Adding Edges between One Node and Every Other Node with the Same Depth in a Complete K-ary Tree

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Abstract—This paper proposes a model of adding relations between members of the same level in a pyramid organization structure which is a complete K-ary tree such that the communication of information between every member in the organization becomes the most efficient. When edges between one node and every other node with the same depth N in a complete K-ary tree of height H are added, an optimal depth $N^* = H$ is obtained by minimizing the total path length which is the sum of lengths of shortest paths between every pair of all nodes.

Keywords—complete K-ary tree, organization structure, shortest path

I. INTRODUCTION

IN the pyramid organization based on the principle of unity of command [1] there exist relations only between each superior and his subordinates [2]. However, it is desirable to have formed additional relations other than that between each superior and his subordinates in advance in case they need communication with other departments in the organization. In companies, the relations with other departments are built by meetings, group training, internal projects, and so on. Personal relations exceeding departments are also considered to be useful for the communication of information in the organization.

The pyramid organization structure can be expressed as a rooted tree, if we let nodes and edges in the rooted tree correspond to members and relations between members in the organization respectively [3], [4]. Then the path between each node in the rooted tree is equivalent to the route of communication of information between each member in the organization. Moreover, adding edges to the rooted tree is equivalent to forming additional relations other than that between each superior and his subordinates.

The purpose of our study is to obtain an optimal set of additional relations to the pyramid organization such that the communication of information between every member in the organization becomes the most efficient. This means that we obtain a set of additional edges to the rooted tree minimizing the sum of lengths of shortest paths between every pair of all nodes. We have obtained an optimal set for each of the following three models of adding relations in the same level to an organization structure which is a complete K-ary tree of

height H: (i) a model of adding an edge between two nodes with the same depth, (ii) a model of adding edges between every pair of nodes with the same depth and (iii) a model of adding edges between every pair of siblings with the same depth [5]. A complete K-ary tree is a rooted tree in which all leaves have the same depth and all internal nodes have K ($K = 2, 3, \cdots$) children [6].

This paper proposes a model of adding edges between one node and every other node with the same depth N ($N = 1, 2, \cdots, H$) in a complete K-ary tree of height H ($H = 1, 2, \cdots$). This model is equivalent to forming additional relations between a representative and the other members in the same level.

If $l_{i,j} (= l_{j,i})$ denotes the path length, which is the number of edges in the shortest path from a node v_i to a node v_j ($i, j = 1, 2, \cdots, (K^{H+1} - 1)/(K - 1)$) in the complete K-ary tree of height H, then $\sum_{i<j} l_{i,j}$ is the total path length. Furthermore, if $l'_{i,j}$ denotes the path length from v_i to v_j after adding edges in the above models, $l_{i,j} - l'_{i,j}$ is called the shortening path length between v_i and v_j, and $\sum_{i<j}(l_{i,j} - l'_{i,j})$ is called the *total shortening path length.*

In Section 2 the total shortening path length is formulated when new edges between one node and every other node with the same depth N in a complete K-ary tree of height H are added. In Section 3 an optimal depth N^* which maximizes the total shortening path length is obtained.

II. FORMULATION OF TOTAL SHORTENING PATH LENGTH

Let $S_H(N)$ denote the total shortening path length, when we add edges between one node and every other node with the same depth N ($N = 1, 2, \cdots, H$) in a complete K-ary tree of height H ($H = 1, 2, \cdots$).

The total shortening path length $S_H(N)$ can be formulated by adding up the following three sums of shortening path lengths: (i) the sum of shortening path lengths between every pair of nodes whose depths are equal to or more than N, (ii) the sum of shortening path lengths between every pair of nodes whose depths are less than N and those whose depths are equal to or more than N and (iii) the sum of shortening path lengths between every pair of nodes whose depths are less than N.

The sum of shortening path lengths between every pair of

nodes whose depths are equal to or more than N is given by

$$
\begin{aligned}
A_H(N) &= \{M(H-N)\}^2(K^N-1) \\
&\quad + \{M(H-N)\}^2(K-1)K^N\sum_{i=1}^{N-1}iK^i, (1)
\end{aligned}
$$

where $M(h)$ denotes the number of nodes of a complete K-ary tree of height h $(h=0,1,2,\cdots)$, and we define

$$
\sum_{i=1}^{0}\cdot = 0 . \tag{2}
$$

The sum of shortening path lengths between every pair of nodes whose depths are less than N and those whose depths are equal to or more than N is given by

$$
\begin{aligned}
B_H(N) &= M(H-N)\{M(N-1)-N\} \\
&\quad + M(H-N)(K-1)\sum_{i=1}^{N-1}iK^i \\
&\quad + M(H-N)(K-1)K^N\sum_{i=1}^{N-2}2i(N-i-1)K^i,
\end{aligned}
\tag{3}
$$

and the sum of shortening path lengths between every pair of nodes whose depths are less than N is given by

$$
\begin{aligned}
C_H(N) &= (K-1)\sum_{i=1}^{N-2}i(N-i-1)K^i \\
&\quad + (K-1)K^N\sum_{i=1}^{N-3}\sum_{j=1}^{i}j(i-j+1)K^j, (4)
\end{aligned}
$$

where we define

$$
\sum_{i=1}^{-1}\cdot = 0 , \tag{5}
$$

$$
\sum_{i=1}^{-2}\cdot = 0 . \tag{6}
$$

From these equations, the total shortening path length $S_H(N)$ is given by

$$
\begin{aligned}
S_H(N) &= A_H(N) + B_H(N) + C_H(N) \\
&= \{M(H-N)\}^2(K^N-1) \\
&\quad + \{M(H-N)\}^2(K-1)K^N\sum_{i=1}^{N-1}iK^i \\
&\quad + M(H-N)\{M(N-1)-N\} \\
&\quad + M(H-N)(K-1)\sum_{i=1}^{N-1}iK^i \\
&\quad + M(H-N)(K-1)K^N\sum_{i=1}^{N-2}2i(N-i-1)K^i \\
&\quad + (K-1)\sum_{i=1}^{N-2}i(N-i-1)K^i
\end{aligned}
$$

$$
+ (K-1)K^N\sum_{i=1}^{N-3}\sum_{j=1}^{i}j(i-j+1)K^j . \tag{7}
$$

Since the number of nodes of a complete K-ary tree of height h is

$$
M(h) = \frac{K^{h+1}-1}{K-1} , \tag{8}
$$

$S_H(N)$ of Equation (7) becomes

$$
\begin{aligned}
&S_H(N) \\
&= \frac{1}{2(K-1)^3}\Bigg\{\Big[-2(K-1)K^{-2N+2} \\
&\quad + 2(2K-1)K^{-N+2}+2(N-1)K^3-2NK^2\Big]K^{2H} \\
&\quad + \Big[-4K^{N+2}-2(K-1)\{(N-1)K-N-1\} \\
&\quad \times K^{-N+1}+2K\{(3N-1)K^2-2(2N-1)K \\
&\quad + N+1\}\Big]K^H \\
&\quad + (K-1)\{N(N-1)K-N^2+N-2\}K^{N+1} \\
&\quad + 2(K-1)(NK^2+K-N)\Bigg\} . \tag{9}
\end{aligned}
$$

III. AN OPTIMAL ADDING DEPTH

In this section, $N = N^*$ which maximizes $S_H(N)$ in Equation (9) is sought.

Let $\Delta S_H(N) \equiv S_H(N+1) - S_H(N)$, so that we have

$$
\begin{aligned}
&\Delta S_H(N) \\
&\equiv S_H(N+1) - S_H(N) \\
&= \frac{1}{2(K-1)^2}\Bigg\{\Big[2(K-1)(K+1)K^{-2N} \\
&\quad - 2(2K-1)K^{-N+1}+2K^2\Big]K^{2H} \\
&\quad + \Big[-4K^{N+2}+2(K-1)\{(N-1)K-N-2\}K^{-N} \\
&\quad + 2K(3K-1)\Big]K^H \\
&\quad + (K-1)\{N(N+1)K-N^2+N-2\}K^{N+1} \\
&\quad + 2(K-1)(K+1)\Bigg\} , \tag{10}
\end{aligned}
$$

for $N = 1, 2, \cdots, H-1$. Let us define a continuous variable x which depends on H as

$$
x = K^H , \tag{11}
$$

then $\Delta S_H(N)$ becomes

$$
\begin{aligned}
&T_N(x) \\
&= \frac{1}{2(K-1)^2}\Bigg\{\Big[2(K-1)(K+1)K^{-2N} \\
&\quad - 2(2K-1)K^{-N+1}+2K^2\Big]x^2 \\
&\quad + \Big[-4K^{N+2}+2(K-1)\{(N-1)K-N-2\}K^{-N} \\
&\quad + 2K(3K-1)\Big]x \\
&\quad + (K-1)\{N(N+1)K-N^2+N-2\}K^{N+1} \\
&\quad + 2(K-1)(K+1)\Bigg\} , \tag{12}
\end{aligned}
$$

which is a quadratic function of x.

By differentiating $T_N(x)$ with respect to x, we obtain

$$
\begin{aligned}
T'_N(x) \\
= \ \frac{1}{(K-1)^2}\Big[&\big\{ 2(K-1)(K+1)K^{-2N} \\
&- 2(2K-1)K^{-N+1} + 2K^2 \big\}x - 2K^{N+2} \\
&+ (K-1)\big\{(N-1)K - N - 2\big\}K^{-N} \\
&+ K(3K-1) \Big].
\end{aligned}
\tag{13}
$$

Since $T_N(x)$ is convex downward from $2(K-1)(K+1)K^{-2N} - 2(2K-1)K^{-N+1} + 2K^2 > 0$, and

$$
\begin{aligned}
T_N(K^{N+1}) \\
= \ \frac{1}{2(K-1)^2}\Big[&2(K-2)K^{2N+3} \\
&+ \big\{ -4K^3 + (N^2+N+8)K^2 - 2(N^2+2)K \\
&\quad + N^2 - N + 2 \big\}K^{N+1} \\
&+ 2(K-1)\big\{ K^3 + NK^2 - (N+1)K + 1 \big\} \Big] \\
> \ 0 &
\end{aligned}
\tag{14}
$$

and

$$
\begin{aligned}
T'_N(K^{N+1}) \\
= \ \frac{1}{K-1}\Big[&2K^{N+2} + \big\{ 2K^2 + (N+1)K - N - 2 \big\} \\
&\times K^{-N} - K(4K-1) \Big] \\
> \ 0 \, ,
\end{aligned}
\tag{15}
$$

we have $T_N(x) > 0$ for $x \geq K^{N+1}$. Hence, we have $\Delta S_H(N) > 0$ for $H = N+1, N+2, \cdots$; that is, $N = 1, 2, \cdots, H-1$.

From the above results, we obtain that the optimal adding depth is $N^* = H$.

IV. CONCLUSIONS

This study considered the addition of relations to an organization structure such that the communication of information between every member in the organization becomes the most efficient. For a model of adding edges between one node and every other node with the same depth N in a complete K-ary tree of height H which can describe the basic type of a pyramid organization, we obtained an optimal depth $N^* = H$ which maximizes the total shortening path length. This result indicates the most efficient way of adding relations between a representative and every other member at the same level is to use the lowest level in the organization structure.

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