\Omega}^2). \quad (18)$$

$$\begin{aligned} I_3 =& \tau(\rho_t^n + div\theta^n + q\rho^n, \varepsilon_t^n + div\pi^n + q\varepsilon^n)_{0,\Omega} \\ &+ (\theta^n - \theta^{n-1} + a(x)\nabla(\rho_t^n - \rho_t^{n-1}) \\ &+ \tau b(x)\nabla\rho_t^n + \tau d(x)\nabla\rho^n, \pi^n + a(x)\nabla\varepsilon_t^n \\ &+ \tau b(x)\nabla\varepsilon_t^n + \tau d(x)\nabla\varepsilon^n)_{0,\Omega} \end{aligned}$$

$$\begin{aligned} =& \tau(\rho_t^n, \varepsilon_t^n)_{0,\Omega} + \tau(\rho_t^n, div\pi^n)_{0,\Omega} + \tau(\rho_t^n, q\varepsilon^n)_{0,\Omega} \\ &+ \tau(div\theta^n, \varepsilon_t^n)_{0,\Omega} + \tau(div\theta^n, div\pi^n)_{0,\Omega} \\ &+ \tau(div\theta^n, q\varepsilon^n)_{0,\Omega} + \tau(q\rho^n, \varepsilon_t^n)_{0,\Omega} \\ &+ \tau(q\rho^n, div\pi^n)_{0,\Omega} + \tau(q\rho^n, q\varepsilon^n)_{0,\Omega} \\ &+ (\theta^n - \theta^{n-1}, \pi^n)_{0,\Omega} + (\theta^n - \theta^{n-1}, a(x)\nabla\varepsilon_t^n)_{0,\Omega} \\ &+ (\theta^n - \theta^{n-1}, \tau b(x)\nabla\varepsilon_t^n)_{0,\Omega} + (\theta^n - \theta^{n-1}, \tau d(x)\nabla\varepsilon^n)_{0,\Omega} \\ &+ (a(x)\nabla(\rho_t^n - \rho_t^{n-1}), \pi^n)_{0,\Omega} \\ &+ (a(x)\nabla(\rho_t^n - \rho_t^{n-1}), a(x)\nabla\varepsilon_t^n)_{0,\Omega} \\ &+ (a(x)\nabla(\rho_t^n - \rho_t^{n-1}), \tau b(x)\nabla\varepsilon_t^n)_{0,\Omega} \\ &+ (a(x)\nabla(\rho_t^n - \rho_t^{n-1}), \tau d(x)\nabla\varepsilon^n)_{0,\Omega} \\ &+ (\tau b(x)\nabla\rho_t^n, \pi^n)_{0,\Omega} + (\tau b(x)\nabla\rho_t^n, a(x)\nabla\varepsilon_t^n)_{0,\Omega} \\ &+ (\tau b(x)\nabla\rho_t^n, \tau b(x)\nabla\varepsilon_t^n)_{0,\Omega} \\ &+ (\tau b(x)\nabla\rho_t^n, \tau d(x)\nabla\varepsilon^n)_{0,\Omega} + (\tau d(x)\nabla\rho^n, \pi^n)_{0,\Omega} \\ &+ (\tau d(x)\nabla\rho^n, a(x)\nabla\varepsilon_t^n)_{0,\Omega} + (\tau d(x)\nabla\rho^n, \tau b(x)\nabla\varepsilon_t^n)_{0,\Omega} \\ &+ (\tau d(x)\nabla\rho^n, \tau d(x)\nabla\varepsilon^n)_{0,\Omega}. \end{aligned}$$

From (15) and applying Green's formula, we know that

$$\begin{aligned} &(\nabla\rho_t^n, \pi^n)_{0,\Omega} = -(\rho_t^n, div\pi^n)_{0,\Omega}, \\ &(\tau d(x)\nabla\rho^n, a(x)\nabla\varepsilon_t^n)_{0,\Omega} = 0, \end{aligned}$$

$$\begin{aligned} &(\tau d(x)\nabla\rho^n, \tau b(x)\nabla\varepsilon_t^n)_{0,\Omega} = 0, \\ &(\tau d(x)\nabla\rho^n, \tau d(x)\nabla\varepsilon^n)_{0,\Omega} = 0. \end{aligned}$$

So we obtain

$$\begin{aligned} I_3 =& \tau(\rho_t^n, \varepsilon_t^n)_{0,\Omega} + \tau(\rho_t^n, q\varepsilon^n)_{0,\Omega} + \tau(div\theta^n, \varepsilon_t^n)_{0,\Omega} \\ &+ \tau(div\theta^n, div\pi^n)_{0,\Omega} + \tau(div\theta^n, q\varepsilon^n)_{0,\Omega} \\ &+ \tau(q\rho^n, \varepsilon_t^n)_{0,\Omega} + \tau(q\rho^n, div\pi^n)_{0,\Omega} + \tau(q\rho^n, q\varepsilon^n)_{0,\Omega} \\ &+ (\theta^n - \theta^{n-1}, \pi^n)_{0,\Omega} + (\theta^n - \theta^{n-1}, a(x)\nabla\varepsilon_t^n)_{0,\Omega} \\ &+ (\theta^n - \theta^{n-1}, \tau b(x)\nabla\varepsilon_t^n)_{0,\Omega} + (\theta^n - \theta^{n-1}, \tau d(x)\nabla\varepsilon^n)_{0,\Omega} \\ &+ (a(x)\nabla(\rho_t^n - \rho_t^{n-1}), \pi^n)_{0,\Omega} \\ &+ (a(x)\nabla(\rho_t^n - \rho_t^{n-1}), a(x)\nabla\varepsilon_t^n)_{0,\Omega} \\ &+ (a(x)\nabla(\rho_t^n - \rho_t^{n-1}), \tau b(x)\nabla\varepsilon_t^n)_{0,\Omega} \\ &+ (a(x)\nabla(\rho_t^n - \rho_t^{n-1}), \tau d(x)\nabla\varepsilon^n)_{0,\Omega} \\ &+ (\tau b(x)\nabla\rho_t^n, a(x)\nabla\varepsilon_t^n)_{0,\Omega} \\ &+ (\tau b(x)\nabla\rho_t^n, \tau b(x)\nabla\varepsilon_t^n)_{0,\Omega} \\ &+ (\tau b(x)\nabla\rho_t^n, \tau d(x)\nabla\varepsilon^n)_{0,\Omega} + (\tau d(x)\nabla\rho^n, \pi^n)_{0,\Omega}. \end{aligned}$$

We obtain

$$\begin{aligned} \sum_{n=0}^{N} I_4 =& c(\|\pi^N\|_{0,\Omega}^2 + \|\nabla\varepsilon_t^N\|_{0,\Omega}^2) + c\sum_{n=0}^{N}(\|\pi^n\|_{0,\Omega}^2 \\ &+ \|\nabla\varepsilon_t^n\|_{0,\Omega}^2 + \tau^2\|\nabla\varepsilon^n\|_{0,\Omega}^2 + \tau^2\|\nabla\varepsilon_t^n\|_{0,\Omega}^2) \\ &\le \sum_{i=1}^{20} T_i. \end{aligned}$$

where

$$\begin{aligned} &T_1 = \sum_{n=0}^{N} \tau(\rho_t^n, \varepsilon_t^n)_{0,\Omega}, T_2 = \sum_{n=0}^{N} \tau(\rho_t^n, q\varepsilon^n)_{0,\Omega}, \\ &T_3 = \sum_{n=0}^{N} \tau(div\theta^n, \varepsilon_t^n)_{0,\Omega}, T_4 = \sum_{n=0}^{N} \tau(div\theta^n, div\pi^n)_{0,\Omega}, \\ &T_5 = \sum_{n=0}^{N} \tau(div\theta^n, q\varepsilon^n)_{0,\Omega}, T_6 = \sum_{n=0}^{N} \tau(q\rho^n, \varepsilon_t^n)_{0,\Omega}, \\ &T_7 = \sum_{n=0}^{N} \tau(q\rho^n, div\pi^n)_{0,\Omega}, T_8 = \sum_{n=0}^{N} \tau(q\rho^n, q\varepsilon^n)_{0,\Omega}, \\ &T_9 = \sum_{n=0}^{N} (\theta^n - \theta^{n-1}, \pi^n)_{0,\Omega}, \\ &T_{10} = \sum_{n=0}^{N} (\theta^n - \theta^{n-1}, a(x)\nabla\varepsilon_t^n)_{0,\Omega}, \\ &T_{11} = \sum_{n=0}^{N} (\theta^n - \theta^{n-1}, \tau b(x)\nabla\varepsilon_t^n)_{0,\Omega}, \\ &T_{12} = \sum_{n=0}^{N} (\theta^n - \theta^{n-1}, \tau d(x)\nabla\varepsilon^n)_{0,\Omega}, \end{aligned}$$

$$T_{13}=\sum_{n=0}^{N}(a(x)\nabla(\rho_t^n-\rho_t^{n-1}),\pi^n)_{0,\Omega},$$
$$T_{14}=\sum_{n=0}^{N}(a(x)\nabla(\rho_t^n-\rho_t^{n-1}),a(x)\nabla\varepsilon_t^n)_{0,\Omega},$$
$$T_{15}=\sum_{n=0}^{N}(a(x)\nabla(\rho_t^n-\rho_t^{n-1}),\tau b(x)\nabla\varepsilon_t^n)_{0,\Omega},$$
$$T_{16}=\sum_{n=0}^{N}(a(x)\nabla(\rho_t^n-\rho_t^{n-1}),\tau d(x)\nabla\varepsilon^n)_{0,\Omega},$$
$$T_{17}=\sum_{n=0}^{N}(\tau b(x)\nabla\rho_t^n,a(x)\nabla\varepsilon_t^n)_{0,\Omega},$$
$$T_{18}=\sum_{n=0}^{N}(\tau b(x)\nabla\rho_t^n,\tau b(x)\nabla\varepsilon_t^n)_{0,\Omega},$$
$$T_{19}=\sum_{n=0}^{N}(\tau b(x)\nabla\rho_t^n,\tau d(x)\nabla\varepsilon^n)_{0,\Omega},$$
$$T_{20}=\sum_{n=0}^{N}(\tau d(x)\nabla\rho^n,\pi^n)_{0,\Omega},$$

From the properties cited in [18] and applying Green's formula, we obtain estimates of T_i:

$$|T_1|\le\sum_{n=0}^{N}\tau\|\varepsilon_t^n\|_{0,\Omega}^2+ch^{2r},|T_2|\le\sum_{n=0}^{N}\tau\|\varepsilon^n\|_{0,\Omega}^2+ch^{2r},$$
$$|T_3|\le\sum_{n=0}^{N}\tau\|\varepsilon_t^n\|_{0,\Omega}^2+ch^{2r},|T_4|\le\sum_{n=0}^{N}\tau\|div\pi^n\|_{0,\Omega}^2+ch^{2r},$$

$$|T_5|\le\sum_{n=0}^{N}\tau\|\varepsilon^n\|_{0,\Omega}^2+ch^{2r},|T_6|\le\sum_{n=0}^{N}\tau\|\varepsilon_t^n\|_{0,\Omega}^2+ch^{2r},$$
$$|T_7|\le\sum_{n=0}^{N}\tau\|div\pi^n\|_{0,\Omega}^2+ch^{2r},|T_8|\le\sum_{n=0}^{N}\tau\|\varepsilon^n\|_{0,\Omega}^2+ch^{2r},$$
$$|T_9|\le\sum_{n=0}^{N}\|\pi^n\|_{0,\Omega}^2+ch^{2r},|T_{10}|\le\sum_{n=0}^{N}\|\nabla\varepsilon_t^n\|_{0,\Omega}^2+ch^{2r},$$

$$|T_{11}|\le\sum_{n=0}^{N}\tau\|\nabla\varepsilon_t^n\|_{0,\Omega}^2+ch^{2r},$$
$$|T_{12}|\le\sum_{n=0}^{N}\tau\|\nabla\varepsilon^n\|_{0,\Omega}^2+ch^{2r},$$
$$|T_{13}|\le\sum_{n=0}^{N}\|div\pi^n\|_{0,\Omega}^2+ch^{2r},$$
$$|T_{14}|\le\sum_{n=0}^{N}\|\triangle\varepsilon_t^n\|_{0,\Omega}^2+ch^{2r},$$
$$|T_{15}|\le\sum_{n=0}^{N}\tau\|\triangle\varepsilon_t^n\|_{0,\Omega}^2+ch^{2r},$$
$$|T_{16}|\le\sum_{n=0}^{N}\tau\|\triangle\varepsilon^n\|_{0,\Omega}^2+ch^{2r},$$
$$|T_{17}|\le\sum_{n=0}^{N}\tau\|\triangle\varepsilon_t^n\|_{0,\Omega}^2+ch^{2r},$$
$$|T_{18}|\le\sum_{n=0}^{N}\tau^2\|\triangle\varepsilon_t^n\|_{0,\Omega}^2+ch^{2r},$$
$$|T_{19}|\le\sum_{n=0}^{N}\tau^2\|\triangle\varepsilon^n\|_{0,\Omega}^2+ch^{2r},$$
$$|T_{20}|\le\sum_{n=0}^{N}\tau\|div\pi^n\|_{0,\Omega}^2+ch^{2r},$$

So we have

$$\sum_{n=0}^{N}I_3\le c(\tau^2+h^{2r}).$$

Note that

$$\begin{aligned}\|u^N\|_{0,\Omega}^2=&\sum_{n=1}^{N}\frac{\|u^n\|_{0,\Omega}^2-\|u^{n-1}\|_{0,\Omega}^2}{\tau}\tau\\ \le&c\sum_{n=1}^{N}(\tau\|u^n\|_{0,\Omega}^2+\|\omega^n\|_{0,\Omega}^2),\end{aligned}$$

where $\omega^n=\frac{u^n-u^{n-1}}{\tau}$.

So we can obtain

$$\begin{aligned}&\max_{0\le n\le t/\tau}(\|\sigma^n-\sigma_h^n\|_{0,\Omega}^2+\|u^n-u_h^n\|_{0,\Omega}^2+\|\omega^n-\omega_h^n\|_{0,\Omega}^2)\\ \le&c(\tau^2+h^{2r})\end{aligned}$$

Thus, we obtain the following theorem

Theorem 5.2: For $\tau>0$, we have

$$\begin{aligned}&\max_{0\le n\le t/\tau}(\|\sigma^n-\sigma_h^n\|_{0,\Omega}^2+\|u^n-u_h^n\|_{0,\Omega}^2+\|\omega^n-\omega_h^n\|_{0,\Omega}^2)\\ &=O(\tau^2+h^{2r}).\end{aligned}\tag{19}$$

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