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A Two-Species Model for a Fishing System with Marine Protected Areas

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***Abstract*—**A model of a system concerning one species of demersal (inshore) fish and one of pelagic (offshore) fish undergoing fishing restricted by marine protected areas is proposed in this paper. This setup was based on the FISH-BE model applied to the Tabina fishery in Zamboanga del Sur, Philippines. The components of the model equations have been adapted from widely-accepted mechanisms in population dynamics. The model employs Gompertz's law of growth and interaction on each type of protected and unprotected subpopulation. Exchange coefficients between protected and unprotected areas were assumed to be proportional to the relative area of the entry region. Fishing harvests were assumed to be proportional to both the number of fishers and the number of unprotected fish. An extra term was included for the pelagic population to allow for the exchange between the unprotected area and the outside environment. The systems were found to be bounded for all parameter values. The equations for the steady state were unsolvable analytically but the existence and uniqueness of non-zero steady states can be proven. Plots also show that an MPA size yielding the maximum steady state of the unprotected population can be found. All steady states were found to be globally asymptotically stable for the entire range of parameter values.

***Keywords*—**fisheries modelling, marine protected areas, sustainable fisheries, Gompertz Law

I. Introduction

MARINE protected areas (MPAs) are areas of seas that have harvest regulations to prevent over-fishing. The usual restrictions are quota limits or total bans in fishing within an MPA. This style of management has the advantage of protecting the marine habitats as well as providing a shelter for juvenile fish to replenish the population [1]. Some famous MPAs can be found flourishing in the United States, Caribbean islands and Australia [2]. In the Philippines, arguably one of the fishing hotspots of the world, MPAs have proven to be a successful fishery management tool that can be effectively managed by local communities and municipal governments [3], [4], [5], [6]. They have also been found to promote tourism and scuba-diving in these areas thus providing a source of income to locals alternate from fishing [7].

In this study, the fishing system was based on the descriptions of the Tabina fishery in Zamboanga del Sur, Philippines as used in the FISH-BE (Fisheries Information for Sustaining Harvests Bio-Economic) Model [4], [8]. A simple model of a two-species fishing system regulated by MPAs is now proposed. The focus of this study is on analyzing the dynamics of this model and the stability of the stable states. Economic factors were not considered but numerical results were used to show how the population of the protected and unprotected subpopulations vary with the size of the MPA. The stability of this equilibrium was also considered using a Lyapunov function. The fishery system consists of one demersal population which remains close to the shore, and one pelagic which lives offshore. This being the case, the habitats of the two groups (and their respective MPAs) were considered separately, although they may overlap in practice. Each type of fish has a protected and unprotected subpopulation. Asymmetric exchange is allowed between the subpopulations, with the exchange coefficient being proportional to the relative area of the entry region. In addition, the pelagic fish population is allowed to have fish coming in from outside the system.

Two types of fishers were taken into account. Municipal fishers focus on catching demersal fish while commercial fleets target pelagic fish. Some amount of crossover between their catches were allowed but this was always assumed to be small. It was also assumed that the distribution of resources throughout the fishery is uniform so that the carrying capacity of any fraction of the entire area is equal to that fraction multiplied by that the total carrying capacity. Thus the carrying capacity of an MPA that is a fraction M of the total area is MK, where K is the total carrying capacity. For the case of uneven distributions, one might instead define M to be the fraction of the total resources encompassed by the MPA.

The rate of growth and intra-specific interactions of the fish populations was controlled by the Gompertz law of growth and interaction. This was chosen instead of the simpler logistic equation because it allows for the slow growth of very small populations. This choice is further discussed in the section on Related Work.

The systems were found to be bounded for all parameter values (Theorem 1). The equations for the steady states turned out to be unsolvable analytically but the existence of a unique non-zero steady state can be proven (Theorem 2). All steady states were found to be globally asymptotically stable for the entire range of parameter values (Theorem 7).

Plots also show that the there is an optimal MPA size that would yield the maximum steady state for the unprotected population. This optimal MPA size can be found by taking the derivative of the unprotected population size with respect to MPA size and setting this to zero. Although not shown, the result would be a system of equations that can be numerically solved for a given set of parameter values.

Russ and Alcala (1996) have suggested that in the case of large predatory fish in the Apo Island reserve of Central Philippines, fish do not display significantly varied distributions outside of the MPA during the early years of protection

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[5]. During the 9th to 11th years however, it was found that the populations were denser the closer they are to the MPAs suggesting significant effects of spillover. This will not be considered in this model but it would be an interesting aspect to investigate further in future studies. It might be taken into account for instance by considering a non-homogenous distribution of the unprotected subpopulation which is denser when it is close to the MPA.

II. Related Work

The FISH-BE model has been applied to the Tabina fishery in studies by Licuanan et al. [4] and Castillo et al. [8]. This FISH-BE model implements graphically-oriented modelling software and can be generalized to other fishery situations in the Philippines. The FISH-BE model may be used to determine optimal MPA size as well as good governance practices when it comes to managing the reserves. Baskett et al. [9] also provides comprehensive information on monitoring and designing reserves based on a compilation of theoretical models.

The dynamics of MPAs has been the subject of many studies. Dubey et al. [10] presented a model of MPAs with logistic growth and constant coefficient of exchange between protected and unprotected subpopulations. This model discussed optimal harvesting policy using Pontryagin's Maximum Principle. Kar and Misra [11] modified this model to include a predatory fish species that is confined to the unprotected region. Greenville and MacAulay [12] also tackled a two-species fishing model with a predator-prey relationship but using a different approach. They also allowed for stochastic effects concluding that the variation in fishing harvest is lowered by the use of MPAs.

In the studies by Kar [13] and Pradhan and Chaudhuri [14] the populations were assumed to follow the Gompertz law of growth and interaction. Pradhan and Chaudhuri explains how this describes fish populations better than the logistic equations and compares their results with the Schaefer model. This paper also employs Gompertz law to describe the population growth for the same reasons and because the use of this law in biological populations merits further study.

The fishing model by Kar [13] also investigates the influences of environmental noises on the growth parameters and the inclusion of delay in the governing equations. Delay accounts for the regulation on capturing juvenile fish, a rule that has been implemented in many MPAs. These factors are not included in the current study but may be incorporated in future work.

Shirai and Harada [15] used an asymmetric exchange between protected and unprotected subpopulations in their model. This study follows their assumption that the coefficient of exchange between regions is proportional to the relative area of the entry region. Armstrong and Skonhoft [16] also assumes asymmetric exchanges but instead of relative areas these were proportional to the relative population densities. This study also allowed for different growth rates between the two subpopulations.

Many studies based on mathematical models have found that MPAs have the potential to be very effective at increasing the fishery yields [12], [15], [17], [18]. There are fewer studies on the comparison of this method with other fishery management tools. One was that by Nowlis [19] in which he compared the short and long-term effects of three major methods of fishery regulation: the use of MPAs, the temporary closures of entire fisheries and the maintenance of the fish levels above a minimum size limit. Nowlis used mathematically based computer simulations and found that give certain conditions, there is a wide range of circumstances in which the use of MPAs is the method that maximizes the harvests [19].

III. System Description

For each type of fish (demersal and pelagic) the subpopulation currently living in protected areas and those living in unprotected areas will be considered separately. Crossover of fish catch between the municipal and commercial fishers is allowed but this is kept very small. Table I gives the list of variables and parameters that will be considered.

TABLE I
Description of Variables and Parameters

Variables and Parameters	
X	demersal population in non-protected area
X_M	demersal population in protected area
Y	pelagic population in non-protected area
Y_M	pelagic population in protected area
Y_E	pelagic population of exterior
M_x, M_y	proportion of protected area *
K_x, K_y	carrying capacity *
r_x, r_y	rate of growth *
α	exchange coefficient between X and X_M
β	exchange coefficient between Y and Y_M
γ	exchange coefficient between Y and Y_E
N_1	number of municipal fishers
N_2	number of commercial fishers
p_1, p_2	fishing coefficient of X **
q_1, q_2	fishing coefficient of Y **

* Subscripts x and y correspond to demersal and pelagic respectively
** Subscripts 1 and 2 correspond to municipal and commercial respectively

A. Governing Equations

The governing equations are given in (1)-(4).

$$\frac{\mathrm{d}X_M}{\mathrm{d}t} = r_x X_M \ln\left(\frac{M_x K_x}{X_M}\right) - \alpha((1-M_x)X_M - M_x X) \tag{1}$$

$$\frac{\mathrm{d}X}{\mathrm{d}t} = r_x X \ln\left(\frac{(1-M_x)K_x}{X}\right) + \alpha((1-M_x)X_M - M_x X) - p_1 N_1 X - p_2 N_2 X \tag{2}$$

$$\frac{\mathrm{d}Y_M}{\mathrm{d}t} = r_y Y_M \ln\left(\frac{M_y K_y}{Y_M}\right) - \beta((1-M_y)Y_M - M_y Y) \tag{3}$$

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = r_y Y \ln\left(\frac{(1-M_y)K_y}{Y}\right) + \beta((1-M_y)Y_M - M_y Y) - q_1 N_1 Y - q_2 N_2 Y + \gamma(Y_E - Y) \tag{4}$$

Equations (1)-(4) might be simplified to include fewer parameters, and the systems for demersal (X_MX) and pelagic (Y_MY) might actually be analyzed separately. In fact, the two systems actually have the same form when $\gamma = 0$. The analysis will be made on the two separate systems however and using all of the given parameters because of the physical significance of the values. This is also done to keep the model open to extensions such as maximizing the total catch of N_1 or the N_2 fishers.

In the governing equations the first term comes from Gompertz's law [13], [14]. It is assumed that the carrying capacity of any fraction of a population habitat behaves as described in the Introduction. It should be noted that because of this, each subpopulation follows Gompertz law with their respective carrying capacities but the total population does not.

The exchange from protected to unprotected subpopulations is assumed to be proportional to the relative area of the unprotected region, while that from unprotected to protected is proportional to the relative area of the protected region [15]. The fishing terms in the unprotected populations are based on a predator-prey model of the total catch being proportional to the total possible meetings between the two populations. The only structural difference between the demersal and pelagic equations is the extra exchange term for Y with the surroundings. The system is illustrated in Figure 1.

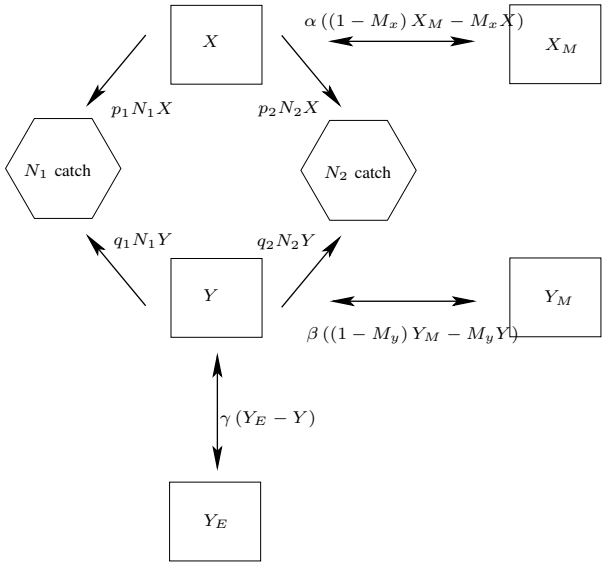


Fig. 1. Illustration of the flow

B. Boundedness

It is natural to expect the populations to be bounded between zero and its carrying capacity. As it turns out however, A is greater than one because of the contribution of Y_E.

Theorem 1. *The XX_M and YY_M systems are bounded as follows.*

$$0 < X_M < M_x K_x \,, \quad 0 < X < (1 - M_x) K_x$$

$$0 < Y_M < A M_y K_y \,, \quad 0 < Y < A (1 - M_y) K_y$$

where $A \geq 1$.

The theorem will be proven by defining the rectangular trapping regions and finding an $A \geq 1$ such that an orbit that begins in the region cannot leave it.

Proof for demersal fish:

Let $\vec{F} = \begin{pmatrix} \dot{X}_M \\ \dot{X} \end{pmatrix}$ and let $\hat{n}$ be a unit normal outwards of the region shown in Figure 2.

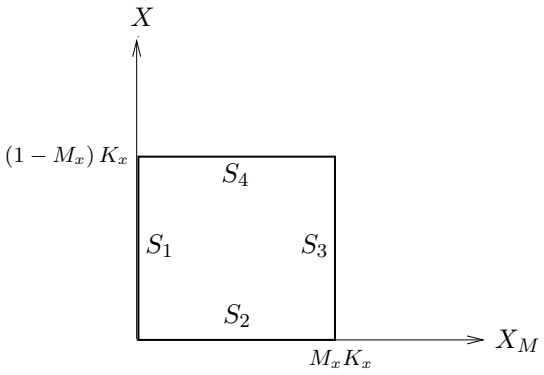


Fig. 2. Illustration of the trapping region

$$\vec{F} \cdot \hat{n} = \begin{cases} -\alpha M_x X & , \mathrm{S}_1 \\ -\alpha (1 - M_x) X_M & , \mathrm{S}_2 \\ -\alpha M_x ((1 - M_x) K_x - X) & , \mathrm{S}_3 \\ -\begin{bmatrix} \alpha (M_x K_x - X_M) \\ \ldots + (p_1 N_1 + p_1 N_2) K_x \end{bmatrix} (1 - M_x) & , \mathrm{S}_4 \end{cases}$$

Since $\vec{F} \cdot \hat{n} \leq 0$ at each boundary of the region, this proves that this is a trapping region.

■

Proof for pelagic fish:

Let $\vec{F} = \begin{pmatrix} \dot{Y}_M \\ \dot{Y} \end{pmatrix}$ and let $\hat{n}$ be a unit normal once again.

$$\vec{F} \cdot \hat{n} = \begin{cases} -\beta M_y Y & , \mathrm{S}_1 \\ -\beta (1 - M_y) Y_M - \gamma Y_E & , \mathrm{S}_2 \\ -A M_y K_y \begin{pmatrix} \beta (1 - M_y) \\ \ldots + r_y \ln A \end{pmatrix} + \beta M_y Y & , \mathrm{S}_3 \\ -A \begin{bmatrix} \beta M_y + q_1 N_1 + q_2 N_2 \\ \ldots + \gamma + r_y \ln A \end{bmatrix} (1 - M_y) K_y \\ \ldots + \gamma Y_E + \beta (1 - M_y) Y_M & , \mathrm{S}_4 \end{cases}$$

Evidently $F \cdot \hat{n} < 0$ for S_1 and S_2. For S_3 it is not as clear because of the positive Y term. This has a maximum value when Y attains its maximum of $A (1 - M_y) K_y$.

$$\vec{F} \cdot \hat{n} \text{ at } \mathrm{S}_3 \leq -A M_y K_y r_y \ln A \tag{5}$$

This is always negative as long as $A > 1$. Now for S_4, $F \cdot \hat{n}$ is maximum at maximum Y_M which occurs at AM_yK_y.

$$\vec{F}\cdot\hat{n} \text{ at } S_3 \leq -A \begin{bmatrix} q_1N_1 + q_2N_2 \\ \ldots + \gamma + r_y \ln A \end{bmatrix} (1 - M_y) K_y + \gamma Y_E \tag{6}$$

Setting that to be less than zero is equivalent to writing

$$A \begin{bmatrix} q_1N_1 + q_2N_2 \\ \ldots + \gamma + r_y \ln A \end{bmatrix} (1 - M_y) K_y > \gamma Y_E \tag{7}$$

This is automatically true if the following is true

$$A (q_1N_1 + q_2N_2 + \gamma) (1 - M_y) K_y > \gamma Y_E \tag{8}$$

$$A > \frac{\gamma Y_E}{(q_1N_1 + q_2N_2 + \gamma) (1 - M_y) K_y} \tag{9}$$

Thus for a given Y_E, the Y_MY system is bounded as given in Theorem 1. A has to be chosen to be greater than one and also satisfying (9).

■

The value of A is also useful in proving that the special case where one or both of the initial values of Y and Y_M are greater than their carrying capacities is still bounded. In this case one can choose A such that the initial point falls within the trapping region and satisfies (9). The use of A can also be extended to the X_MX system to show that the system is still bounded when initial values exceed the carrying capacities. In this case $\gamma = 0$ so A need only be chosen such that the initial point falls within the trapping region.

C. Stable States

After boundedness, the next question to consider is the existence of steady states within the boundaries and whether or not these are unique. These questions are answered by the Theorem 5. The following Lemmas are necessary to prove the Theorem.

Lemma 2. *Any steady state $\left(\overline{X_M}, \overline{X}\right)$ of the X_MX system has to satisfy the following relationships.*

$$\overline{X_M} = \frac{M_x \alpha \overline{X}}{r_x W\left(\frac{\alpha \overline{X} \exp\left(\frac{\alpha(1-M_x)}{r_x}\right)}{K_x r_x}\right)} \tag{10}$$

where $W(\cdot)$ is the Lambert W-function. The relationship between $\overline{X_M}$ and $\overline{X}$ is strictly increasing and continuous over the interval of $\overline{X}$ as given by the range of X in Theorem 1.

Similarly, any steady state $\left(\overline{Y_M}, \overline{Y}\right)$ of the Y_MY system has to satisfy the following relationship.

$$\overline{Y_M} = \frac{M_y \beta \overline{Y}}{r_y W\left(\frac{\beta \overline{Y} \exp\left(\frac{\beta(1-M_y)}{r_y}\right)}{K_y r_y}\right)} \tag{11}$$

The relationship between $\overline{Y_M}$ and $\overline{Y}$ is also strictly increasing and continuous over the interval of $\overline{Y}$ as given by the range of Y.

Proof:

It is sufficient to prove this for the X_MX system since the proof for the Y_MY system is very similar.

Let the range of $\overline{X}$ be the range of X given in Theorem 1. Setting (1) to zero and solving for X_M leads to the expression in (10). Since all parameters are positive, the continuity of the equation is guaranteed for all $\overline{X_M} > 0$ and $\overline{X} > 0$.

To prove the strictly increasing relationship, equation (10) can be simplified to

$$\overline{X_M} = \frac{C_1 \overline{X}}{W\left(C_2 \overline{X}\right)} \tag{12}$$

where $C_1 > 0$ and $C_2 > 0$. The derivative of this with respect to X is

$$\frac{C_1}{W\left(C_2 \overline{X}\right)} \left(1 - \frac{1}{1 + W\left(C_2 \overline{X}\right)}\right) \tag{13}$$

Since $W\left(C_2 \overline{X}\right) > 0$ as long as $\overline{X} > 0$, then the derivative is always positive. This proves that the expression $\overline{X_M}$ as a function of $\overline{X}$ is strictly increasing.

■

Note that in the limit $\overline{X} \to 0$, $\overline{X_M}$ also approaches zero yielding the extinction steady state $\left(\overline{X_M}, \overline{X}\right) = (0, 0)$. In the Y_MY system, $(0, 0)$ satisfies (11) but this is obviously not a steady state.

Lemma 3. *The function*

$$g(X) = C_1 \ln\left(\frac{C_2}{X}\right) + \frac{C_3}{W(C_4 X)} - C_5 \tag{14}$$

where

$$\begin{aligned} C_1 &= r_x \\ C_2 &= (1 - M_x) K_x \\ C_3 &= \frac{\alpha^2 (1 - M_x) M_x}{r_x} \\ C_4 &= \frac{\alpha \exp(\alpha (1 - M_x)/r_x)}{K_x r_x} \\ C_5 &= \alpha M_x + p_1 N_1 + p_2 N_2 \end{aligned}$$

has exactly one root.

Proof:

This function comes from substituting (10) into the right-hand-side of (2) and dividing by X. It is continuous in the range of X given in Theorem 1. The C_i's are all positive because all of the parameters are also positive.

$$\lim_{X \to 0^+} g(X) = \frac{s(C_3)}{s(C_4)} \infty = +\infty \tag{15}$$

Here $s(\cdot)$ is the signum function. Thus the limit of $g(X)$ is $+\infty$ and g starts off with positive values.

It is a bit more involved to show that g attains a negative value in the other limit $X \to (1-M_x)\,K_x$.

$$\lim_{X\to(1-M_x)K_x} g(X) < 0$$
$$\Leftrightarrow C_3 < C_5 \cdot W\left(\frac{C_3}{\alpha M_x}\exp\left(\frac{C_3}{\alpha M_x}\right)\right)$$
$$\Leftrightarrow \frac{C_3}{\alpha M_x} < \left(1+\frac{p_1N_1+p_2N_2}{\alpha M_x}\right)\cdot W\left(\frac{C_3}{\alpha M_x}\exp\left(\frac{C_3}{\alpha M_x}\right)\right)$$
$$\Leftrightarrow \frac{C_3}{\alpha M_x} < \left(1+\frac{p_1N_1+p_2N_2}{\alpha M_x}\right)\frac{C_3}{\alpha M_x}$$

This last line is obviously true thus g attains both positive and negative values in the range of X. Since g is continuous in this range then it must at some point pass through $g=0$.

The derivative of g is

$$g'(X) = -\frac{C_1}{X} - \frac{C_3}{W(C_4X)(1+W(C_4X))X} \tag{16}$$

This is always negative for $X>0$. Since g is monotonically decreasing then it has only one root then it has only one root. ■

Lemma 4. *The function*

$$h(Y) = C_1\ln\left(\frac{C_2}{Y}\right) + \frac{C_3}{W(C_4Y)} - C_5 + \frac{C_6}{Y} \tag{17}$$

where

$$C_1 = r_y$$
$$C_2 = (1-M_y)\,K_y$$
$$C_3 = \frac{\beta^2(1-M_y)M_y}{r_x}$$
$$C_4 = \frac{\beta\exp(\beta(1-M_y)/r_y)}{K_y r_y}$$
$$C_5 = \alpha M_y + q_1N_1 + q_2N_2$$
$$C_6 = \gamma Y_E$$

has exactly one root in the range $Y \in (0, A(1-M_y)K_y)$.

Proof: The proof of this is very similar to that for Lemma 3. It also requires using $A>1$ and (9). ■

Theorem 5. *There exists unique non-zero steady states* $\left(\overline{X_M},\overline{X}\right)$ *and* $\left(\overline{Y_M},\overline{Y}\right)$ *within the boundaries given in Theorem 1. These steady states satisfy the relationships in Lemma 2 and*

$$g\left(\overline{X}\right) = 0 \tag{18}$$

$$h\left(\overline{Y}\right) = 0 \tag{19}$$

Proof:

The roots of the functions g and h, together with the corresponding X_M and Y_M from Lemma 2 coincide with the zeros of the governing equations (1)-(4). According to Lemmas 3 and 4), $\overline{X}$ and $\overline{Y}$ are unique and lie on the intervals given in Theorem 1.

From Lemma 2 the maximum possible values of $\overline{X_M}$ and $\overline{Y_M}$ occur at the maximum possible values of $\overline{X}$ and $\overline{Y}$ respectively. For the X_MX system this occurs at $\overline{X} = (1-M_x)\,K_x$ yielding $\overline{X_M} = M_xK_x$. For the Y_MY system this occurs at $\overline{Y} = (A(1-M_x)\,K_x$.

■

For a given set of parameter values the steady states can be computed. Table II shows the values that were assigned to the parameters for the graphs in Figures 3 and 4. The plots show that there is an MPA size that maximizes the population of the unprotected subpopulations. This optimal size can be found by taking the derivative of $\overline{X}$ with respect to M_x and setting this to zero. The optimal M_x can then be found by numerically solving for it.

TABLE II
ASSUMED VALUES OF PARAMETERS

List of Parameters	
Y_E	1×10^6 units of fish
K_x	1×10^8 units of fish
K_y	1×10^6 units of fish
α	0.5
β	0.5
γ	0.05
N_1	1350 fishers
N_2	350 fishers
p_1	0.0005 per fisher-unit time
p_2	0.0001 per fisher-unit time
q_1	0.00001 per fisher-unit time
q_2	0.002 per fisher-unit time

D. Stability Requirements

Convert the X_MX and Y_MY systems to x_Mx and y_My systems by mapping the steady states to zero.

$$X_M = \overline{X_M} + x_M \qquad X = \overline{X} + x$$
$$Y_M = \overline{Y_M} + y_M \qquad Y = \overline{Y} + y$$

Corollary 6. *The ranges of the new system are given by*

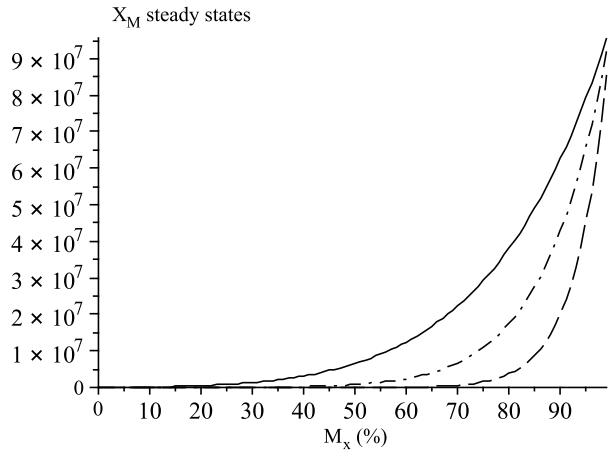
$$-\overline{X_M} < x_M < M_xK_x - \overline{X_M}$$
$$-\overline{X} < x < (1-M_x)\,K_x - \overline{X}$$
$$-\overline{Y_M} < y_M < AM_yK_y - \overline{Y_M}$$
$$-\overline{Y} < y < A(1-M_y)\,K_y - \overline{Y}$$

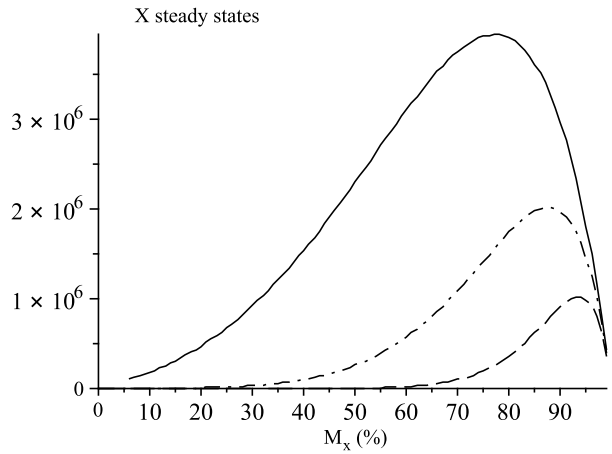
Proof: This follows directly from Theorem 1. ■

Writing the demersal system (1)-(2) in terms of the perturbations and setting the derivatives of the steady states to zero yields

$$\begin{aligned}\frac{\mathrm{d}x_M}{\mathrm{d}t} =& r_x\ln\left(\frac{\overline{X_M}}{\overline{X_M}+x_M}\right)\overline{X_M} \\ &+ r_x\ln\left(\frac{M_xK_x}{\overline{X_M}+x_M}\right)x_M \\ &- \alpha(1-M_x)x_M + \alpha M_x x\end{aligned} \tag{20}$$

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} =& r_x\ln\left(\frac{\overline{X}}{\overline{X}+x}\right)\overline{X} \\ &+ r_x\ln\left(\frac{(1-M_x)K_x}{\overline{X}+x}\right)x \\ &+ \alpha(1-M_x)x_M - \alpha M_x x \\ &- p_1N_1 - p_2N_2\end{aligned} \tag{21}$$





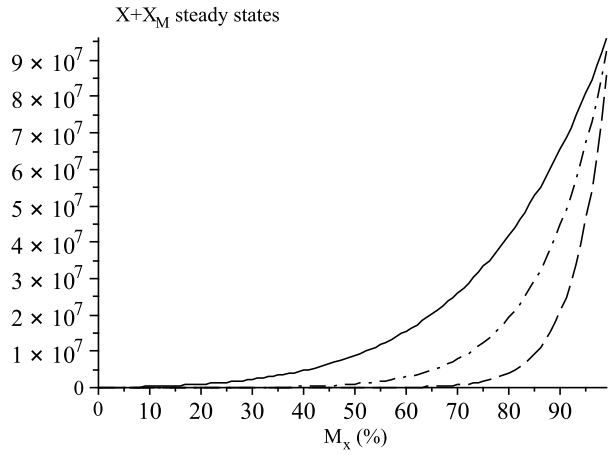
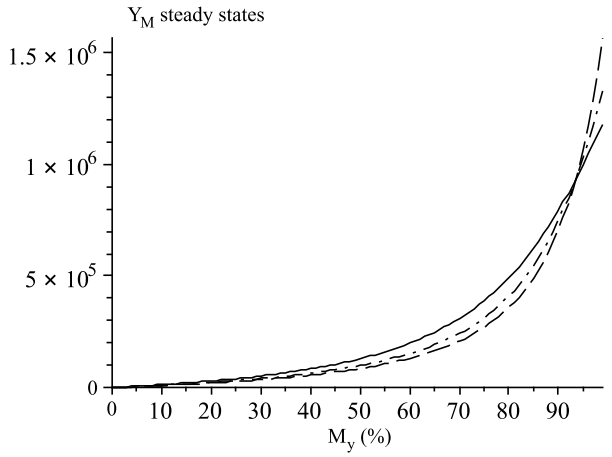
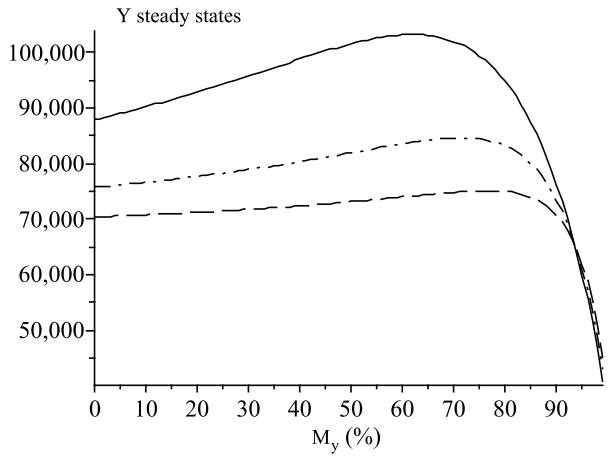


Fig. 3. Steady state values for r_x equal to 0.02, 0.04 and 0.08 yr^{-1} (dash, dash-dot and solid line respectively).





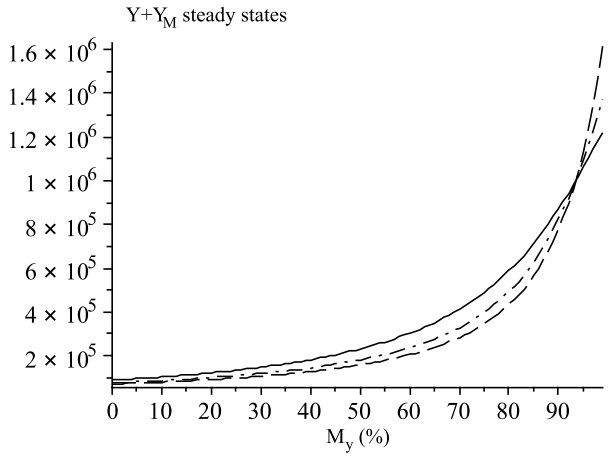
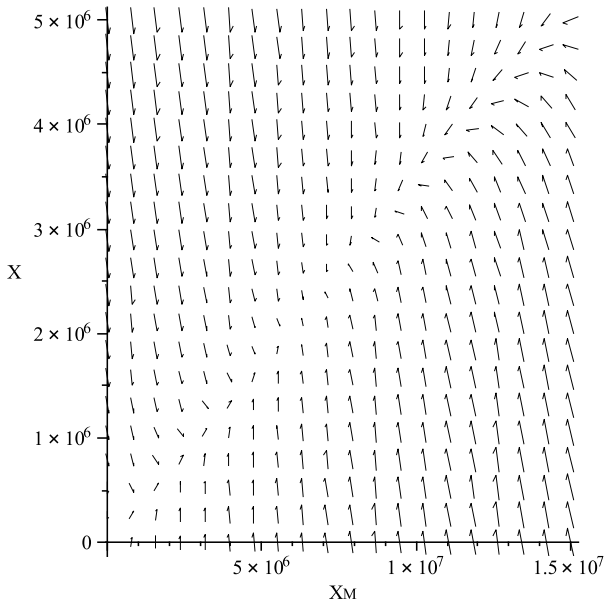
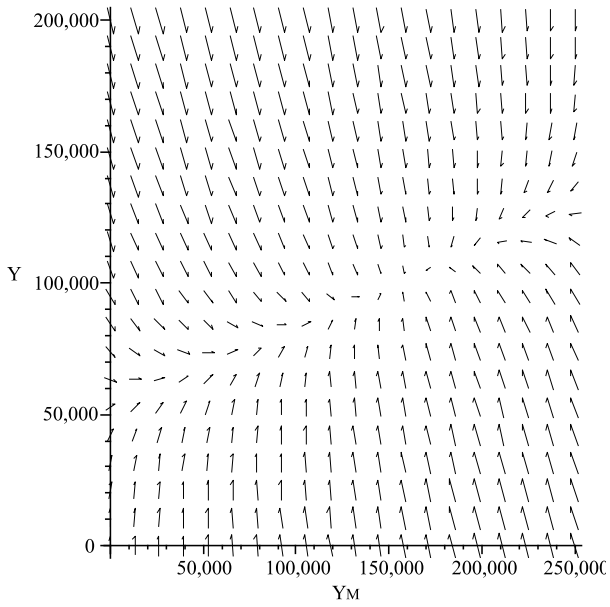


Fig. 4. Steady state values for r_y equal to 0.02, 0.04 and 0.08 yr^{-1} (dash, dash-dot and solid line respectively).



(a) X and X_M



(b) Y and Y_M

Fig. 5. Phase plots using Maple. Parameter values from Table II with $r_x = r_y = 0.08$ and $M_x = M_y = 0.5$

Theorem 7. *The stable states at $\overline{X_M}$ and $\overline{X}$ ($x_M = 0$ and $x = 0$) are globally stable.*

Proof:

Introduce the following Lyapunov function candidate for the $X_M X$ system.

$$V = \frac{1}{2}(1 - M_x)x_M^2 + \frac{1}{2}M_x x^2 \tag{22}$$

Using (20) and (21), the derivative of the function is

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} =& r_x(1 - M_x)\left[\ln\left(\frac{\overline{X_M}}{\overline{X_M} + x_M}\right)\overline{X_M}x_M \right.\\ &\left. + \ln\left(\frac{M_x K_x}{\overline{X_M} + x_M}\right)x_M^2\right] \\ &+ r_x M_x\left[\ln\left(\frac{\overline{X}}{\overline{X} + x}\right)\overline{X}x + \ln\left(\frac{(1 - M_x)K_x}{\overline{X} + x}\right)x^2\right] \\ &- \alpha(1 - M_x)^2 x_M^2 - \alpha M_x^2 x^2 - M_x(p_1 N_1 + p_2 N_2)x^2\end{aligned} \tag{23}$$

This is obviously zero at x_M=x=0. What is not obvious is if this expression is negative at all other values as required for V to be a Lyapunov function. The terms involving $X_M x_M$ and Xx terms are always negative for all nonzero values of x_M and x. The other logarithmic terms however, are always positive because of Theorem 1. The remaining quadratic terms are always negative. There are no mixed terms of $x_M x$. Since these are independent perturbations two conditions can be found to determine if the derivative is negative-definite.

For all $x_M \neq 0$ and $x \neq 0$, the requirements for stability are

$$\begin{aligned}\ln\left(\frac{\overline{X_M}}{\overline{X_M} + x_M}\right)\frac{\overline{X_M}}{x_M} + \ln\left(\frac{M_x K_x}{\overline{X_M} + x_M}\right) \\ - \frac{\alpha(1 - M_x)}{r_x} < 0\end{aligned} \tag{24}$$

$$\begin{aligned}\ln\left(\frac{\overline{X}}{\overline{X} + x}\right)\frac{\overline{X}}{x} + \ln\left(\frac{(1 - M_x)K_x}{\overline{X} + x}\right) \\ - \frac{(\alpha M_x + p_1 N_1 + p_2 N_2}{r_x} < 0\end{aligned} \tag{25}$$

These conditions can be re-written as (26)-(27)

$$M_x K_x \exp\left(-\frac{\alpha\,(1 - M_x)}{r_x}\right)\left(\frac{\overline{X_M}}{\overline{X_M} + x_M}\right)^{\frac{\overline{X_M}}{x_M}+1} < \overline{X_M} \tag{26}$$

$$\begin{aligned}(1 - M_x)\,K_x \exp\left(-\frac{\alpha M_x + p_1 N_1 + p_2 N_2}{r_x}\right) \\ \cdot\left(\frac{\overline{X}}{\overline{X} + x}\right)^{\frac{\overline{X}}{x}+1} < \overline{X}\end{aligned} \tag{27}$$

A similar Lyapunov function can be found for the pelagic fish with similar stability requirements. In this case, for all $y_M \neq 0$ and $y \neq 0$, the requirements for stability are given by (28)-(29)

$$M_y K_y \exp\left(-\frac{\beta\,(1 - M_y)}{r_y}\right)\left(\frac{\overline{Y_M}}{\overline{Y_M} + y_M}\right)^{\frac{\overline{Y_M}}{y_M}+1} < \overline{Y_M} \tag{28}$$

$$\begin{aligned}(1 - M_y)\,K_y \exp\left(-\frac{\beta M_y + q_1 N_1 + q_2 N_2 + \gamma}{r_y}\right) \\ \cdot\left(\frac{\overline{Y}}{\overline{Y} + y}\right)^{\frac{\overline{Y}}{y}+1} < \overline{Y}\end{aligned} \tag{29}$$

As it turns out, the two sets of conditions given by (26) to (29) are always satisfied. This can be shown by using the following Lemma.

Lemma 8. *The following expressions are true for the $X_M X$ and $Y_M Y$ systems.*

$$\overline{X_M} = M_x K_x \exp\left(-\frac{\alpha(1-M_x)}{r_x}\right) \exp\left(\frac{\alpha M_x \overline{X}}{r_x \overline{X_M}}\right) \quad (30)$$

$$\overline{X} = (1-M_x)K_x \exp\left(-\frac{\alpha M_x + p_1 N_1 + p_2 N_2}{r_x}\right) \cdot \exp\left(\frac{\alpha(1-M_x)\overline{X_M}}{r_x \overline{X}}\right) \quad (31)$$

$$\overline{Y_M} = M_y K_y \exp\left(-\frac{\beta(1-M_y)}{r_y}\right) \exp\left(\frac{\beta M_y \overline{Y})}{r_y \overline{Y_M}}\right) \quad (32)$$

$$\overline{Y} = (1-M_y)K_y \exp\left(-\frac{\beta M_y + q_1 N_1 + q_2 N_2 + \gamma}{r_y}\right) \cdot \exp\left(\frac{\beta(1-M_y)\overline{Y_M} + \gamma Y_E)}{r_y \overline{Y}}\right) \quad (33)$$

Proof:
These expressions can be derived from the governing equations (1) to (4).

■

Now consider the (26) condition for stability. Comparing this with (30), it can be shown that the (26) is true if the following is true.

$$\left(\frac{\overline{X_M}}{\overline{X_M} + x_M}\right)^{\frac{\overline{X_M}}{x_M}+1} < \exp\left(\frac{\alpha M_x \overline{X}}{r_x \overline{X_M}}\right) \quad (34)$$

The term on the left side is always less than or equal to one for the ranges in Corollary 6. The exponential term on the right side is always greater than one since the steady states and parameters are always positive. Thus this inequality is always satisfied. A similar argument proves that all of the other conditions for stability (27)-(29) are also satisfied. Thus by method of Lyapunov functions we have shown that the equilibrium is globally asymptotically stable.

■

To confirm that the steady states are indeed stable, a time graph was generated for each population (protected and unprotected) using a simple Euler method algorithm on the derivatives given in equations (1) to (4). These plots are shown in Figures 6-7 and were generated using Maple.

IV. Conclusion

This paper has shown that marine protected areas described by the model in (1)-(4) are successful at providing for sustainable harvests. The systems were found to be bounded and non-zero steady states have been found to exist for the entire range of parameter values. These steady states have been shown to be globally asymptotically stable.

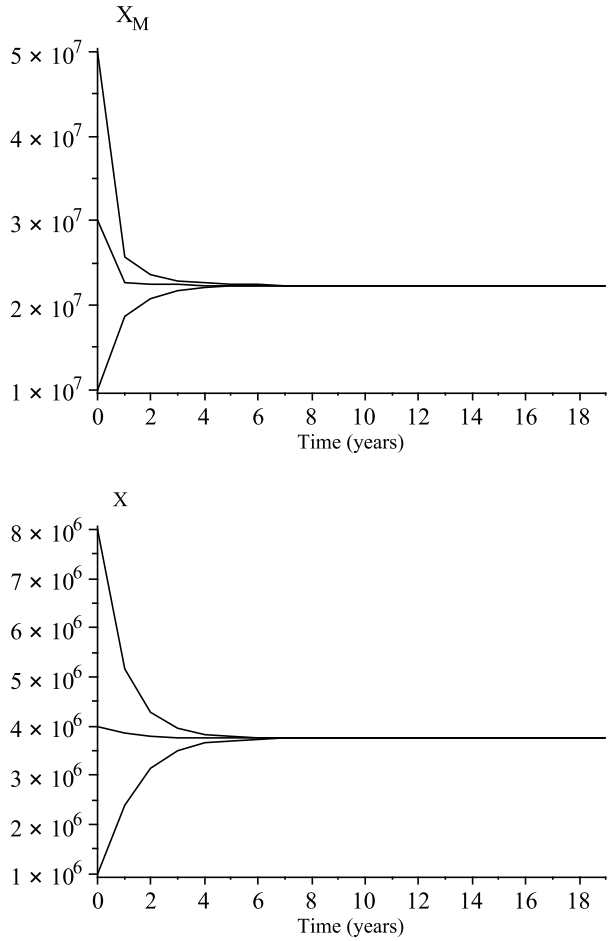


Fig. 6. Time plots of demersal fish population.

Figures 3 to 4 also show that that there is an optimum MPA size for each species of fish. This can be determined by taking the derivative of $\overline{X}$ with respect to M_x and setting this to zero. Although there is no analytic formula for this, this optimal size can still be found using numerical methods

Future work should include finding data to compare with the model. More factors can be considered such as time delay for the fish spawn to grow to catchable size, inter-species competition, pollution and perhaps a predator-prey relationship between the demersal and pelagic fish. The distribution of fish in the unprotected area should also be considered as evidence have shown that this may not be homogenous [5].

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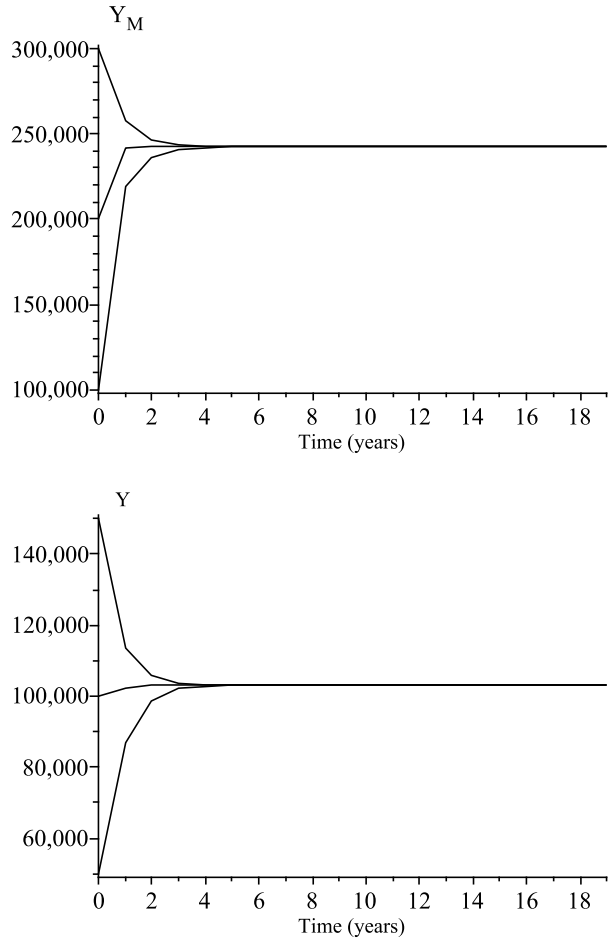

Fig. 7. Time plots of pelagic fish population.

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