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A Sum Operator Method for Unique Positive Solution to a Class of Boundary Value Problem of Nonlinear Fractional Differential Equation

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Abstract—By using a fixed point theorem of a sum operator, the existence and uniqueness of positive solution for a class of boundary value problem of nonlinear fractional differential equation is studied. An iterative scheme is constructed to approximate it. Finally, an example is given to illustrate the main result.

Keywords—Fractional differential equation, Boundary value problem, Positive solution, Existence and uniqueness, Fixed point theorem of a sum operator.

I. Introduction

FRACTIONAL differential equations are various used in mechanics, physics, chemistry, engineering, economics and biological sciences, etc.; see [1]-[9] and the references therein. In recent years, the existence and multiplicity of positive solutions for nonlinear fractional differential equation boundary value problem have been of great interest. Their analysis relies on Leray-Shauder theory, fixed-point theorems, etc., see [10]-[15]. However, there are few papers consider the existence of unique positive solution for nonlinear fractional differential equation boundary value problem, see [16]-[18].

In particular, by means of a sum operator method, [18] consider the existence and uniqueness of positive solution for the following fractional boundary value problem given by

$$\begin{cases} -D^{\alpha}_{0+}u(t) = f(t,u(t)) + g(t,u(t)), 0 < t < 1, 3 < \alpha \leq 4 \\ u(0) = u'(0) = u''(0) = u''(1) = 0, \end{cases} \tag{1}$$

where D^{α}_{0+} is the standard Riemann-Liouville fractional derivative.

Motivated by the work mentioned above, in this paper, by using of a fixed point theorem for a sum operator, we obtain the existence of unique positive solution for the following nonlinear fractional differential equation boundary value problem:

$$\begin{cases} -D^{v}_{0+}u(t) = f(t,u(t)), 0 < t < 1, n-1 < v \leq n \\ u(0) = u'(0) = u''(0) = \cdots = u^{(n-2)}(0) = 0, \\ [D^{\alpha}_{0+}u(t)]_{t=1} = 0, 1 \leq \alpha \leq n-2, \end{cases} \tag{2}$$

where $f(t,u(t)) = g(t,u(t)) + h(t,u(t))$ and D^{v}_{0+} is the standard Riemann-Liouville fractional derivative of order v. Moreover, we can construct an iterative scheme to approximate the unique positive solution, which is important for evaluation and application.

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II. Preliminaries and Previous Results

In this section, we present some definitions, lemmas and basic results that will be used in the proof of our main result.

Definition 1 [9] The integral

$$I^{\alpha}_{0+}f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}\mathrm{d}t, \quad x > 0$$

is called the Riemann-Liouville fractional integral of order α, where $\alpha > 0$ and $\Gamma(\alpha)$ denotes the gamma function.

Definition 2 [9] For a function $f(x)$ given in the interval $[0,\infty)$, the expression

$$D^{\alpha}_{0+}f(x) = \frac{1}{\Gamma(n-\alpha)}(\frac{\mathrm{d}}{\mathrm{d}x})^n \int_0^x \frac{f(t)}{(x-t)^{\alpha-n+1}}\mathrm{d}t$$

is called the Riemann-Liouville fractional derivative of order α, where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α.

Lemma 1 [11] Let $y \in C^n[0,1]$ and $n-1 < v \leq n$. The unique solution of problem

$$\begin{cases} -D^{v}_{0+}u(t) = y(t), 0 < t < 1 \\ u(0) = u'(0) = u''(0) = \cdots = u^{(n-2)}(0) = 0, \\ [D^{\alpha}_{0+}u(t)]_{t=1} = 0, 1 \leq \alpha \leq n-2, \end{cases} \tag{3}$$

is

$$u(t) = \int_0^1 G(t,s)y(s)\mathrm{d}s, \ \ t \in [0,1],$$

where

$$G(t,s) = \begin{cases} \frac{t^{v-1}(1-s)^{v-\alpha-1} - (t-s)^{v-1}}{\Gamma(v)}, & 0 \leq s \leq t \leq 1, \\ \frac{t^{v-1}(1-s)^{v-\alpha-1}}{\Gamma(v)}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{4}$$

Here $G(t,s)$is called the Green function of boundary value problem (3).

Lemma 2 [17] The Green function $G(t,s)$ defined by (4) has the following property:

$$\begin{aligned} &\tfrac{1}{\Gamma(v)}t^{v-1}[1-(1-s)^{\alpha}](1-s)^{v-\alpha-1} \leq G(t,s) \\ &\leq \tfrac{1}{\Gamma(v)}t^{v-1}(1-s)^{v-\alpha-1}, \forall t,s \in [0,1]. \end{aligned} \tag{5}$$

In the sequel, we present some basic concepts in ordered Banach spaces for completeness and a fixed point theorem which will be used later. For convenience of readers, we suggest that one refer to [19] for details.

Suppose $(E, ||\cdot||)$ is a real Banach space which is partially ordered by a cone $P \subset E$, i.e. $x \leq y$ if and only if $y - x \in P$. If $x \leq y$ and $x \neq y$, then we denote $x < y$. We denote the zero element of E by θ. Recall that a non-empty closed convex set $P \subset E$ is a cone if it satisfies (i)$x \in P, \lambda \geq 0 \Rightarrow \lambda x \in P$; (ii)$x \in P, -x \in P \Rightarrow x = \theta$.

Putting $P^0 = \{x \in P | x \text{ is an interior point of } P\}$, a cone P is said to be solid if P^0 is non-empty. Moreover, P is called normal if there exists a constant $N > 0$ such that, for all $x, y \in E, \theta \leq x \leq y$ implies $||x|| \leq N||y||$; in this case N is called the normality constant of P. We say that an operator $A : E \to E$ is increasing if $x \leq y$ implies $Ax \leq Ay$.

For all $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \leq y \leq \mu x$. Clearly $\sim$ is an equivalence relation. Given $w > \theta$ (i.e. $w \geq \theta$ and $w \neq \theta$), we denote the set $P_w = \{x \in E | x \sim w\}$ by P_w. It is easy to see that $P_w \subset P$ for $w \in P$.

Definition 3 [18] Let $D = P$ or $D = P^0$ and γ be a real number with $0 \leq \gamma < 1$. An operator $A : P \to P$ is said to be γ-concave if it satisfies

$$A(tx) \geq t^\gamma Ax, \qquad \forall t \in (0,1), \;\; x \in D. \tag{6}$$

Definition 4 [18] An operator $A : E \to E$ is said to be homogeneous if it satisfies

$$A(tx) = tAx, \qquad \forall t \in (0,1), \;\; x \in E. \tag{7}$$

An operator $A : P \to P$ is said to be sub-homogeneous if it satisfies

$$A(tx) \geq tAx, \qquad \forall t \in (0,1), \;\; x \in P. \tag{8}$$

In recent paper, Zhai and Anderson [20] considered the following sum operator equation

$$Ax + Bx + Cx = x,$$

where A is an increasing γ-concave operator, B is an increasing sub-homogeneous operator and C is a homogeneous operator. They established the existence and uniqueness of positive solutions for the above equation, and when C is a null operator, they present the following interesting result.

Lemma 3 [20] Let P be a normal cone in a real Banach space E, $A : P \to P$ be an increasing γ-concave operator and $B : P \to P$ be an increasing sub-homogeneous operator. Assume that

(i) there is $w > \theta$ such that $Aw \in P_w$ and $Bw \in P_w$;

(ii) there exists a constant $\delta_0 > 0$ such that $Ax \geq \delta_0 Bx, \forall x \in P$.

Then operator equation $Ax + Bx = x$ has a unique solution x^* in P_w. Moreover, constructing successively the sequence $y_n = Ay_{n-1} + By_{n-1}, n = 1, 2, \cdots$ for any initial value $y_0 \in P_w$, we have $y_n \to x^*$ as $n \to \infty$.

Remark 1 [20] When B is a null operator, lemma 3 also holds.

In this paper, we will work in the Banach space $C[0,1]$ with the standard norm $\|x\| = \sup\{|x(t)| : t \in [0,1]\}$. Notice that this space can be endowed with a partial order given by $x, y \in C[0,1], \;\; x \leq y \Leftrightarrow x(t) \leq y(t)$ for $t \in [0,1]$.

Let $P = \{x \in C[0,1] | x(t) \geq 0, t \in [0,1]\}$ be the standard cone. Evidently, P is a normal cone in $C[0,1]$ and the normality constant is 1.

III. Main Results

In this section, we apply lemma 3 to investigate the problem (2), and obtain the new result on the existence and uniqueness of positive solution.

Theorem 1 Assume that

(H1) $g, h : [0,1] \times [0,\infty) \to [0,\infty)$ are continuous and increasing with respect to the second argument, $h(t,0) \not\equiv 0$;

(H2) there exists a constant $\gamma \in (0,1)$ such that $g(t, \lambda x) \geq \lambda^\gamma g(t,x), \forall t \in [0,1], \lambda \in (0,1), x \in [0,\infty)$, and $h(t, \mu x) \geq \mu h(t,x)$ for $\mu \in (0,1)$,$t \in [0,1]$,$x \in [0,\infty)$;

(H3) there exists a constant $\delta_0 > 0$ such that $g(t,x) \geq \delta_0 h(t,x), t \in [0,1], x \geq 0$. Then the problem (2) has a unique positive solution u^* in P_w, where $w(t) = t^{v-1}, t \in [0,1]$. Moreover, for any initial value $u_0 \in P_w$, constructing successively the iterative scheme

$$u_{n+1}(t) = \int_0^1 G(t,s) f(s, u_n(s)) \mathrm{d}s, \quad n = 0, 1, 2, \cdots,$$

we have $u_n(t) \to u^*(t)$ as $n \to \infty$, where $G(t,s)$ is given as (4).

Proof: To begin with, from Lemma 1, the problem (2) has an integral formulation given by

$$\begin{aligned} u(t) &= \textstyle\int_0^1 G(t,s) f(s, u(s)) \mathrm{d}s \\ &= \textstyle\int_0^1 G(t,s)[g(s,u(s)) + h(s,u(s))] \mathrm{d}s \end{aligned}$$

where $G(t,s)$ is given as (4).

Define two operators $A : P \to E$ and $B : P \to E$ by

$$Au(t) = \textstyle\int_0^1 G(t,s) g(s,u(s)) \mathrm{d}s,$$

$$Bu(t) = \textstyle\int_0^1 G(t,s) h(s,u(s)) \mathrm{d}s.$$

It is easy to prove that u is the solution of the problem (2) if and only if $u = Au + Bu$.By assumption (H1) and Lemma 2, we know that $A : P \to P$ and $B : P \to P$. In the sequel we check that A, B satisfy all assumptions of Lemma 3.

Firstly, we prove that A and B are two increasing operators.

In fact, from assumption (H1) and Lemma 2, for $u, v \in P$ with $u \geq v$, we know that $u(t) \geq v(t), \; t \in [0,1]$ and obtain

$$\begin{aligned} Au(t) &= \textstyle\int_0^1 G(t,s) g(s,u(s)) \mathrm{d}s \\ &\geq \textstyle\int_0^1 G(t,s) g(s,v(s)) \mathrm{d}s \\ &= Av(t) \end{aligned}$$

That is $Au \geq Av$. Similarly, $Bu \geq Bv$.

Next we show that A is a γ-concave operator and B is a sub-homogeneous operator.

In fact, for any $\lambda \in (0,1)$ and $u \in P$, from (H2) we know that

$$\begin{aligned} A(\lambda u)(t) &= \textstyle\int_0^1 G(t,s) g(s, \lambda u(s)) \mathrm{d}s \\ &\geq \lambda^\gamma \textstyle\int_0^1 G(t,s) g(s,u(s)) \mathrm{d}s \\ &= \lambda^\gamma Au(t) \end{aligned}$$

That is, $A(\lambda u) \geq \lambda^\gamma Au$ for $\lambda \in (0,1), u \in P$. So the operator A is a γ-concave operator. Also, for any $\mu \in (0,1)$ and $u \in P$, by (H2) we obtain

$$\begin{aligned} B(\mu u)(t) &= \int_0^1 G(t,s)h(s,\mu u(s))\mathrm{d}s \\ &\geq \mu \int_0^1 G(t,s)h(s,u(s))\mathrm{d}s \\ &= \mu Bu(t) \end{aligned}$$

That is, $B(\mu u) \geq \mu BAu$ for $\mu \in (0,1), u \in P$. So the operator B is a sub-homogeneous operator.

Now, we show that $Aw \in P_w$ and $Bw \in P_w$, where $w(t) = t^{v-1}$. By (H1) and Lemma 2,

$$\begin{aligned} &\tfrac{1}{\Gamma(v)} w(t) \int_0^1 [1-(1-s)^\alpha](1-s)^{v-\alpha-1} g(s,0)\mathrm{d}s \leq Aw(t) \\ &\leq \tfrac{1}{\Gamma(v)} w(t) \int_0^1 (1-s)^{v-\alpha-1} g(s,1)\mathrm{d}s \end{aligned}$$

From (H1) and (H3), we have

$$g(s,1) \geq g(s,0) \geq \delta_0 h(s,0) \geq 0$$

Since $h(t,0) \not\equiv 0$, we can get

$$\int_0^1 g(s,1)\mathrm{d}s \geq \int_0^1 g(s,0)\mathrm{d}s \geq \delta_0 \int_0^1 h(s,0)\mathrm{d}s > 0,$$

and in consequence,

$$\begin{aligned} l_1 &:= \tfrac{1}{\Gamma(v)} \int_0^1 [1-(1-s)^\alpha](1-s)^{v-\alpha-1} g(s,0)\mathrm{d}s > 0 \\ l_2 &:= \tfrac{1}{\Gamma(v)} \int_0^1 (1-s)^{v-\alpha-1} g(s,1)\mathrm{d}s > 0 \end{aligned}$$

So $l_1 w(t) \leq Aw(t) \leq l_2 w(t), t \in [0,1]$; and hence we have $Aw \in P_w$.

Similarly,

$$\begin{aligned} &\tfrac{1}{\Gamma(v)} w(t) \int_0^1 [1-(1-s)^\alpha](1-s)^{v-\alpha-1} h(s,0)\mathrm{d}s \leq Bw(t) \\ &\leq \tfrac{1}{\Gamma(v)} w(t) \int_0^1 (1-s)^{v-\alpha-1} h(s,1)\mathrm{d}s \end{aligned}$$

from $h(t,0) \not\equiv 0$, we easily prove $Bw \in P_w$. Hence the condition (i) of lemma 3 is satisfied. In the following we show that the condition (ii) of lemma 3 is satisfied. For $u \in P$, by (H3),

$$\begin{aligned} Au(t) &= \int_0^1 G(t,s)g(s,u(s))\mathrm{d}s \\ &\geq \delta_0 \int_0^1 G(t,s)h(s,u(s))\mathrm{d}s \\ &= \delta_0 Bu(t) \end{aligned}$$

Then we get $Au \geq \delta_0 Bu,\ u \in P$.

Finally, by means of lemma 3, the operator equation $Au + Bu = u$ has a unique positive solution u^* in P_w. Moreover, constructing successively the iterative scheme

$$u_n = Au_{n-1} + Bu_{n-1}, n = 1,2,\cdots$$

for any initial value $u_0 \in P_w$, we have $u_n \to u^*$ as $n \to \infty$. That is, the problem (2) has a unique positive solution u^* in P_w. For any initial value $u_0 \in P_w$, constructing successively the iterative scheme

$$u_{n+1}(t) = \int_0^1 G(t,s)f(s,u_n(s))\mathrm{d}s, \quad n = 0,1,2,\cdots,$$

we have $u_n \to u^*$ as $n \to \infty$.

Corollary 1 When $h(t,u(t)) \equiv 0$, assume that

(H4) $g : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous and increasing with respect to the second argument, $g(t,0) \not\equiv 0$;

(H5) there exists a constant $\gamma \in (0,1)$ such that

$$g(t,\lambda x) \geq \lambda^\gamma g(t,x), \forall t \in [0,1], \lambda \in (0,1), x \in [0,\infty).$$

Then problem

$$\begin{cases} -D_{0+}^v u(t) = f(t,u(t)), 0 < t < 1, n-1 < v \leq n \\ u(0) = u'(0) = u''(0) = \cdots = u^{(n-2)}(0) = 0, \\ [D_{0+}^\alpha u(t)]_{t=1} = 0, 1 \leq \alpha \leq n-2, \end{cases}$$

has a unique positive solution u^* in P_w, where $w(t) = t^{v-1}, t \in [0,1]$. Moreover, for any initial value $u_0 \in P_w$, constructing successively the iterative scheme

$$u_{n+1}(t) = \int_0^1 G(t,s)f(s,u_n(s))\mathrm{d}s, \quad n = 0,1,2,\cdots,$$

we have $u_n(t) \to u^*(t)$ as $n \to \infty$, where $G(t,s)$ is given as (4).

Remark 2 By Remark 1 and Theorem 1, Corollary 1 is obvious. Comparing Corollary 1 with main result in [11], the uniqueness of positive solution is not treated in [11]; Corollary 1 gives the existence and uniqueness of positive solution. Moreover, the unique positive solution u^* we obtain satisfies: (i) there exist $\lambda > \mu > 0$ such that $\mu t^{v-1} \leq u* \leq \lambda t^{v-1}, t \in [0,1]$, (ii) we can take any initial value in P_w and then construct an iterative scheme which can approximate the unique solution.

Remark 3 In particular, by a similar method used in Theorem 1 and Corollary 1, when $n = 3,\ \ \alpha = 1$, Theorem 1 and Corollary 1 hold. Comparing our main result with main result in [15], the uniqueness of positive solution is not treated in [15]; we give the existence and uniqueness of positive solution, which is similar with remark 2.

Remark 4 When $n = 4,\ \ \alpha = 2$,, Theorem 1 and Corollary 1 also hold. The corresponding result in [18] turn out to be special cases of our main result, see [[18 Theorem 3.1 and Corollary 3.2]].

IV. Example

We present one example to illustrate our main result.

Example 1 Consider the following problem:

$$\begin{cases} -D_{0+}^{7.3} u(t) = u^{\frac{1}{3}}(t) + \arctan u(t) + t^3 + t + \frac{\pi}{2}, 0 < t < 1, \\ u(0) = u'(0) = u''(0) = \cdots = u^{(6)}(0) = 0, \\ [D_{0+}^{5.1} u(t)]_{t=1} = 0, \end{cases} \tag{9}$$

In this example, the problem (9) fits the framework of the problem (2) with $v = 7.3, \alpha = 5.1$.

Let

$$g(t,u) = u^{\frac{1}{3}}(t) + t + \frac{\pi}{2},\ h(t,u) = \arctan u(t) + t^3,\ \gamma = \frac{1}{3}.$$

Obviously, $g, h : [0,1] \times [0,\infty) \to [0,\infty)$ are continuous and increasing with respect to the second argument, $h(t,0) = t^3 \not\equiv 0$.

Besides, for $t \in [0,1], \lambda \in (0,1), x \in [0,\infty)$, we have

$$\begin{aligned} g(t,\lambda u) &= \lambda^{\frac{1}{3}} u^{\frac{1}{3}}(t) + t + \tfrac{\pi}{2} \geq \lambda^{\frac{1}{3}} u^{\frac{1}{3}}(t) + \lambda^{\frac{1}{3}}(t + \tfrac{\pi}{2}) \\ &= \lambda^{\frac{1}{3}}(u^{\frac{1}{3}}(t) + t + \tfrac{\pi}{2}) = \lambda^\gamma g(t,u) \end{aligned}$$

and for $t \in [0,1], \mu \in (0,1), x \in [0,\infty)$, we have

$$\arctan(\mu u) \geq \mu \arctan u$$

thus

$$h(t,\mu u) \geq \mu h(t,u).$$

Moreover, if we take $\delta_0 \in (0,1]$, then we obtain

$$\begin{aligned} g(t,u) &= u^{\frac{1}{3}}(t) + t + \tfrac{\pi}{2} \geq t + \tfrac{\pi}{2} \geq t^3 + \arctan u \\ &\geq \delta_0(t^3 + \arctan u) = \delta_0 h(t,u) \end{aligned}$$

Hence all the conditions of Theorem 1 are satisfied. An application of Theorem 1 implies that problem (9) has a unique positive solution in P_w, where $w(t) = t^{v-1} = t^{6.3}$,$t \in [0,1]$.

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