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Advanced Gronwall-Bellman-Type Integral Inequalities and Their Applications

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Abstract—In this paper, some new nonlinear generalized Gronwall-Bellman-Type integral inequalities with mixed time delays are established. These inequalities can be used as handy tools to research stability problems of delayed differential and integral dynamic systems. As applications, based on these new established inequalities, some p-stable results of a integro-differential equation are also given. Two numerical examples are presented to illustrate the validity of the main results.

Keywords—Gronwall-Bellman-Type integral inequalities, integro-differential equation, p-exponentially stable, mixed delays.

I. Introduction

As an important basic tool, inequality technique is extensively applied in diversity areas including global existence, uniqueness, stability, boundary value problem, and other properties. In the past decades, various inequalities and their generalized forms have been established, such as Halanay-type inequality [1], [2], impulsive integral inequality [3], impulsive differential inequalities [4], [5], and so on. As pointed out in [6], since Gronwall-Bellman inequality provides an explicit bound to the unknown function, it has been a powerful tool in the study of quantitative properties and stability of solutions of differential and integral equations. In [7]-[9], by using Gronwall-Bellman inequality, projective or feedback neural networks for solving program problems were investigated and some stability criteria were obtained. Based on Riccati-equations and Gronwall-Bellman inequality, bounded input bounded output (BIBO) problems of delayed system were studied in [10]. In [6], Cheung and Zhao established some new nonlinear Gronwall-Bellman-Type inequalities. These new established inequalities can be used to solve boundary value problems. Recently, the research on Gronwall-Bellman-Type inequality attracts considerable attention, and all kinds of new generalized forms are derived in terms of various practical applications (see [6], [11]- [15]).

However, these previous established Gronwall-Bellman-Type inequalities can not be applied to the stability problems of integro-differential equations with mixed time delays. For solving this problem, it is necessary to established some new generalized Gronwall-Bellman-Type inequalities.

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Motivated by the above discussions, the objective of this paper is to establish some new advanced Gronwall-Bellman-Type inequalities. Applying mathematical analysis method, some new Gronwall-Bellman-Type inequalities with mixed delays are established. The new inequalities generalize some previous results. In addition, some stability results of a class of integro-differential equations are also given by using these new established inequalities. Finally, two numerical examples are also provided to illustrate the validity of the proposed results.

Notations. The notations are used in our paper except where otherwise specified. $|\cdot|$ denotes the Euclidean norm; $\|\cdot\|$ denotes a vector or a matrix norm; The notation $\|\cdot\|^p$ is used to denote a vector norm defined by $\|\cdot\|^p = \sum_{i=1}^{n} |x_i|^p$; $\|\cdot\|^p_\Delta \triangleq \sup_{-\infty<t\le 0} |\cdot|^p$, $\mathscr{R}, \mathscr{R}^n$ are real and n-dimension real number sets respectively.

II. Advanced Gronwall-Bellman-Type Integral Inequalities

Theorem 2.1: If there exist positive scalars a, b, h, τ, γ_1, γ_2, γ_3, nonnegative continuous functions $m(t)$, $k(t)$ and nonnegative continuous differentiable function $u(t)$ on interval $[t_0-\tau, +\infty)$ such that the following conditions hold:

$$\begin{cases} m(t) \le u(t)h + \gamma_1 \displaystyle\int_{t_0}^{t} u(t-s)m(s)ds \\ \qquad + \gamma_2 \displaystyle\int_{t_0}^{t} u(t-s)m(s-\tau(s))ds \\ \qquad + \gamma_3 \displaystyle\int_{t_0}^{t} u(t-s)\int_{-\infty}^{s} k(s-\xi)m(\xi)d\xi ds, \\ u'(t) \le -au(t),\ u(0)=b, \\ \qquad a > b\gamma_1 + b\gamma_2 + kb\gamma_3,\ k \triangleq \displaystyle\int_0^\infty e^{\varepsilon s}k(s)ds \end{cases}$$

where $t_0 \ge 0, 0 \le \tau(t) \le \tau$, then as $t \ge t_0$, we have

$$m(t) \le bhe^{-\varepsilon(t-t_0)}, \tag{1}$$

where ε is the unique positive solution of the following equation

$$\varepsilon = a - b\gamma_1 - b\gamma_2 e^{\varepsilon\tau} - kb\gamma_3.$$

Proof. Set

$$y(t) = u(t)h + \gamma_1 \int_{t_0}^{t} u(t-s)m(s)ds + \gamma_2 \int_{t_0}^{t} u(t-s)m(s-\tau(s))ds$$

$$+\gamma_3 \int_{t_0}^{t} u(t-s)\int_{-\infty}^{s} k(s-\xi)m(\xi)d\xi ds.$$

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In views of $0 \le m(t) \le y(t)$, we have

$$\begin{aligned} y'(t) &= u'(t)h + \gamma_1 \int_{t_0}^{t} u'(t-s)m(s)ds \\ &\quad +\gamma_2 \int_{t_0}^{t} u'(t-s)m(s-\tau(s))ds \\ &\quad +\gamma_3 \int_{t_0}^{t} u'(t-s) \int_{-\infty}^{s} k(s-\xi)m(\xi)d\xi \\ &\quad +\gamma_1 u(0)m(t) + \gamma_2 u(0)m(t-\tau(t)) \\ &\quad +\gamma_3 u(0) \int_{-\infty}^{t} k(t-s)m(s)ds \\ &\le -ahu(t) - a\gamma_1 \int_{t_0}^{t} u(t-s)m(s)ds \\ &\quad -a\gamma_2 \int_{t_0}^{t} u(t-s)m(s-\tau(s))ds \\ &\quad -a\gamma_3 \int_{t_0}^{t} u(t-s) \int_{-\infty}^{s} k(s-\xi)m(\xi)d\xi + b\gamma_1 m(t) \\ &\quad +b\gamma_2 m(t-\tau(t)) + b\gamma_3 \int_{-\infty}^{t} k(t-s)m(s)ds \\ &= b\gamma_1 m(t) + b\gamma_2 m(t-\tau(t)) \\ &\quad +b\gamma_3 \int_{-\infty}^{t} k(t-s)m(s)ds - ay(t) \\ &\le (b\gamma_1 - a)y(t) + b\gamma_2 y(t-\tau(t)) \\ &\quad +b\gamma_3 \int_{-\infty}^{t} k(t-s)y(s)ds. \end{aligned} \tag{2}$$

Set $\widetilde{y}(t) = \{\sup_{-\infty<\theta\le 0} bhe^{\varepsilon(t_0+\theta)}\}e^{-\varepsilon t} = bhe^{-\varepsilon(t-t_0)}$, we first prove that $y(t) \le bhe^{-\varepsilon(t-t_0)}$. For arbitrary positive scalar $l > 1$, we claim that $y(t) \le lbhe^{-\varepsilon(t-t_0)}$. If it is not true, since $y(t) \le \widetilde{y}(t) = bhe^{-\varepsilon(t-t_0)} < lbhe^{-\varepsilon(t-t_0)} = l\widetilde{y}(t)$ for all $t \le t_0$, there must exist $t^* > t_0$ such that

$$y(t) < l\widetilde{y}(t), \forall t < t^*; \qquad y(t^*) = l\widetilde{y}(t^*).$$

Namely

$$y'(t^*) - l\widetilde{y}'(t^*) \ge 0. \tag{3}$$

On the other hand, from inequality (2) and the conditions of Theorem 2.1, we have

$$\begin{aligned} y'(t^*) &\le -(a-b\gamma_1)y(t^*) + b\gamma_2 y(t^*-\tau(t^*)) \\ &\quad +b\gamma_3 \int_{-\infty}^{t^*} k(t^*-s)y(s)ds \\ &= -l(a-b\gamma_1)\widetilde{y}(t^*) + b\gamma_2 y(t^*-\tau(t^*)) \\ &\quad +b\gamma_3 \int_{-\infty}^{t^*} k(t^*-s)y(s)ds \\ &< -l(a-b\gamma_1)\widetilde{y}(t^*) + b\gamma_2 l\widetilde{y}(t^*-\tau(t^*)) \\ &\quad +b\gamma_3 l \int_{-\infty}^{t^*} k(t^*-s)\widetilde{y}(s)ds \\ &= -l(a-b\gamma_1)bhe^{-\varepsilon(t^*-t_0)} + b^2\gamma_2 lhe^{-\varepsilon(t^*-\tau(t^*)-t_0)} \\ &\quad +b^2\gamma_3 l \int_{-\infty}^{t^*} k(t^*-s)he^{-\varepsilon(s-t_0)}ds \\ &= -l(a-b\gamma_1)bhe^{-\varepsilon(t^*-t_0)} + b^2\gamma_2 lhe^{-\varepsilon(t^*-\tau-t_0)} \\ &\quad +b^2\gamma_3 lhe^{-\varepsilon(t^*-t_0)} \int_0^{+\infty} e^{\varepsilon s}k(s)ds \\ &= [-(a-b\gamma_1) + kb\gamma_3 + b\gamma_2 e^{\varepsilon\tau}]lbhe^{-\varepsilon(t^*-t_0)} \\ &= -\varepsilon lbhe^{-\varepsilon(t^*-t_0)} = l\widetilde{y}'(t^*). \end{aligned} \tag{4}$$

This contradicts to inequality (3), thus, $y(t) \le lbhe^{-\varepsilon(t-t_0)}$. Let $l \to 1$, we can obtain that $y(t) \le bhe^{-\varepsilon(t-t_0)}$. Noting that $m(t) \le y(t)$, we have $m(t) \le bhe^{-\varepsilon(t-t_0)}$, which complete the proof.

Theorem 2.2: If there exist positive scalars a, b, h, τ, γ_1, γ_2, γ_3, nonnegative continuous functions $m(t)$, $k(t)$ and nonnegative continuous differentiable functions $u(t)$ on interval $[t_0-\tau, +\infty)$ such that the following conditions hold:

$$\begin{cases} m(t) \le u(t)h + \gamma_1 \displaystyle\int_{t_0}^{t} u(t-s)m(s)ds \\ \qquad + \gamma_2 \displaystyle\int_{t_0}^{t} u(t-s)m(s-\tau(s))ds \\ \qquad + \gamma_3 \displaystyle\int_{t_0}^{t} u(t-s) \int_{-\infty}^{s} k(s-\xi)m(\xi)d\xi ds \\ u'(t) \le -au(t), \ u(0) = b, \\ \qquad a > b\gamma_1 + b\gamma_2 + b\gamma_3, \ \displaystyle\int_0^{\infty} k(s)ds = 1 \end{cases}$$

where $t_0 \ge 0, 0 \le \tau(t) \le \tau$, then as $t \ge t_0$, we have

$$m(t) \le \{\sup_{-\infty<\theta\le 0} bhe^{a(t_0+\theta)}\} = bhe^{at_0}, and \lim_{t\to+\infty} m(t) = 0.$$

Proof. We will complete the proof in two steps. In step 1, we will prove that $m(t) \le y_{t_0} \triangleq bhe^{at_0} = \sup_{-\infty<\theta\le 0} bhe^{a(t_0+\theta)}$. In step 2, we will prove that $\lim_{t\to+\infty} m(t) = 0$.

Step 1: we first prove that for any positive constant $d > 1$, the following inequality holds

$$y(t) < d \cdot y_{t_0}, \ t \ge t_0, \tag{5}$$

where $y(t)$ is the same as defined in Theorem 2.1. Since for any $t \in (-\infty, t_0)$, $y(t) \le \sup_{-\infty<\theta\le 0} bhe^{a(t_0+\theta)} = y_{t_0}$. If $y_{t_0} = 0$, then we get $0 \le y(t) \le 0$, namely $y(t) \equiv 0$. Thus, we always assume that $y_{t_0} > 0$. When $t \le t_0$, we have $y(t) \le y_{t_0} < d \cdot y_{t_0}$. If inequality (5) is not true, there must exist $t_1 > t_0$ such that

$$y(t_1) = d \cdot y_{t_0}, \ y(t) < d \cdot y_{t_0}, \ \forall t < t_1,$$

which implies that $y'(t_1) \ge 0$. From inequality (2), we have

$$\begin{aligned} y'(t_1) &\le -(a-b\gamma_1)y(t_1) + b\gamma_2 y(t_1-\tau(t_1)) \\ &\quad +b\gamma_3 \int_{-\infty}^{t_1} k(t_1-s)y(s)ds \\ &< -(a-b\gamma_1)d \cdot y_{t_0} + b\gamma_2 d \cdot y_{t_0} \\ &\quad +b\gamma_3 \int_{-\infty}^{t_1} d \cdot k(t_1-s)y_{t_0}ds \\ &= [-(a-b\gamma_1) + b\gamma_2 + b\gamma_3 \int_{-\infty}^{t_1} k(t_1-s)ds]d \cdot y_{t_0} \\ &= [-(a-b\gamma_1) + b\gamma_2 + b\gamma_3 \int_0^{+\infty} k(s)ds]d \cdot y_{t_0} \\ &= [-(a-b\gamma_1) + b\gamma_2 + b\gamma_3]d \cdot y_{t_0} < 0. \end{aligned} \tag{6}$$

This contradicts to $y'(t_1) \ge 0$, namely (5) holds. According to the arbitrary property of positive constant d, we have $y(t) \le bhe^{at_0}$. In views of $m(t) \le y(t)$, we get

$$m(t) \le bhe^{at_0}, \ \forall t \ge t_0. \tag{7}$$

Step 2: In what follows, we will prove $\lim_{t\to+\infty} m(t) = 0$.

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From inequality (5), we know that $y(t)$ is a bounded continuous function, thus when $t \to +\infty$, the upper limit(noted by p) of $y(t)$ exists, namely

$$\overline{\lim}_{t\to+\infty} y(t) = p, p \geq 0. \quad (8)$$

The remaining proof is to prove $p = 0$.

If it is not true, there must exist arbitrary positive constant $\varepsilon > 0$, and constant $T_1 > t_0$ such that

$$y(t-\tau(t)) < p+\varepsilon, \quad y(t) < p+\varepsilon, \quad \forall t > T_1.$$

On the other hand, since $\int_0^\infty k(s)ds = 1$, there must exist $T_2 > t_0$ such that

$$\int_t^{+\infty} k(s)ds < \varepsilon, \forall t \geq T_2.$$

Set $T = \max\{T_1, T_2\}$, when $t \geq 2T$, we have

$$\begin{aligned}
y'(t) \quad &\leq \quad -(a-b\gamma_1)y(t) + b\gamma_2 y(t-\tau(t)) \\
&\quad +b\gamma_3 \int_{-\infty}^t k(t-s)y(s)ds \\
&= \quad -(a-b\gamma_1)y(t) + b\gamma_2 y(t-\tau(t)) \\
&\quad +b\gamma_3 \int_{-\infty}^{t-T} k(t-s)y(s)ds + b\gamma_3 \int_{t-T}^t k(t-s)y(s)ds \\
&< \quad -(a-b\gamma_1)y(t) + b\gamma_2 y(t-\tau(t)) \\
&\quad +b\gamma_3 y_{t_0} \int_{-\infty}^{t-T} k(t-s)ds + (p+\varepsilon)b\gamma_3 \int_{t-T}^t k(t-s)ds \\
&= \quad -(a-b\gamma_1)y(t) + b\gamma_2 y(t-\tau(t)) \\
&\quad +b\gamma_3 y_{t_0} \int_T^{+\infty} k(s)ds + (p+\varepsilon)b\gamma_3 \int_0^T k(s)ds \\
&< -(a-b\gamma_1)y(t) + b\gamma_2(p+\varepsilon) + b\gamma_3\varepsilon y_{t_0} + (p+\varepsilon)b\gamma_3. \quad (9)
\end{aligned}$$

By direct calculation, we get

$$y(t) \leq y(t_0)\exp\{-(a-b\gamma_1)(t-t_0)\}$$
$$+\frac{1}{(a-b\gamma_1)}[(p+\varepsilon)b\gamma_2 + \varepsilon b\gamma_3 y_{t_0} + pb\gamma_3 + \varepsilon b\gamma_3].$$

From (8), we get

$$p \leq \frac{1}{(a-b\gamma_1)}[b\gamma_2\varepsilon + b\varepsilon\gamma_3 y_{t_0} + pb\gamma_3 + pb\gamma_2 + \varepsilon b\gamma_3].$$

In views of the arbitrary property of ε, we have $p \leq \frac{pb\gamma_3 + pb\gamma_2}{(a-b\gamma_1)}$, namely $(a-b\gamma_1) \leq b\gamma_2 + b\gamma_3$, which contradicts to $a > b\gamma_1 + b\gamma_2 + b\gamma_3$, thus $\lim_{t\to+\infty} y(t) = 0$ holds, namely, $\lim_{t\to+\infty} m(t) = 0$. The proof is completed.

Similar to the proof of Theorem 2.1, Theorem 2.2, we can easily obtain the following Corollaries.

Corollary 2.1: If there exist positive scalars a, b, h, τ, γ_1, γ_2, γ_3, nonnegative continuous functions $m(t)$, $k(t)$ and nonnegative continuous differentiable functions $u(t)$ on interval $[t_0-\tau, +\infty)$ such that the following conditions hold:

$$\begin{cases}
m(t) \leq u(t)h + \gamma_1 \int_{t_0}^t u(t-s)m(s)ds \\
\qquad + \gamma_2 \int_{t_0}^t u(t-s)m(s-\tau(s))ds \\
\qquad + \gamma_3 \int_{t_0}^t u(t-s) \int_{-\infty}^s k(s-\xi)m(\xi)d\xi ds, \\
u'(t) \leq -au(t), \ u(0) = b, \\
\qquad a > b\gamma_1 + b\gamma_2 + b\gamma_3, \ 1 = \int_0^\infty e^{\varepsilon s}k(s)ds
\end{cases}$$

where $t_0 \geq 0, 0 \leq \tau(t) \leq \tau$, then as $t \geq t_0$, we have

$$m(t) \leq bhe^{-\varepsilon(t-t_0)},$$

where ε is the unique positive solution of the following equation

$$\varepsilon = a - b\gamma_1 - b\gamma_2 e^{\varepsilon\tau} - b\gamma_3.$$

Corollary 2.2: If there exist positive scalars a, b, h, τ, γ_1, γ_2, γ_3, ε, nonnegative continuous functions $m(t)$, $k(t)$ and nonnegative continuous differentiable functions $u(t)$ on interval $[t_0-\tau, +\infty)$ such that the following conditions hold:

$$\begin{cases}
m(t) \leq u(t)h + \gamma_1 \int_{t_0}^t u(t-s)m(s)ds \\
\qquad + \gamma_2 \int_{t_0}^t u(t-s)m(s-\tau(s))ds \\
\qquad + \gamma_3 \int_{t_0}^t u(t-s) \int_{-\infty}^s k(s-\xi)m(\xi)d\xi ds \\
u'(t) \leq -au(t), \ u(0) = b, \\
\qquad a > b\gamma_1 + b\gamma_2 + bk\gamma_3, \ \int_0^\infty k(s)ds = k
\end{cases}$$

where $t_0 \geq 0, 0 \leq \tau(t) \leq \tau$, then as $t \geq t_0$, we have

$$m(t) \leq \{\sup_{-\infty<\theta\leq 0} bhe^{a(t_0+\theta)}\} = bhe^{at_0}, and \lim_{t\to+\infty} m(t) = 0.$$

III. Applications

The inequalities obtained in Section 2 can be widely applied to research the stability of delayed integral and differential dynamic systems. To illustrate the validity, consider the following integro-differential dynamic system:

$$\begin{cases}
x_i'(t) = -c_i x_i(t) + \sum_{j=1}^n a_{ij} f_j(x_j(t)) + \sum_{j=1}^n b_{ij} g_j(x_j(t-\tau(t))) \\
\qquad + \sum_{j=1}^n d_{ij} \int_{-\infty}^t k_{ij}(t-s) f_j(x_j(s))ds, \\
x_i(t) = \varphi(t), \ t \leq 0
\end{cases} \quad (10)$$

where $x(t) \in \mathscr{R}^n$ is state vector; $c_i > 0$, a_{ij}, b_{ij} and d_{ij} represent the connection weight and the delayed connection weight respectively; f_i, g_i are continuous functions satisfying $|f_i(x) - f_i(y)| \leq l_i|x-y|, |g_i(x) - g_i(y)| \leq l_i'|x-y|, \forall x, y \in \mathscr{R}$, where $l_i, l_i' (i = 1, 2, \cdots, n)$ are Lipschitz constant; $f(x(t)) = (f_1(x_1(t)), f_2(x_2(t)), \ldots, f_n(x_n(t)))^T$, $g(x(t)) = (g_1(x_1(t)), g_2(x_2(t)), \ldots, g_n(x_n(t)))^T$. $0 < \tau(t) \leq \tau$ is transmission delay. Kernel functions $k_{ij}(t)(i, j = 1, 2, \ldots, n)$, are real-valued nonnegative continuous functions defined on $[0, \infty)$. $\varphi(t)$ is initial condition satisfying $\varphi(t) \in C((-\infty, 0], \mathscr{R}^n)$ and $\sup_{-\infty<t\leq 0} |\varphi(t)|^p < \infty$, where $C((-\infty, 0], \mathscr{R}^n)$ denote the family of all continuous $\mathscr{R}^n-$valued functions $\phi(t)$ on $(-\infty, 0]$ with the norm $\|\varphi(t)\|_\Delta^p = \sup_{-\infty<t\leq 0} |\varphi(t)|^p$. For the further discussion, the following standard hypothesis, definition and lemmas are needed.

(H_1) Assume that $f(0) \equiv 0, g(0) \equiv 0$.

(H_2) $\int_0^\infty k_{ij}(t)dt = 1$, $i, j = 1, 2, \cdots, n$.

(H_3) There exists an $\varepsilon > 0$ such that $\int_0^\infty e^{\varepsilon t} k_{ij}(t)dt \triangleq \overline{k}_{ij} < \infty$. $k(t) \triangleq \sup_{1\leq i,j\leq n}\{k_{ij}(t)\}, k' \triangleq \max_{1\leq i,j\leq n}(\overline{k}_{ij})$.

Definition 3.1: The trivial solution of system (10) is said to be p-exponentially stable if there exists a pair of positive constants λ and α such that

$$\|x(t)\|^p \leq \alpha\|\varphi\|_\Delta^p e^{-\lambda t}, t \geq 0.$$

Lemma 3.1: (Holder inequality)[16]) Assume that there exist two continuous functions $f(x), g(x)$ and a set Ω, p and q satisfying $1/q + 1/p = 1$, for any $p > 0, q > 0$, if $p > 1$, then the following inequality holds

$$\int_\Omega |f(x)g(x)|dx \leq (\int_\Omega |f(x)|^p dx)^{1/p} (\int_\Omega |g(x)|^q dx)^{1/q}.$$

Lemma 3.2: [17] *Assume that there exist constants* $a_k \geq 0, k = 1, 2, \ldots, n, p$ *and* q *satisfying* $1/q + 1/p = 1$,*for any* $p \geq 0, q \geq 0$,*if* $p > 1$,*then the following inequality holds*

$$(\sum_{k=1}^n a_k)^p \leq n^{p-1} \sum_{k=1}^n a_k^p.$$

Applying the inequalities obtained in Section 2, we can obtain the following stability results.

Theorem 3.1: Under the assumptions $(H_1), (H_3)$, the trivial solution of system (10) is p-exponentially stable($p \geq 2$), if

$$\gamma_1 + \gamma_2 + k'\gamma_3 < c, \tag{11}$$

where

$$\begin{aligned}
\gamma_1 &= [c^{-\frac{p}{q}} \sum_{j=1}^n [\sum_{i=1}^n |a_{ji}|^q |l_i|^q]^{\frac{p}{q}}] 4^{p-1}, c = \min\{c_1, c_2, \cdots, c_n\}, \\
\gamma_2 &= [c^{-\frac{p}{q}} \sum_{j=1}^n [\sum_{i=1}^n |b_{ji}|^q |l_i'|^q]^{\frac{p}{q}}] 4^{p-1}, \\
\gamma_3 &= 4^{p-1} (\frac{c}{k'})^{-\frac{p}{q}} \sum_{j=1}^n (\sum_{i=1}^n |d_{ji}|^q |l_i|^q)^{\frac{p}{q}}, \\
k' &= \max_{1 \leq i,j \leq n} \{\overline{k}_{ij}\}, q = \frac{p}{p-1}.
\end{aligned} \tag{12}$$

Proof. For system (10), by using the method of variation of parameters, we have

$$\begin{aligned}
|x_i(t)| &\leq e^{-c_i t}|x_i(0)| + \int_0^t e^{-c_i(t-s)} |\sum_{j=1}^n a_{ij} f_j(x_j(s))| ds \\
&\quad + \int_0^t e^{-c_i(t-s)} |\sum_{j=1}^n b_{ij} g_j(x_j(s - \tau_j(s)))| ds \\
&\quad + \int_0^t e^{-c_i(t-s)} |\sum_{j=1}^n d_{ij} \int_{-\infty}^s k_{ij}(s-v) f_j(x_j(v - \tau_j(s))) dv| ds \\
&\leq e^{-ct}|x_i(0)| + \int_0^t e^{-c(t-s)} |\sum_{j=1}^n a_{ij} f_j(x_j(s))| ds \\
&\quad + |\int_0^t e^{-c(t-s)} |\sum_{j=1}^n b_{ij} g_j(x_j(s - \tau_j(s)))| ds \\
&\quad + \int_0^t e^{-c(t-s)} |\sum_{j=1}^n d_{ij} \int_{-\infty}^s k_{ij}(s-v) f_j(x_j(v - \tau_j(s))) dv| ds \\
&\triangleq I_{1i} + I_{2i} + I_{3i} + I_{4i}.
\end{aligned} \tag{13}$$

In views of Lemma 3.2, the following inequality holds

$$\sum_{i=1}^n |x_i(t)|^p \leq 4^{p-1} \sum_{i=1}^n (I_{1i}^p + I_{2i}^p + I_{3i}^p + I_{4i}^p).$$

By Lemma 3.1, we can obtain

$$\begin{aligned}
\sum_{i=1}^n I_{2i}^p &= \sum_{i=1}^n [\int_0^t e^{-c(t-s)} |\sum_{j=1}^n a_{ij} f_j(x_j(s))| ds]^p \\
&= \sum_{i=1}^n [\int_0^t e^{-c(t-s)} (\sum_{j=1}^n |a_{ij}||f_j(x_j(s))|) ds]^p \\
&= \sum_{i=1}^n [\int_0^t e^{\frac{-c(t-s)}{q}} e^{\frac{-c(t-s)}{p}} (\sum_{j=1}^n |a_{ij}||f_j(x_j(s))|) ds]^p \\
&\leq \sum_{i=1}^n \{[\int_0^t e^{-c(t-s)} ds]^{\frac{p}{q}} \int_0^t e^{-c(t-s)} [\sum_{j=1}^n |a_{ij}||f_j(x_j(s))|]^p ds\} \\
&\leq c^{-\frac{p}{q}} \sum_{i=1}^n \{\int_0^t e^{-c(t-s)} [\sum_{j=1}^n |a_{ij}|^q |l_j|^q]^{\frac{p}{q}} [\sum_{j=1}^n |x_j(s)|^p] ds\} \\
&= c^{-\frac{p}{q}} \sum_{i=1}^n \{[\sum_{j=1}^n |a_{ij}|^q |l_j|^q]^{\frac{p}{q}} \int_0^t e^{-c(t-s)} \sum_{j=1}^n |x_j(s)|^p ds\} \\
&= c^{-\frac{p}{q}} \sum_{j=1}^n \{[\sum_{i=1}^n |a_{ji}|^q |l_i|^q]^{\frac{p}{q}} \int_0^t e^{-c(t-s)} \sum_{i=1}^n |x_i(s)|^p ds\} \\
&= c^{-\frac{p}{q}} \sum_{j=1}^n [\sum_{i=1}^n |a_{ji}|^q |l_i|^q]^{\frac{p}{q}} \int_0^t e^{-c(t-s)} \sum_{i=1}^n |x_i(s)|^p ds.
\end{aligned} \tag{14}$$

Similarly, for I_{3i}^p, I_{4i}^p, we have

$$\sum_{i=1}^n I_{3i}^p \leq c^{-\frac{p}{q}} \sum_{j=1}^n [\sum_{i=1}^n |b_{ji}|^q |l_i'|^q]^{\frac{p}{q}} \int_0^t e^{-c(t-s)} \sum_{i=1}^n |x_i(s - \tau_i(s))|^p ds. \tag{15}$$

$$\begin{aligned}
\sum_{i=1}^n I_{4i}^p &= \sum_{i=1}^n [\int_0^t e^{-c(t-s)} \\
&\quad \times |\sum_{j=1}^n d_{ij} \int_{-\infty}^s k_{ij}(s-v) f_j(x_j(v)) dv| ds]^p \\
&\leq \sum_{i=1}^n [\int_0^t e^{-c(t-s)} \\
&\quad \times (\sum_{j=1}^n |d_{ij}| \int_{-\infty}^s k_{ij}(s-v) |f_j(x_j(v))| dv) ds]^p \\
&\leq \sum_{i=1}^n \{[\int_0^t e^{-c(t-s)} ds]^{\frac{p}{q}} [\int_0^t e^{-c(t-s)} \\
&\quad \times (\sum_{j=1}^n |d_{ij}| \int_{-\infty}^s k_{ij}(s-v) |f_j(x_j(v))| dv)^p ds]\} \\
&= \sum_{i=1}^n \{[\frac{1 - e^{-ct}}{c}]^{\frac{p}{q}} [\int_0^t e^{-c(t-s)} \\
&\quad \times (\sum_{j=1}^n |d_{ij}| \int_{-\infty}^s k_{ij}(s-v) |f_j(x_j(v)| dv)^p ds]\} \\
&\leq c^{-\frac{p}{q}} \sum_{i=1}^n \{\int_0^t e^{-c(t-s)} \\
&\quad \times (\sum_{j=1}^n |d_{ij}| \int_{-\infty}^s k_{ij}(s-v) |f_j(x_j(v))| dv)^p ds\}
\end{aligned}$$

$$\begin{aligned}
&\leq c^{-\frac{p}{q}}\sum_{i=1}^{n}\{\int_0^t e^{-c(t-s)}\\
&\quad\times(\sum_{j=1}^{n}|d_{ij}||l_j|\int_{-\infty}^{s}k_{ij}(s-v)|x_j(v)|dv)^p ds\}\\
&\leq c^{-\frac{p}{q}}\sum_{i=1}^{n}\{\int_0^t e^{-c(t-s)}\\
&\quad\times(\sum_{j=1}^{n}|d_{ij}|^q|l_j|^q)^{\frac{p}{q}}\sum_{j=1}^{n}(\int_{-\infty}^{s}k_{ij}(s-v)|x_j(v)|dv)^p ds\}\\
&= c^{-\frac{p}{q}}\sum_{i=1}^{n}\{(\sum_{j=1}^{n}|d_{ij}|^q|l_j|^q)^{\frac{p}{q}}\{\int_0^t e^{-c(t-s)}\\
&\quad\times\sum_{j=1}^{n}(\int_{-\infty}^{s}k_{ij}(s-v)|x_j(v)|dv)^p ds\}\}\\
&= c^{-\frac{p}{q}}\sum_{i=1}^{n}\{(\sum_{j=1}^{n}|d_{ij}|^q|l_j|^q)^{\frac{p}{q}}\{\int_0^t e^{-c(t-s)}\\
&\quad\times\sum_{j=1}^{n}(\int_{-\infty}^{s}k_{ij}^{\frac{1}{q}}(s-v)k_{ij}^{\frac{1}{p}}(s-v)|x_j(v)|dv)^p ds\}\}\\
&\leq c^{-\frac{p}{q}}\sum_{i=1}^{n}\{(\sum_{j=1}^{n}|d_{ij}|^q|l_j|^q)^{\frac{p}{q}}\{\int_0^t e^{-c(t-s)}\\
&\quad\times\sum_{j=1}^{n}k'^{\frac{p}{q}}\int_{-\infty}^{s}k(s-v)|x_j(v)|^p dvds\}\}\\
&= (\frac{c}{k'})^{-\frac{p}{q}}[\sum_{j=1}^{n}(\sum_{i=1}^{n}|d_{ji}|^q|l_i|^q)^{\frac{p}{q}}]\\
&\quad\times[\int_0^t e^{-c(t-s)}\int_{-\infty}^{s}k(s-v)\sum_{i=1}^{n}|x_i(v)|^p dvds]. \qquad (16)
\end{aligned}$$

Set $u(t)=e^{-ct}$, one can easily obtains that $u'(t)\leq -cu(t)$, $u(0)=1$. From inequalities (12)-(15), and Theorem 2.1, there exists an $\varepsilon>0$ such that $\sum_{i=1}^{n}|x_i(t)|^p\leq\sum_{i=1}^{n}|\varphi_i|^p e^{-\varepsilon t}$ ($t_0=0$), namely

$$\|x(t)\|^p\leq\|\varphi\|_{\Delta}^p e^{-\varepsilon t},\ t\geq 0.$$

The proof is completed.

Theorem 3.2: Under the assumptions $(H_1),(H_2)$, the trivial solution of system (10) is p-asymptotically stable($p\geq 2$), if

$$\gamma_1+\gamma_2+\gamma_3'<c, \qquad (17)$$

where $\gamma_3'=4^{p-1}c^{-\frac{p}{q}}\sum_{j=1}^{n}(\sum_{i=1}^{n}|d_{ji}|^q|l_i|^q)^{\frac{p}{q}}, q=\frac{p}{p-1}$.

Proof. In views of (H_2), similar to the proof of Theorem 3.1, inequality (15) becomes the following form

$$\sum_{i=1}^{n}I_{4i}^p\leq(c)^{-\frac{p}{q}}[\sum_{j=1}^{n}(\sum_{i=1}^{n}|d_{ji}|^q|l_i|^q)^{\frac{p}{q}}] \qquad (18)$$

$$\times[\int_0^t e^{-c(t-s)}\int_{-\infty}^{s}k(s-v)\sum_{i=1}^{n}|x_i(v)|^p dvds]. \qquad (19)$$

From inequalities (12)-(14), (17) and Theorem 2.2, we can get that the trivial solution of system (10) is p-asymptotically stable.

Theorem 3.3: Under the assumptions $(H_1)-(H_3)$, the trivial solution of system (10) is p-exponentially stable($p\geq 2$), if

$$\gamma_1+\gamma_2+k'\gamma_3'<c. \qquad (20)$$

Proof.

From inequalities (12)-(14), (17) and Theorem 2.1, we can easily obtain this result.

Remark 1. In some previous literature, the time-varying delay $\tau(t)$ is assumed to be differential and it's derivative is simultaneously required to be not greater than 1 or a positive constant, and may impose a very strict constraint on model because time delays sometimes vary dramatically with time in real circuits. In our results, we only require $0<\tau(t)\leq\tau$.

Remark 2. In [18], [19], kernel functions are assumed to satisfy $\int_0^\infty k(s)ds=1$, $\int_0^\infty e^{\varepsilon s}k(s)ds<\infty$, and $\int_0^\infty se^{\varepsilon s}k(s)ds<\infty$. In our results, they are only assumed to satisfy one or two above conditions, thus our results enlarge the selection of kernel functions, which will be shown in three examples provided later (Details see example 1,2).

IV. Numerical examples

In this section, two numerical examples will be presented to show the validity of our results.

Example 1. Consider the following two-dimensional integral-differential equation with mixed delays.

$$\begin{cases} dx(t)=[-Cx(t)+Af(x(t))+Bg(x(t-\tau(t)))\\ \qquad +D\int_{-\infty}^{t}K(t-s)f(x(s))ds]dt,\ t>0\\ x(t)=\eta(t),t\leq 0.\end{cases} \qquad (21)$$

where

$$C=\begin{bmatrix}2&0\\0&2\end{bmatrix},A=\begin{bmatrix}0.21&0.1\\0.3&0.1\end{bmatrix},B=\begin{bmatrix}-0.31&0.11\\0.21&-0.31\end{bmatrix},$$

$$D=\begin{bmatrix}-0.51&0.31\\0.3&0.25\end{bmatrix},K(t)=\begin{bmatrix}e^{-2t}&e^{-2t}\\e^{-2t}&e^{-2t}\end{bmatrix}.$$

$f(x)=g(x)=\tanh(x),\tau(t)=2+0.02|\sin t|,\eta(t)=[-3.4,5.6]^T$,

By direct calculation, we get that $l_1=l_2=l_1'=l_2'=1$. Set $\varepsilon=0.1$, we can obtain $k'=\max_{1\leq i,j\leq 2}\{\bar{k}_{ij}\}\approx 1.1$. Let $p=2$, we can get $\gamma_1=0.3082$, $\gamma_2=0.4846$, $k'\gamma_3=0.9249$, $\gamma_2+\gamma_2+k'\gamma_3=1.8102<c=2$. In views of Theorem 3.1, the equilibrium point $(0,0)^T$ of the given system (19) is exponentially stable.

Remark 3. From system (19), we can see that kernel function $k_{ij}(t)=e^{-2t}$ (i,j=1,2). It obviously satisfies (H_3), but does not satisfy (H_2). Time-varying delay $2-0.02|\sin t|$ is non-differentiable, thus previous results in the literature cited therein can not be used to judge the stability of this system.

Example 2. In example 1, set

$$C=\begin{bmatrix}1.3&0\\0&1.3\end{bmatrix},A=\begin{bmatrix}0.11&0.1\\0.3&0.1\end{bmatrix},B=\begin{bmatrix}-0.21&0.11\\0.21&-0.11\end{bmatrix},$$

$$D=\begin{bmatrix}-0.21&0.11\\0.2&0.25\end{bmatrix},K(t)=[k_{ij}(t)]_{2\times 2},$$

$\eta(t)=[-4.4,6]^T$, $k_{ij}(t)=\frac{\pi}{2+2t^2}$ (i,j=1,2). We can verify that $k_{ij}(t)$ satisfies (H_2), but dose not satisfy (H_3). By simple calculation, we get that $l_1,l_2,l_1',l_2'=1$. Let $p=2$, we have $\gamma_1=0.3757$, $\gamma_2=0.3458$, $\gamma_3=0.3995,\gamma_1+\gamma_2+\gamma_3=1.2098<c=1.3$. In views of Theorem 3.2, the equilibrium point $(0,0)^T$ of the given system (19) is asymptotically stable. However, the results obtained in the literature cited therein can not be used to judge the stability of this system.

V. Conclusion

In this paper, some new Gronwall-Bellman-Type inequalities with mixed delays are established. Applying these new established inequalities, some new sufficient conditions ensuring p-exponential stability of a integro-differential equation are obtained. The results improve and generalize some previous works. Numerical examples show that our results are valid.

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