International Journal of Business, Human and Social Sciences
ISSN: 2517-9411
Vol:15, No:8, 2021

Advances on the Understanding of Sequence Convergence Seen from the Perspective of Mathematical Working Spaces

Paula Verdugo-Hernández, Patricio Cumsille

Abstract**—**We analyze a first-class on the convergence of real number sequences, named hereafter sequences, to foster exploration and discovery of concepts through graphical representations before engaging students in proving. The main goal was to differentiate between sequences and continuous functions-of-a-real-variable and better understand concepts at an initial stage. We applied the analytic frame of Mathematical Working Spaces, which we expect to contribute to extending to sequences since, as far as we know, it has only developed for other objects, and which is relevant to analyze how mathematical work is built systematically by connecting the epistemological and cognitive perspectives, and involving the semiotic, instrumental, and discursive dimensions.

Keywords**—**Convergence, graphical representations, Mathematical Working Spaces, paradigms of real analysis, real number sequences.

I. Introduction

THERE is an increasing demand from society to improve learning in the Science, Technology, Engineering, and Mathematics (STEM) disciplines since the number of people working in STEM fields is increasing continuously. There are various calculus courses tailored to meet university students' needs in STEM fields [1]. To have a suitable basis on the convergence notion, among others, is fundamental, particularly that of real number sequences and series of functions is essential for the foundation of real analysis, which is the basis for natural sciences applications. Hereafter, by simplicity, real number sequences will be named as sequences.

This work is about teaching sequences convergence, particularly its graphical aspect, usually neglected in university students' training in STEM fields. Our first hypothesis consists of precisely that, this is not addressed in university, i.e., graphical representations of notions associated with convergence are rarely considered and discussed by instructors. Our second hypothesis is that no differentiation is made between sequences and real-valued functions of a real variable after that, named merely as functions. By differentiation, we mean that sequences, even if they can be plotted in two-dimensional graphs, have another way to be represented graphically, its own limit's formal definition and rules for limits calculation, and convergence results that functions do not have. Also, sequences can describe processes in discrete time, widely used in applied mathematics, whereas functions can generally express the dependence between two arbitrary variables, including continuous ones. In this regard, do not distinguish sequences from functions could induce students to use them without considering the previous aspects, which is not necessarily suitable for solving problems.

The general goal consists of studying the teaching of sequence convergence, fostering its graphical study and differentiation from functions, from the perspective of *Mathematical Working Spaces (MWS)* [2] as an analytic framework. For over 15 years, the MWS model has been the object of collaborative research among several researchers, generally coming from French and Spanish speaking countries such as France, Spain, and Chile. One of the MWS model's main strengths is investigating the interactions between the epistemological and cognitive perspectives, providing a tool for the specific study of mathematical work in which students and teachers are engaged. The concept of *mathematical working space* refers to a structure organized to analyze individuals' mathematical activity when solving problems. Thus, analyzing mathematical work through MWS allows capturing how it is systematically built, connecting the epistemological and cognitive perspectives. This is done according to linked genetic developments, each one identified as a *genesis* that accounts for a specific dimension in the MWS model: *semiotic, instrumental,* and *discursive.* On the other hand, according to every mathematical field's historical-epistemological development, mathematical work is guided by *working paradigms* in which it is framed. In the calculus case, the MWS model considers the *paradigms of real analysis* derived in [3] from historical-epistemological and education viewpoints.

We investigated how a university teacher addressed sequence convergence graphically in the first class on the subject. In terms of the MWS model, the instructor guided and fostered mathematical concept building in the *Geometrical-Analysis (GA) working paradigm* [3] and reasoning framed on the semiotic and instrumental dimensions [2]. Both the GA paradigm and the semiotic and instrumental dimensions will be explained in Section III. The goal was that concepts were internalized through the symbolic notation representing them,

Paula Verdugo-Hernández works at Escuela de Pedagogía en Ciencias Naturales y Exactas, Facultad de Ciencias de la Educación, Universidad de Talca, Linares. CP: 3580000, Chile (e-mail: pauverdugo@utalca.cl).

Patricio Cumsille works at Department of Basic Sciences, University of Bío-Bío, Chillán, Chile (e-mail: pcumsille@ubiobio.cl).

then discovered through graphical registers (work within the semiotic dimension), and finally constructed from these registers' gained information (work within the instrumental dimension).

II. Literature Review

We present a brief review of the literature to frame the research proposed. We restrict ourselves to discussing works directly related to this proposition to construct students' conceptions of sequence convergence; specific studies about series are few [4].

In [4], the authors pointed out that the notion of limit has been studied in mathematical education for around 40 years, including sequences and functions; see, e.g., [5]-[9]. The authors reviewed, among others, the difficulties for students' learning, its importance in real analysis, and its relationship to other topics such as continuity, differentiability in such a way that the last aspect could also have an impact on the learning of such concepts.

Despite the considerable qualitative gap between the intuitive and formal conceptions of limits [10], the interplay between both aspects is crucial in this case [4] since success in convergence learning is more likely to occur when understanding the formal aspects built upon students' spontaneous conceptions [11]. Indeed, many researchers consider that the passage from a dynamic to a static formulation of limit is at the core of the difficulties experienced by many students [12], together with the algebraic notation and quantifiers involved in the formal definition [13], [14].

In [7], it is studied and classified the different forms of conceptualization of students' descriptions about sequence convergence in models through speeches, examples, and representations. The author obtained three main types of models about the limit in university students, which consequently became the subject of several research studies: dynamic, static, and monotone. In the first one, the convergence is associated with the idea of approximation (the terms of a sequence, seen as a process, approach a certain real number, as n grows). The static model translates the formal (ε, N)-definition to the natural language (all the terms of a sequence, except the firsts N, are contained in an epsilon-neighborhood of a given real number). Finally, the monotone model states that the distance between the terms of a sequence and its limit decreases, which constitutes a particular case (monotone and bounded sequences). The author investigated the productions of students in a test and observed that the three types of representations found performed quite differently: students that evidenced a static model were successful, those that answered with a monotone model failed, and among those who showed a dynamic model, half succeeded, and half failed. Accordingly, students may reach a suitable formalism level if they are taught under the static model, although the idea of approximation (under the dynamic model) is also essential.

In [15], it is pointed out that little research has addressed students' understanding of the limit's formal definition and conducted experiments to construct this concept by leveraging their intuitive ideas. The work was based on [16], who also modeled and provided detailed characterizations of students' spontaneous reasoning about the concept. In [17], it is extended and identified five that better represented the conceptualization, from which the strongest one is the approximation that is close to formal reasoning and guide students' exploration. This one conceives the limit as an unknown value that must estimate by using the sequence terms, where the epsilon has the role of a bound or a predefined tolerance of the error, i.e., the accuracy of the approximation.

Insights on successful convergence models (static and dynamic) can be used to better understand concepts from graphical representations, which relies mainly on the use of two-dimensional representations of a given sequence, considering a strip of width 2ε around its limit (if it exists) to depict the inequality $|s_n - l| < \varepsilon$ in the graph. This graphical register is used widely by calculus textbooks and research about this topic as a tool to help students to build the right mental image on convergence, e.g., see [18], [19].

III. The Mathematical Working Space Model

As explained in Section I, we analyze the teaching of sequence convergence by the MWS model. Schematically, the MWS model conceives two articulated horizontal planes: one of epistemological nature-related near to the mathematical content according to the field of study; the other of a cognitive nature, related to the individual's thinking solving problems.

To describe the mathematical work in its epistemological dimension, the plane named in this way consists of three interacting components organized according to purely mathematical criteria: a set of concrete and tangible objects, a set of artifacts such as drawing instruments or software, and a theoretical framework based on definitions, properties, and theorems. These components must be organized to model the teaching and learning process of a given content within a mathematical field.

The MWS model uses the term *sign* or *representamen* to summarize the concrete and tangible objects component. This term stands to somebody in some respect or capacity and may encompass geometrical images, algebraic symbols, graphics, or even photos in the case of modeling problems; see Fig. 1.

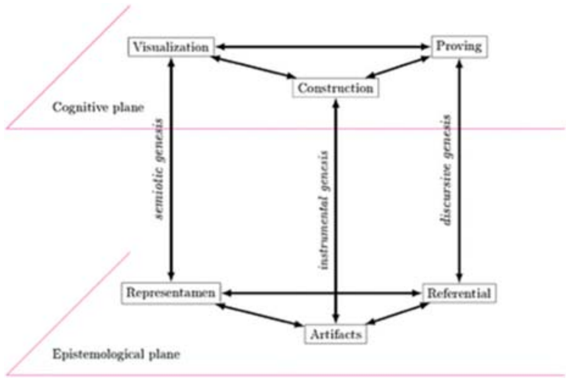

Fig. 1 Scheme of the MWS model [2]

For the MWS model, the notion of *artifact* includes everything that has transformed, although small, of human origin. It is not restricted to material objects but also includes symbolic systems. To avoid confusion, the MWS model considers as artifacts the symbolic systems (algorithms) that are connected, either to material artifacts (e.g., abacus, logarithmic or trigonometric tables) or techniques of calculation or construction (e.g., Euclidian division, constructions with ruler and compasses).

The MWS model names as *referential* (Fig. 1) to the set of properties, definitions, theorems, and axioms that compose the mathematical work's theoretical part, which supports the discourse of proof specific to mathematics, which should be coherently organized, adapted to the proposed tasks, and whose statements have been the object of a form of institutionalization.

It is essential to understand how individuals—or individuals' communities—acquire, develop, and use mathematical knowledge through their practice of the discipline. Thus, the MWS model considers a second dimension centered on the cognitive subject and related to the epistemological dimension. In one-to-one correspondence with the epistemological components, as suggested by Fig. 1, the three cognitive components—understood in a vast scope—are *visualization* related to deciphering and interpreting signs, and to internally building a representation of the involved objects and relations; *construction* that depends on the used artifacts and the associated techniques; *proving* produced through validation processes based on the theoretical frame of reference.

Visualization is associated with diagrams and operations to decipher and use signs, which does not necessarily involve visualizing speaking strictly, and must be extended beyond the simple vision or perception of signs. It can be considered a means for structuring the information provided by signs and encompasses *deciphering, interpreting, establishing relationships.*

Construction is related to actions triggered by tools (artifacts within the epistemological plane), which do not necessarily have to result in tangible productions such as drawings or writings. However, it may encompass (instrumentally guided) observation, exploration, or even (more systematic and technically supported) experimentation.

Proving is related to the mathematical proof that must be beyond a mere empirical validation, more pertinent to construction. It must lead to argumentations organized into propositions that encompass definitions, hypotheses, conjectures, or counterexamples' enunciation.

The MWS model conceptualizes the semiotic and instrumental dimensions through geneses. The semiotic genesis is when an individual decodes and interprets signs and encodes or instantiates a sign to construct or specify it. The instrumental genesis describes adaptations that the user would make to manipulate the tools *(instrumentation)* and a tool's suitable choice, with possible adaptations of the artifact to the required actions *(instrumentalization).* One assumes that the mathematical knowledge is more involved and developed in this last. Both geneses articulate in the *Sem-Ins vertical plane* within the MWS that focuses on the conceptualization and understanding of a particular notion, without any formal validation goal, through the information provided by artifacts, particularly by the exploration from graphical representations. Also, we consider the *discursive genesis* that is the process by which the properties and results organized in the theoretical frame of reference, according to the subject studied, are activated for mathematical reasoning and discursive validations (proving), i.e., those that go beyond graphic, empirical, or instrumented proofs, even if could be triggered by the latter. For a complete review of the MWS model and its applications to diverse mathematical fields, see [2], [20]. Also, we consider the *paradigms of real analysis* derived from historical-epistemological and education viewpoints in [3]. The *Arithmetic/Geometric Analysis (GA) paradigm* involves a perceptive work based on graphs or numbers, which supports the first stage of teaching objects such as equations or functions. It also allows interpretations based on geometry, arithmetic calculation, or the real world; particularly, property discovery and concept construction are fostered through explorations from graphical registers. The *Calculation Analysis (CA) paradigm* considers a work based on calculation rules, defined explicitly or implicitly, and applied independently to justify or reflect on the involved objects' existence and nature. Finally, the *Real Analysis (RA) paradigm* delves deeper into this last aspect using the completeness axiom to justify calculation rules. For instance, usually one proves that the range of a function is within some interval by a work located in the CA paradigm; conversely, to show that any point of that interval is within the range of the function, one often requires the Intermediate Value Theorem whose justification relies on the completeness of the real numbers set.

IV. Method

This research frames within a methodology based on the Didactic Engineering (DE) [21], [22], which consists of the conception, application, observation, and analysis of tasks or successions of teaching, using a validation based on the comparison between analyses *a priori* and *a posteriori*. We aim to foster a better understanding of sequence convergence by engaging the instructor to teach by associating graphical representations with relevant concepts. Therefore, we omitted *a priori* analysis since the components, geneses, vertical planes, and paradigms of the real analysis within the MWS model where work would locate are prefixed. More precisely, the components of the epistemological plane that the instructor would activate are symbols and graphical representations of some concepts and properties within the referential of the sequences' convergence; the components of the cognitive plane would be the visualization and construction since proving in this stage would be premature. The reasoning would focus on the semiotic and instrumental genesis, i.e., within the Sem-Ins vertical plane, and the work would be developed within the GA paradigm in the real analysis field.

The teacher's lecture was filmed to analyze her/his mathematical work to foster students with a better understanding of concepts based on graphical representations.

V. Results and Discussion

The semiotic genesis is predominant in the first part of the class since the teacher's symbolic registers must be decoded and interpreted by students. Also, because she/he emphasized different notations after giving sequence formal definition, we observe a first differentiating element between sequences and functions. The professor then introduced the first example of a sequence: (a_n) defined by $a_n = 1/n$ for every $n \in \mathbb{N}, n \geq 1$, and depicted by two types of graphs (unidimensional and two-dimensional). Given that students are more familiar with two-dimensional graphical representations since they have previously learned functions, the professor first showed the students the unidimensional graph, making them differentiate both mathematical objects. Thus, she/he depicted the sequence defined previously, as in Fig. 2.

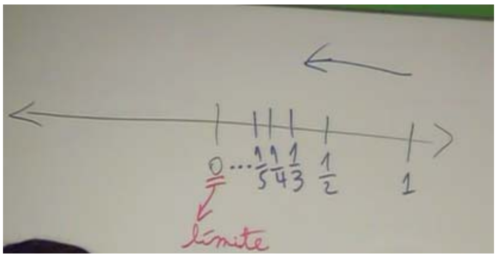

Fig. 2 Limit's 1-D representation of $a_n = 1/n$

As depicted in Fig. 2, she/he tried to foster reasoning within the Sem-Ins vertical plane, with a work located at the GA paradigm by the following argumentation: "the terms go in that direction […] they asymptotically approach their limit".

Teacher (T): […] Beginning with 1, continue with ½, 1/3, 1/4, … they decrease and asymptotically, as we mathematicians say, approach […] zero, which is the limit. The word asymptotically means that they will converge to zero, right? Since they decrease, the values become increasingly smaller, and, as we can see here, they approach zero.

Later, the teacher drew a two-dimensional graph, as depicted in Fig. 3, providing the students with the possibility to use two types of graphical representations for sequences, no expressing preference for any of them.

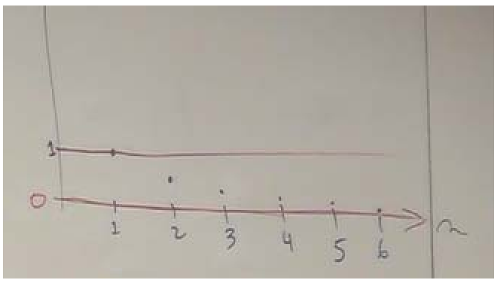

Fig. 3 2-D representation of $a_n = 1/n$ and their bounds

To discuss the difference between sequences and functions, the teacher asked students the difference between the two graphs, which generated dialogue with a student who responded interestingly that one of the graphical representations is discrete.

T: What is the difference between the two graphs?

Student (ST): Well, this one is discrete.

T: Right? That one is discrete. What does "discrete" mean?

ST: That it is not continuous. That there is a jump between the values.

T: That there is a jump between the values, right? Because from 1, jump to 2, and between 1 and 2, there is nothing. From 2, jump to 3, and between 2 and 3, there is nothing. Why should this (the graph) be like that?

ST: Because the natural numbers do not have [inaudible].

T: Right, because the natural numbers are discrete by essence and because this function, which is a sequence defined by the general term that is 1/n, is discrete by nature because its domain is the set of natural numbers. A continuous line cannot be drawn since it would be a conceptual error, right?

Although the teacher only wanted to explain that there are two types of graphical representations for the same purpose, which constitutes one of the differences between sequences and functions, she/he took advantage of the opportunity to thoroughly explain why the graphs for the firsts must be discrete. In this sense, we observe the implicit use of elements of the RA paradigm since the completeness of the real numbers set is used, in this case, to justify that the functions should depict by continuous-dense plots, in contrast to the sequences that have to be graphically represented by a discrete set of points. Afterwards, the professor introduces the informal definition of a convergent sequence:

Definition (Informal definition of convergence): Let (s_n) be a sequence and $l \in \mathbb{R}$. We say that (s_n) converges to l or that the terms (s_n) tend toward l (denoted $s_n \to l$) if given any closed interval $[l - \varepsilon, l + \varepsilon]$ with $\varepsilon > 0$, only a finite quantity of terms of (s_n) remain outside of it, i.e., all the rest remains within the interval.

The teacher then explained the content of the definition supported by the graph depicted in Fig. 4 by arguing what follows.

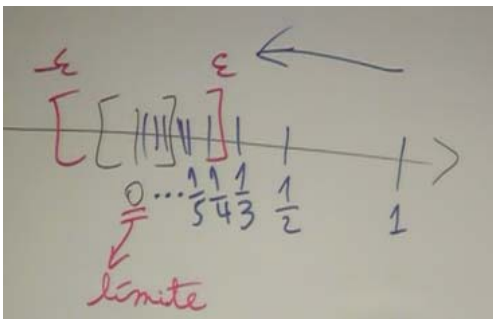

Fig. 4 Convergence's 1-D representation of $a_n = 1/n$

T: […] So, let us do here a neighborhood […] of radius epsilon, I mean, this will be minus epsilon (the bottom end), and the top end will be plus epsilon, right? Then, the convergence means graphically that only a finite quantity of terms is excluded from this interval and actually, here in the graph […] I picked an epsilon […], that is I did a neighborhood of zero, such that starting from the fourth term of the sequence […] all of them remain in the interval and […] ultimately […] only the first three terms exclude.

Later, the professor provided the same explanation, but this time based on a two-dimensional graph, as depicted in Fig. 5.

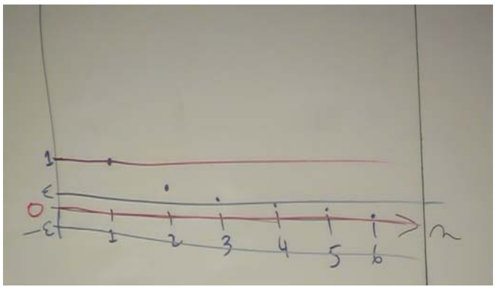

Fig. 5 2-D convergence representation of $a_n = 1/n$

With the support from the graph, the professor introduced the "band" concept, represented as the region of the plane encompassed between the two blue lines y = ±ε, to indicate that there are infinite terms of the sequence within it, meaning this last converges to 0 (represented as the band center, i.e., the line y = 0). This idea, whose graph appears thereby in calculus textbooks, has been suggested by several authors (e.g., [18], [19]) to support teaching the convergence concept.

With the argumentations based on Figs. 4 and 5, respectively, the professor attempts to provide an interpretation of the convergence through instrumented work (by visualization from a graphical register) located in the GA paradigm via reasoning in the Sem-Ins vertical plane, introducing implicitly elements of the RA paradigm (neighborhood) to relate them, before presenting the formal definition of convergence. Also, we infer a work located within the CA paradigm since it seems to focus on the sequence' range, a concept pertinent to the referential of the functions. Once again, the teacher used both graph types to foster a differentiation between sequences and functions.

Before introducing the formal definition of convergence, the teacher provided the following discourse: To formalize the previous definition, one must establish mathematically that only a finite quantity of terms of the sequence remains outside from [l - ε, l + ε]. To make that, one could say that, starting at a given term, all those that follow are within the interval. In that way, the professor affirms that the previous statement mathematically is formalized as:

$$(\exists n_0 \in \mathbb{N})(\forall n \geq n_0)\; s_n \in [l - \varepsilon, l + \varepsilon].$$

This latter gives way to the following:

Definition (Convergence). We will say that the sequence $(s_n)_{n \in \mathbb{N}}$ converges to l, or that the terms s_n tend toward l (denoted by $s_n \to l$) if it is satisfied that:

$$(\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n \geq n_0)\; s_n \in [l - \varepsilon, l + \varepsilon].$$

Next, the professor provides the following discourse to discuss the relation between ε and n_0 in the previous definition.

T: […] There are various observations. The first one is that n_0 depends on epsilon since as […] graphically observed, while smaller we take epsilon (in the drawing depicted in Fig. 5), there are more terms of the sequence outside from the interval. [...] If it is convergent, always it will be within it […], it will delay that moment, but it will do it, which is why the n_0 depends on epsilon, right?

In the above transcription, we observe that the teacher tries to explain the relationship between n_0 and ε. We point out that the explanation is within the breakdown of the meaning of convergence concept studied in [13], whose author attributes great importance to relationships study between the variables that constitute the proposition of the definition, its relationship to the central inequality $|s_n - l| < \varepsilon$, and with the use of quantifiers. However, the teacher explanation is not so accurate since it does not consider, for example, the case of a constant sequence. Indeed, for any ε > 0, n_0 only requires to be large enough to obtain that for any $n \geq n_0$, the distance between s_n and its limit is less or equal than ε.

The teacher activated the discursive genesis since the formal definition of convergence was subjected to theoretical consideration. Despite this was made using a graphical representation, it is considered an element of the RA paradigm, reinforcing the mentioned activation. In this sense, the teacher propitiates somehow a circulation of knowledge between Sem-Ins and Ins-Dis vertical planes. He/she tried to enlarge the knowledge obtained from symbolic and graphical registers' relationship using the theoretical consideration made starting from the latter, evidenced by the relation between ε and n_0 acquired from data given by the latter instrument along with inductive reasoning (an experimental proof).

Later, the professor continues with the discussion on the formal definition of convergence by giving the following discourse: The previous definition is equivalent to:

$$(\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n \geq n_0)\; l - \varepsilon \leq s_n \leq l + \varepsilon$$
$$(\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n \geq n_0)\; -\varepsilon \leq s_n - l \leq \varepsilon$$
$$(\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n \geq n_0)\; |s_n - l| \leq \varepsilon.$$

T: So, what does the definition of convergence say? It says that there is a natural number from which the distance between s_n and l is less than or equal to epsilon for every epsilon greater than zero. Since one can take a very small epsilon, the distance between s_n and its limit is as small as one wishes, which means convergence, ok?

Finally, we can observe that the professor uses the distance concept and again the neighborhood one, both of which belong to topology, saying that this has a life of its own in the context of topology, that this is the branch of mathematics that studies the forms of sets, and whose fundamental objective is to introduce convergence in the most general way possible.

To summarize, the reasoning throughout the class centered on the Sem-Ins vertical plane, with a strong emphasis on the semiotic genesis evidenced by decoding and interpreting different signs (symbolic and graphical). On the other hand, the professor activated the instrumental genesis to construct concepts based on graphical representations and a work framed mainly in the GA paradigm. The goal was to favor a smoother transition by using interpretations obtained from the relation of concepts with graphical representations since we believe that the coherent integration of the three paradigms may produce a better understanding of concepts.

VI. Conclusions

We analyzed a first-class on sequence convergence by a university teacher that fostered graphical representations of relevant concepts related. MWS model allowed us to locate in a precise way the graphical study of sequences through a work positioned in the GA paradigm and reasoning within the Sem-Ins vertical plane. In this sense, we expect to have contributed to enlarging the MWS model that, as far as we know, has only developed for other objects and which is relevant for the analysis of mathematical work.

In future works, we expect to implement experimentations consisting of solving graphical nature tasks on sequence convergence. The aim would be to engage students in distinguishing between functions and sequences and better understanding by instrumented work before engaging them in convergence provings. In this sense, we would want to implement specialized learning for sequences, a critical subject encompassing several aspects, including the graphical.

Finally, we would like to delve deeper into the RA paradigm strengthening the discursive genesis to enhance the students' understanding. This would be done by incorporating theoretical tools to perform proofs of convergence with appropriate instructor supervision. In this sense, the ultimate goal would be to delve into and articulate the three paradigms of real analysis, strengthening the GA paradigm, just as we have intended to make in the present study, an issue that we believe is rarely practiced in university instruction about sequences.

Funding Information

Paula Verdugo-Hernández's work was support by Convenio Marco FID-Talca 1856 from the Universidad de Talca (2021).

References

[1] Keene, K. A., Hall, W., & Duca, A. (2014). Sequence limits in calculus: using design research and building on intuition to support instruction. *ZDM - Mathematics Education*. https://doi.org/10.1007/s11858-014-0597-8

[2] Kuzniak, A., Tanguay, D., & Elia, I. (2016). Mathematical Working Spaces in schooling: an introduction. *ZDM - Mathematics Education*. https://doi.org/10.1007/s11858-016-0812-x

[3] Montoya-Delgadillo, E., & Vivier, L. (2016). Mathematical working space and paradigms as an analysis tool for the teaching and learning of analysis. *ZDM - Mathematics Education*, *48*(6), 739–754. https://doi.org/10.1007/s11858-016-0777-9

[4] Oktaç, A., & Vivier, L. (2016). Conversion, Change, Transition… in Research About Analysis. In *The Didactics of Mathematics: Approaches and Issues*. https://doi.org/10.1007/978-3-319-26047-1_5

[5] Cornu, B. (2002). Limits. In David Tall (Ed.), *Advanced Mathematical Thinking* (pp. 153–166). Springer, Dordrecht. https://doi.org/https://doi.org/10.1007/0-306-47203-1_10

[6] Davis, R. B., & Vinner, S. (1986). The notion of limit: Some seemingly unavoidable misconception stages. *Journal of Mathematical Behavior*, *5*(3), 281–303.

[7] Robert, A. (1982). Divers travaux de mathématiques et l'acquisition de la notion de convergence des suites numériques dans l'enseignement supérieur. *Recherches En Didactique Des Mathématiques*, *Thèse d'Et*(3), 307–341

[8] Tall, D., & Schwarzenberger, R. L. E. (1978). Conflicts in the learning of real numbers and limits. *Mathematics Teaching*, *82*(1978), 44–49.

[9] Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12*(2), 151–169. https://doi.org/10.1007/BF00305619

[10] Artigue, M. (1998). Enseñanza y aprendizaje del análisis elemental: ¿qué se puede aprender de las investigaciones didácticas y los cambios curriculares? *RELIME. Revista Latinoamericana de Investigación En Matemática Educativa*, *98*(1), 40–55.

[11] Fernández, E. (2004). The students' take on the epsilon-delta definition of a limit. *PRIMUS*, *14*(1), 43–54. https://doi.org/10.1080/10511970408984076

[12] Tall, D. (1992). The Transition to Advanced Mathematical Thinking: Functions, Limits, Infinity and Proof. In *Handbook of Research on Mathematics Teaching and Learning* (pp. 495–511).

[13] Mamona-Downs, J. (2001). Letting the intuitive bear on the formal; a didactical approach for the understanding of the limit of a sequence. *Educational Studies in Mathematics*. https://doi.org/10.1023/A:1016004822476

[14] Swinyard, C., & Larsen, S. (2012). Coming to Understand the Formal Definition of Limit: Insights Gained from Engaging Students in Reinvention. *Journal for Research in Mathematics Education*, *43*(4), 465–493. https://doi.org/10.5951/jresematheduc.43.4.0465

[15] Oehrtman, M., Swinyard, C., & Martin, J. (2014). Problems and solutions in students' reinvention of a definition for sequence convergence. *The Journal of Mathematical Behavior*, *33*, 131–148. https://doi.org/10.1016/J.JMATHB.2013.11.006

[16] Oehrtman, M. (2008). Layers of abstraction: Theory and design for the instruction of limit concepts. In *Making the Connection: Research and Teaching in Undergraduate Mathematics Education*. https://doi.org/10.5948/UPO9780883859759.007

[17] Oehrtman, M. (2009). Collapsing dimensions, physical limitation, and other student metaphors for limit concepts. *Journal for Research in Mathematics Education*, *40*(4), 396–426. Retrieved November 27, 2020, from http://www.jstor.org/stable/40539345

[18] Roh, K. H. (2008). Students' images and their understanding of definitions of the limit of a sequence. *Educational Studies in Mathematics*, *69*(3), 217–233. https://doi.org/10.1007/s10649-008-9128-2

[19] Roh, K. H. (2010). An empirical study of students' understanding of a logical structure in the definition of limit via the ε-strip activity. *Educational Studies in Mathematics*, *73*(3), 263–279. https://doi.org/10.1007/s10649-009-9210-4

[20] Gómez-Chacón, I. M., Kuzniak, A., & Vivier, L. (2016). The Teacher's role from the perspective of Mathematical Working Spaces. *Bolema - Mathematics Education Bulletin*. https://doi.org/10.1590/1980-4415v30n54a01

[21] Artigue, M. (1988). Ingénierie didactique. *Recherches En Didactique Des Mathématiques*, *9*(3), 281–308.

[22] Artigue, M. (2011). L'ingénierie didactique comme thème d'étude. In F. V. & F. W. C. Margolinas, M.Abboud-Blanchard, L. Bueno-Ravel, N. Douek, A. Fluckiger, P. Gibel (Ed.), *En amont et en aval des ingénieries didactiques* (pp. 15–25). Grenoble : La pensée sauvage.