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An Active Set Method in Image Inpainting

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Abstract**—**In this paper, we apply a semismooth active set method to image inpainting. The method exploits primal and dual features of a proposed regularized total variation model, following after the technique presented in [4]. Numerical results show that the method is fast and efficient in inpainting sufficiently thin domains.

Keywords**—**Active set method, image inpainting, total variation model.

I. Introduction

THE process of filling-in removed, damaged, or unwanted regions in images is called inpainting. This is synonymous with image interpolation wherein continuously defined data is constructed on a region in such a way that the region blends well with the surrounding features. For a long time, inpainting has been used by artists in restoring artistic paintings. In [1], Bertalmio et al first applied inpainting to digital images by using high order PDE models. Since then, numerous approaches to inpainting have been developed: variational techniques, wavelet-based methods, combination of wavelets and total variation minimization, elastica model, isotropic diffusion, etc..

In [2], Chan and Shen introduced the following total variation model that inpaints non-texture type images:

$$\min_{u\in BV(E\cup D)} \alpha \int_{E\cup D} |\nabla u| dx + \frac{1}{2}\int_E (u-u_0)^2\, dx \tag{1}$$

where the observed image is denoted by u_0, E is any fixed closed domain outside D, and $|\cdot|$ denotes the Euclidean norm. The image domain $E\cup D$ is taken to be a square. The model is closely related to the Rudin, Osher, and Fatemi (ROF) model for image denoising. Since the TV model is nondifferentiable, Chan and Shen introduced a global smoothing parameter to the TV term and a steady solution is obtained using a low pass filter and a Gauss-Jordan iteration scheme.

A variational model for image reconstruction on a rectangular image domain Ω with Lipschitz continuous boundary $\partial\Omega$ is

$$\min_{u\in BV(\Omega)} \frac{\mu}{2}\int_\Omega |\nabla u|_2^2 dx + \frac{1}{2}\int_\Omega |Ku-d|^2\, dx + \alpha\int_\Omega |\nabla u|_2 dx$$

In [4], Hintermueller and Stadler regularized the above model by local smoothing on the TV term, i.e., by replacing $\int_\Omega |\nabla u|_2 dx$ with $\int_\Omega F_\gamma(\nabla u)dx$, where

$$F_\gamma(\nabla u) = \begin{cases} \frac{1}{2\gamma}|\nabla u|_2^2 & \text{if} |\nabla u|_2 < \gamma \\ |\nabla u|_2 - \frac{\gamma}{2} & \text{if} |\nabla u|_2 \geq \gamma \end{cases}$$

and $\gamma > 0$. The resulting model is

$$\min_{u\in BV(\Omega)} \frac{\mu}{2}\int_\Omega |\nabla u|_2^2 dx + \frac{1}{2}\int_\Omega |Ku-d|^2\, dx + \alpha\int_\Omega F_\gamma(\nabla u) \tag{2}$$

Further, they developed a semismooth Newton-type method that solves the resulting regularized version of the TV model using an active set strategy. The method was shown to converge superlinearly.

In this paper, we modify the regularized variational model (2) to make it amenable to image inpainting. The primary change is in the restriction of the fidelity term $\int \frac{1}{2}(u-d)^2$ to the non-inpainting domain E. We develop an active-set approach to solve the resulting model and we show that the method exhibits good inpainting capabilities.

II. Model

In the discrete setting, each (i,j) pixel in the $n\times n$ grid of pixels in $E\cup D$ is represented by $u_{i,j}$. For ease in computation, we stack the image matrix u to an image vector v. The components of the image vector v are described as:

$$v_{(j-1)n+i} = u_{i,j} \text{ for } 1 \leq i,j \leq n$$

where $v\in R^N, N = n^2$. The discrete total variation of v is formulated as

$$TV(v) = \sum_{l=1}^{N} |[\nabla v]_l| = \sum_{l=1}^{N} \sqrt{(\nabla_x v)_l^2 + (\nabla_y v)_l^2} \tag{3}$$

where $[\nabla v]_l = \left[(\nabla_x v)_l, (\nabla_y v)_{l+n}\right]^T$. The gradient components ∇_x and ∇_y are approximated using forward differences.

The discrete total variation image inpainting model for (1) is

$$\min_{v\in R^n} TV(v) + f(v) = \alpha\sum_{l=1}^{N} |[\nabla v]_l| + \frac{1}{2}|v - v^0|_{\{\bar{E}\}}^2 \tag{4}$$

where we define $|x|_{\{W\}} = \sqrt{\sum_{i\in W} x_i^2}$ for an index set W, and

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v_0 is the observed image in stacked form. We propose the following regularized variation model for inpainting:

$$\min_{v\in R^N}\left\{\alpha F_\gamma(\nabla v)+\tfrac{1}{2}|v-v_0|^2_{\{\tilde{E}\}}+\tfrac{\mu}{2}\sum_{l=1}^{N}|[\nabla v]_l|^2\right\} \qquad (P_\gamma)$$

where

$$F_\gamma(\nabla u)=\begin{cases}\frac{\lambda}{2\gamma}\sum_{l=1}^{N}|[\nabla v]_l|^2 & \text{if}\,|[\nabla v]_l|<\gamma\\ |\nabla u|_2-\frac{\gamma}{2} & \text{if}\,|[\nabla v]_l|\geq\gamma\end{cases}$$

For $l=1,2,\ldots,N$. Clearly, model (P_γ) is convex and is guaranteed a unique solution. When the inpainting region D is empty, (P_γ) reverts to the working model in [4].

The active set method that we implement exploits the primal-dual features of (P_γ), whose Fenchel pre-dual is the supremum of

$$-\tfrac{1}{2}|||\operatorname{div}p+v^0|||^2_{\{\tilde{E}\}}+\tfrac{1}{2}\|v^0\|^2_{\{\tilde{E}\}}-\tfrac{1}{2}|||\operatorname{div}p|||^2_{\{\tilde{D}\}} \qquad (P_\gamma)$$

where $p\in R^{2N}$, the Euclidean norm $\|[p]_l\|\leq\lambda$,

$$|||x|||^2_{\{X\}}=\begin{cases}\langle (I_N-\mu\Delta)^{-1}x,x\rangle & \text{if } X=\tilde{E},\\ \langle(-\mu\Delta)^{-1}x,x\rangle & \text{if } X=\tilde{D}.\end{cases}$$

Δ is the discrete Laplacian, and $\operatorname{div}=-\nabla^T$ is the discretized divergence (cf. [5]).

III. Optimality conditions

The solutions to the primal (P_γ) and dual (D_γ) problems, given by $\bar{v}_\gamma$ and $\bar{p}_\gamma$ respectively, satisfy the following optimality conditions ([4],[5]):

$$-\mu\Delta\bar{v}_\gamma+\bar{v}_\gamma-\operatorname{div}\bar{p}_\gamma=v^0 \qquad \text{on } \tilde{E} \qquad (5)$$

$$-\mu\Delta\bar{v}_\gamma=\operatorname{div}\bar{p}_\gamma \qquad \text{on } \tilde{D} \qquad (6)$$

$$\begin{cases}\gamma[\bar{p}_\gamma]_l-\lambda[\nabla\bar{v}_\gamma]_l=0 & \text{if } \left\|[\bar{p}_\gamma]_l\right\|<\lambda,\\ \left\|[\nabla\bar{v}_\gamma]_l\right\|[\bar{p}_\gamma]_l=\lambda[\nabla\bar{v}_\gamma]_l & \text{if } \left\|[\bar{p}_\gamma]_l\right\|=\lambda\end{cases} \text{on } \tilde{E}\cup\tilde{D} \quad (7)$$

for $l=1,2,\ldots,N$.

Let $\kappa\in R^N$ with $\kappa_i=1$ if pixel-index $i\in E$, and 0 otherwise. We combine equations (5) and (6) as

$$-\mu\Delta\bar{v}_\gamma-\operatorname{div}\bar{p}_\gamma+\kappa_{\tilde{E}}(\bar{v}_\gamma-v^0)=0 \qquad (8)$$

where $\kappa_{\tilde{E}}=D(\kappa)$, the $N\times N$ diagonal matrix.

The optimal conditions in (7) can also be combined as:

$$\max(\gamma,\|[\nabla\bar{v}_\gamma]_l\|)\,[\bar{p}_\gamma]_l-\lambda[\nabla\bar{v}_\gamma]_l=0 \qquad (9)$$

for every $l=1,2,\ldots,N$. In the next section, we present a Newton-type solution method based on the optimality conditions presented here.

IV. A Semismooth Method

Using equations (8) and (9), we determine a Newton method that mirrors the active set approach in [4] for image denoising. Results in generalized differentiability and semismoothness (cf. [3]) allow the use of a Newton step to (5),(6), and (9) at the k-th approximations v^k and p^k:

$$\Phi\delta=\begin{pmatrix}\mu\Delta v^k+\operatorname{div}p^k+\kappa_{\tilde{E}}(-v^k+v^0)\\ \lambda\nabla v^k-D(m^k)p^k\end{pmatrix} \qquad (10)$$

where

$$\Phi=\begin{pmatrix}\mu\Delta+\kappa_{\tilde{E}} & -\operatorname{div}\\ G\nabla & D(m^k)\end{pmatrix},$$

$$\delta=\begin{pmatrix}\delta_v\\ \delta_p\end{pmatrix},$$

$$G=-\lambda I_{2N}+\chi_{A_{k+1}}D(p^k)J(\nabla v^k),$$

$$m^k=\max(\gamma I_{2N},\eta(\nabla v^k))\in R^{2N}$$

with the mapping $\eta: R^{2N}\to R^{2N}$ given by

$$(\eta(v))_i=\|v_i\|\,\text{with}\,v\in R^{2N}, i=1,2,\ldots,2N$$

Now, the active set indicator $\chi_{A_{k+1}}=D(t^k)$ which is a $2N\times N$ diagonal matrix with

$$t_i^k:=\begin{cases}1 & \text{if}\,(\eta(\nabla v^k))_i\geq\gamma\\ 0 & \text{if}\,(\eta(\nabla v^k))_i<\gamma\end{cases}$$

determines whether a component is part of the active set A_{k+1} by setting $t_i=1$, or not. The matrix J is the Jacobian of η, that is,

$$J(\nabla v)=\left(D(\eta(\nabla v))\right)^{-1}\begin{pmatrix}D(\nabla_x v) & D(\nabla_y v)\\ D(\nabla_x v) & D(\nabla_y v)\end{pmatrix}$$

With all components of $m^k>0$, this means that the diagonal matrix $D(m^k)$ is invertible. We obtain δ_p and δ_v as

$$\delta_p=\lambda D^{-1}(m^k)\nabla v^k-p^k-D^{-1}(m^k)G\nabla\delta_v \qquad (11)$$

and

$$H_k\delta_v=f_k \qquad (12)$$

with

$$H_k=-\mu\Delta+\kappa_{\tilde{E}}+\operatorname{div}D^{-1}(m^k)G\nabla$$

$$f_k=\mu\Delta v^k+\operatorname{div}\lambda D^{-1}(m^k)\nabla v^k+\kappa_{\tilde{E}}(-v^k+v^0).$$

Whenever H_k is not positive definite, we use the shift modifications in [4] to get a positive definite matrix H_k^+.

We propose the following active set method for inpainting:

Algorithm: Active Set Method

1) Set $k=0$ and initialize $(v^0,p^0)\in R^N\times R^{2N}$.
2) Determine the members of the active set by solving $\chi_{A_{k+1}}\in R^{2N}\times R^{2N}$.
3) Compute H_k^+ if p^k is not feasible for all $i=1,\ldots,N$. Otherwise set $H_k^+=H_k$.
4) Solve for δ_v in $H_k^+\delta_v=f_k$ and compute δ_p.

5) Update $v^{k+1} = v^k + \delta_v$ and $p^{k+1} = p^k + \delta_p$.
6) Stop, or set $k := k + 1$ and go to step 2.

We note that the proposed method for inpainting is analogous to that in [4] for denoising. Numerical implementations of the algorithm are presented next.

V. Numerics

The algorithm is implemented in MATLAB R2010b on a machine with a speed of 2.93 GHz and with 2 GB of RAM. Our test images are square grayscale images which are nearly noise-free and blur-free degraded only by thin lines and text which are the inpainting domains. The goal of inpainting is to reconstruct the inpainting domain by using the image information surrounding these domains.There is no ideal value for γ; however, the smaller γ is, the better the observed inpainting and restoration of edges. With this, we set $\gamma = 10^{-4}$.The value of α is set to 0.01. We observed that in our test runs, there was no need to perturb H_k to make it positive definite.

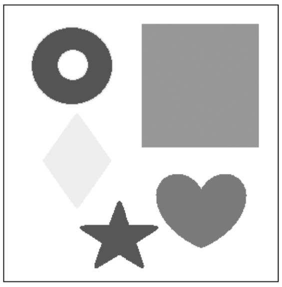

Fig. 1 (a) Original image

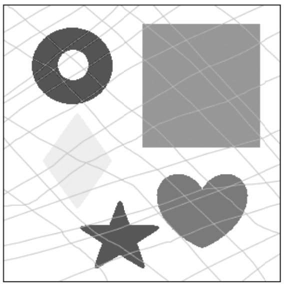

Fig. 2 (b) Image with lines

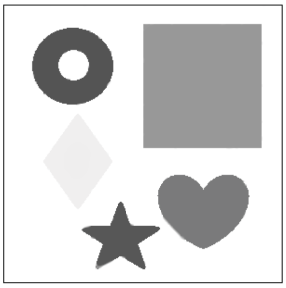

Fig. 2 (c) Active set method

Our first image sample is a256×256 image (figure 1(a)). The image masked with thin lines (about 2 to 4 pixels wide) is figure 2(b). The mask is user-defined and is created using an image editing software. The result obtained using the active set method is shown in figure 2(c). This result is obtained in 3 iterations, with a time of 5.1 seconds. The method converged in 14 iterations. Convergence is determined once the norm of the vector composed of the left-hand-sides of optimality conditions (8) and (9) has sufficiently decreased from its initial value.

Fig. 2 (a) Image with text

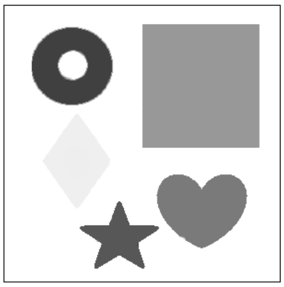

Fig. 2 (b) Active set method

The second masked image has text to be inpainted (figure 2(a)). The reconstruction is given in figure 2(b) obtained in 3 iterations and 6.9 seconds. We see that the method is effective in inpainting both line and text regions.

The next example is a 256×256 grayscale image (figure 3(a). Thin lines constitute the inpainting domain. The reconstruction using the active set method effectively removed the lines in 10 iterations with a time of 12.4 seconds.

Fig. 3 (a) Original image

Fig. 3 (b) Image with lines

Fig. 3 (c) Reconstructed image

VI. Conclusion

We presented a variation model for image inpainting and a semismooth primal-dual active set method to solve it. Our numerical experiments show that the method is very effective in providing good reconstructions within reasonable time. The algorithm is applicable for filling in small domains in non-texture based images.

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