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A Spectral Decomposition Method for Ordinary Differential Equation Systems with Constant or Linear Right Hand Sides

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Abstract**—**In this paper, a spectral decomposition method is developed for the direct integration of stiff and nonstiff homogeneous linear (ODE) systems with linear, constant, or zero right hand sides (RHSs). The method does not require iteration but obtains solutions at any random points of t, by direct evaluation, in the interval of integration. All the numerical solutions obtained for the class of systems coincide with the exact theoretical solutions. In particular, solutions of homogeneous linear systems, i.e. with zero RHS, conform to the exact analytical solutions of the systems in terms of t.

Keywords**—**Spectral decomposition, eigenvalues of the Jacobian, linear RHS, homogeneous linear systems.

I. Introduction

THE Jibunoh numerical spectral decomposition (NSD) developed in [1] obtains direct accurate solutions of any homogeneous linear (ODE) systems, given by,

$$y^1 = f = Ay, \qquad y(t_0) = y_0 \tag{1}$$

where A is the constant Jacobian.

The solutions of the general non homogeneous counterpart, i.e.

$$y^1 = f = Ay + b(t), \qquad y(t_0) = y_0 \tag{2}$$

which are easily obtained by automatic iteration using the QBASIC codes. However, there is an interesting class of linear nonhomogeneous ODE systems to which the NSD shall be applicable so as to obtain accurate solutions at any random step n, without iterations. This class constitutes the systems with linear or constant RHSs which particularly includes the homogeneous linear system whose RHS is constant, zero.

Suppose that a k-dim nonhomogeneous system is given by (2). Then, the RHS, b(t), must be of the linear form

$$b(t) = at + c \tag{3}$$

where a and c are constant column vectors.

If a = 0, then b(t) = c, so RHS is a constant. If a =c = 0, then the system is homogeneous.

There is a vast literature on the methods of integrating ODE systems especially with reference to those of the Euler, the RK method, the Rosenbrock method, and the recent Jibunoh Exponential Method [2] etc., in which the integrations of systems with linear RHS are subsumed under the general integrations. These methods have no special schemes for the direct integration of ODE systems belonging to the class of linear RHSs. The short and quick method to be proposed here, for this class of systems, could show the way for short and quick methods for general ODE integrations. In this paper, therefore, we shall follow the exposition in [1] to derive the required formulas for integrating directly the ODE systems whose RHSs are either linear or constant.

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II. Deriving the Integration Formulas

From [1], the general integration formula for (2) subject to spectral decomposition is given by

$$y_{n+1} = e^{hA}y_n + (e^{hA} - I)A^{-1}b_n \tag{4}$$

where A is the KxK constant Jacobian to be decomposed, h is a constant step size, and $b_n = b(t_n)$ at step n. We may put (4) in the form

$$y_{n+1} = e^{hA}y_n + Mb_n \tag{5}$$

where $M = (e^{hA} - I)A^{-1}$, is a constant KxK matrix.

By iteration of (6), we have

$$y_1 = e^{hA}y_0 + Mb_0$$

$$y_2 = e^{2hA}y_0 + e^{hA}Mb_0 + Mb_1$$

or in general

$$y_n = e^{nhA}y_o + \sum_{j=0}^{n-1} e^{hA(n-j-1)}Mb_j \tag{6}$$

By the spectral decomposition of Jibunoh [1]

$$e^{nhA} = A_1e^{\lambda_1 hn} + A_2e^{\lambda_2 hn} + \ldots A_k e^{\lambda_k hn} \tag{7}$$

and

$$M = A_1\frac{e^{\lambda_1 h} - 1}{\lambda_1} + A_2\frac{e^{\lambda_2 h} - 1}{\lambda_2} + .. + A_k\frac{e^{\lambda_k h} - 1}{\lambda_k} \tag{8}$$

where λ_1, $\lambda_2 \ldots \lambda_k$ are distinct or made to be distinct as in [1] and where

$$\frac{e^{\lambda_1 h} - 1}{\lambda_1} = h, \text{ if } \lambda_j = 0 \tag{9}$$

Also A_i, i= 1 (1) K are KxK constant matrices obtained from the matrix equation

$$A_1 + A_2 + \ldots + A_k = I$$

$$A_1\lambda_1 + A_2\lambda_2 + \ldots + A_k\lambda_k = A \tag{10}$$

$$A_1\lambda_1^2 + A_2\lambda_2^2 + \ldots + A_k\lambda_k^2 = A^2$$

$$A_1\lambda_1^{k-1} + A_2\lambda_2^{k-1} + \ldots + A_k\lambda_k^{k-1} = A^{k-1}$$

where A is the constant Jacobian, and I is the KxK identity matrix. The matrix equation (10) is easily deducible from (7) by substituting for n = 0 to K-1.

From (10), we obtain the values for $A_1, A_2, \ldots A_k$ as

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_k \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_k^2 \\ \vdots & \vdots & \cdots & \vdots \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \cdots & \lambda_k^{k-1} \end{pmatrix}^{-1} \begin{pmatrix} I \\ A_1 \\ A^2 \\ \vdots \\ A^{k-1} \end{pmatrix} \tag{11}$$

Using (7) and (8), we write (6) as;

$$y_n = \left[A_1 y_0, A_2 y_0, \ldots A_k y_0\right] \begin{pmatrix} e^{\lambda_1 hn} \\ e^{\lambda_2 hn} \\ \vdots \\ e^{\lambda_k hn} \end{pmatrix} + \sum_{j=0}^{n-1} \left[A_1 Mb_j, A_2 Mb_j \ldots, A_k Mb_j\right] \begin{pmatrix} e^{\lambda_1 h(n-j-1)} \\ e^{\lambda_2 h(n-j-1)} \\ \vdots \\ e^{\lambda_k h(n-j-1)} \end{pmatrix} \tag{12}$$

where $\left[A_1 y_0, A_2 y_0, \ldots A_k y_0\right]$ is a KxK matrix whose columns are given by $A_i y_0$, i= 1(1)K and $\left[A_1 Mb_j, A_2 Mb_j \ldots, A_k Mb_j\right]$ is also a KxK matrix whose columns are given by

$$A_i Mb_j \quad i = 1(1)K, \quad j = 0(1)n-1$$

By (3),

$$b_j = at_j + c \tag{13}$$

Then, since $t_n = t_0 + nh + \frac{h}{2}$ at step n, where $\frac{h}{2}$ is *Jibunoh correction for continuity* [1], we have

$$t_j = t_0 + jh + \frac{h}{2}$$

$$\text{Thus, } b_j = jah + a(t_0 + \tfrac{h}{2}) + c = jah + u \tag{14}$$

where $u = a(t_0 + \frac{h}{2}) + c$. Therefore, (12) can be expressed in the form

$$y_n = \overline{y_n} + \left[A_1 Mah, A_2 Mah, \ldots A_k Mah\right] \begin{pmatrix} \sum_{j=0}^{n-1} je^{\lambda_1 h(n-j-1)} \\ \sum_{j=0}^{n-1} je^{\lambda_2 h(n-j-1)} \\ \vdots \\ \sum_{j=0}^{n-1} je^{\lambda_k h(n-j-1)} \end{pmatrix} + \left[A_1 Mu, A_2 Mu, \ldots A_k Mu\right] \begin{pmatrix} \sum_{j=0}^{n-1} e^{\lambda_1 h(n-j-1)} \\ \sum_{j=0}^{n-1} e^{\lambda_2 h(n-j-1)} \\ \vdots \\ \sum_{j=0}^{n-1} e^{\lambda_k h(n-j-1)} \end{pmatrix} \tag{15}$$

where $\overline{y_n}$ is the first part of (2.9) for any particular $\lambda_i \neq 0$, i = 1(1)K. At any step n, it is easily derived that

$$\sum_{j=0}^{n-1} je^{\lambda_i h(n-j-1)} = \frac{e^{\lambda_i hn} - ne^{\lambda_i h} + n - 1}{(e^{\lambda_i h} - 1)^2} = r_i \tag{16}$$

and

$$\sum_{j=0}^{n-1} e^{\lambda_i h(n-j-1)} = \frac{e^{\lambda_i hn} - 1}{e^{\lambda_i h} - 1} = s_i$$

If $\lambda_i = 0$, then we obtain

$$\sum_{j=0}^{n-1} je^{\lambda_i h(n-j-1)} = \frac{n(n-1)}{2} = r_i \tag{17}$$

$$\sum_{j=0}^{n-1} e^{\lambda_i h(n-j-1)} = n = s_i$$

By computations, the integration formula (15) is expanded and simplified to obtain the component by component integration formulas $(y_{1n}, y_{2n}, \ldots y_{kn})^T$. This will make

integration easy. During computations, any number or resultant number of the formed $a \times 10^{-r} = 0$, by definition, provided $|a| < 10$ and $r \geq 6$. e.g. $2.561 \times 10^{-6} = 0$ or $-3.1124 \times 10^{-8} = 0$.

We take note of the following;

i. The integration formula (15) is the formula for the general case in which the RHS is b(t) = at + c, where a and c are constant column vectors, a $\neq 0$, c $\neq 0$. It is possible to have a $\neq 0$, but c = 0.

ii. If a =c = 0, then the system is homogeneous and from (16), we have

$$y_n = \overline{y_n} = \left[A_1 y_0, A_2 y_0, \ldots A_k y_0\right] \begin{pmatrix} e^{\lambda_1 hn} \\ e^{\lambda_2 hn} \\ \vdots \\ e^{\lambda_k hn} \end{pmatrix} \quad (18)$$

which translates as in [1] to

$$y_*(t) = \left[A_1 y_0, A_2 y_0, \ldots A_k y_0\right] \begin{pmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \vdots \\ e^{\lambda_k t} \end{pmatrix} \quad (19)$$

iii. If the RHS of the system b(t) = at + c is such that a = 0 but $c \neq 0$, then the integration formula (15) is not very adequate. Therefore, we derive an alternative preferred formula for this case, as follows.

By (4),

$$y_{n+1} = e^{hA} y_n + (e^{hA} - I)A^{-1} b_n$$

Since b_n is a constant, let us denote it by $b_0 = c$, a constant column vector. Then, by iteration

$$y_1 = e^{hA}(y_0 + A^{-1}b_0) - A^{-1}b_0 \quad (20)$$

$$y_2 = e^{2hA}(y_0 + A^{-1}b_0) - A^{-1}b_0 \quad (21)$$

and in general, for all n,

$$y_n = e^{hAn}(y_0 + A^{-1}b_0) - A^{-1}b_0, \quad (22)$$

since $b_j = b_0$, j =0(1)n-1.

$$\text{Let } y_{0*} = y_0 + A^{-1}b_0 \quad (23)$$

$$\text{such that, by (22), } y_n = e^{hA} y_{0*} - A^{-1}b_0 \quad (24)$$

Therefore, applying the NSD steps (7) and (11), we obtain the integration formula

$$y_n = \left[A_1 y_{0*}, A_2 y_{0*}, \ldots A_k y_{0*}\right] \begin{pmatrix} e^{\lambda_1 hn} \\ e^{\lambda_2 hn} \\ \vdots \\ e^{\lambda_k hn} \end{pmatrix} - A^{-1}b_0 \quad (25)$$

or as in the homogeneous case

$$y_*(t) = \left[A_1 y_{0*}, A_2 y_{0*}, \ldots A_k y_{0*}\right] \begin{pmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ \vdots \\ e^{\lambda_k t} \end{pmatrix} - A^{-1}b_0 \quad (26)$$

where b_0 is the constant column vector on the RHS of the system.

The formula (25) or (26) assumes that A^{-1} exists. If A^{-1} does not exist, we revert to the integration formula (15) with a = 0 and $b_0 = u$.

We shall denote the integration formulas (15) and (26) by the acronym (NSDL), i.e. the NSD formulas for systems with constant or linear RHSs.

III. Stepsize for the Integrations and Computational Procedures of the NSDL

The stepsize defined for all integrations is h = 0.001, or h = 10^{-r} where r ≥ 3 is an integer, for stiff and nonstiff systems. On computational procedures, the following sequence is recommended.

i. Obtain the eigenvalues of the Jacobian A, i.e. λ_1, $\lambda_2 \ldots \lambda_k$

ii. For spectral decomposition of e^{hAn} obtain the matrices A_1, A_2,… A_k from the equation

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_k \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_k^2 \\ \vdots & \vdots & \cdots & \vdots \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \cdots & \lambda_k^{k-1} \end{pmatrix}^{-1} \begin{pmatrix} I \\ A_1 \\ A^2 \\ \vdots \\ A^{k-1} \end{pmatrix}$$

provided λ_i, i= 1(1)K are distinct.

iii. Obtain the matrix M given by (8), i.e.

$$M = A_1 \frac{e^{\lambda_1 h} - 1}{\lambda_1} + A_2 \frac{e^{\lambda_2 h} - 1}{\lambda_2} + .. + A_k \frac{e^{\lambda_k h} - 1}{\lambda_k}$$

given that $\frac{e^{\lambda_i h - 1}}{\lambda_i} = h$, if $\lambda_i = 0$

iv. The RHS of the system is $b_j = at_j + c$ as in (13). From this, we saw in (14) that

b_j = jah + u, where u = $a(t_0 + \frac{h}{2}) + c$ is a column vector.

Compute for the column vectors

(a) Ma
(b) Mu

v. Obtain the column vectors;

$$A_1Ma,\ \ A_2Ma, \ldots A_kMa$$
$$A_1Mu,\ A_2Mu, \ldots A_kM$$

vi. Set up the matrices

(a) h[A_1Ma, A_2Ma, … A_kMa], where h is the stepsize.
(b) [A_1Mu, A_2Mu, … A_kMu]

vii. Write down the integration formula at any step n, as the simplified (15).

$$y_n = \left[A_1y_0, A_2y_0, \ldots A_ky_0\right]\begin{pmatrix} e^{\lambda_1 hn} \\ e^{\lambda_2 hn} \\ \vdots \\ e^{\lambda_k hn} \end{pmatrix} +$$

$$h\left[A_1Ma, A_2Ma, \ldots A_kMa\right]\begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_k \end{pmatrix} +$$

$$\left[A_1Mu, A_2Mu, \ldots A_kMu\right]\begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_k \end{pmatrix}$$

where r_i and s_i are defined in (16) or (17) depending on whether $\lambda_i \neq 0$ or $\lambda_i = 0$.

For any randomly chosen value of $t \geq t_0$ in the interval of integration, the corresponding step is $n = \frac{t - t_0}{h}$. Then, y_n is the accurate numerical solution corresponding to y(t).

The integration formula y_n is then expanded and simplified as earlier explained to obtain the component by component integration formulas $(y_{1n}, y_{2n}, \ldots y_{kn})^T$.

The steps of integration outlined above are mainly for the general case in which the RHS of the system is any of the forms

- b(t) = at + c, $a \neq 0, c \neq 0$
- b(t) = at + c, $a \neq 0, c = 0$ (27)
- b(t) = at + c, $a = 0, c = 0$, the homogenous case

For the particular case,

- b(t) = at + c, $a = 0$, $c = b_0$, a constant column vector, (28)

The additional steps after steps (i) and (ii) above are the inverse of the Jacobian and $y_{0*} = y_0 + A^{-1}b_0$. These are important components of the integration formula (26), i.e.

$$y_*(t) = \left[A_1y_{0*}, A_2y_{0*}\right]\begin{pmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \end{pmatrix} - A^{-1}b_0$$

Note: All the eigenvalues of the Jacobian so far under description are assumed to be distinct and real. For cases in which the eigenvalues are complex or not distinct, the mathematical procedures are available in the literature, or in the alternative, see Jibunoh [1] for the necessary exposition.

IV. Numerical Applications

Example I

$$y_1^1 = 32y_1 + 66y_2 + \frac{2}{3}t + \frac{2}{3}$$

$$y_2^1 = -66y_1 - 133y_2 - \frac{1}{3}t - \frac{1}{3}$$

$$y_1(0) = y_2(0) = \frac{1}{3}, \qquad 0 \leq t \leq 1$$

This is a stiff system from Burden and Faires [3] with a linear RHS. The theoretical solutions are given by

$$y_1(t) = \frac{2}{3}t + \frac{2}{3}e^{-t} - \frac{1}{3}e^{-100t}$$

$$y_2(t) = -\frac{1}{3}t - \frac{1}{3}e^{-t} + \frac{2}{3}e^{-100t}$$

The eigenvalues of the Jacobian are $\lambda_1 = -1$, $\lambda_2 = -100$.

By applying the procedures in Section III, with h = 0.001, we obtain the integration formula

$$y_n = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} e^{-.001n} \\ e^{-0.1n} \end{pmatrix} +$$
$$\frac{1}{9}\begin{pmatrix} .000005997 & 0 \\ -.0000029985 & 0 \end{pmatrix}\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} +$$
$$\frac{1}{9}\begin{pmatrix} .0059999995 & 0 \\ -.00299999975 & 0 \end{pmatrix}\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$$

where

$$r_1 = \frac{\left(e^{-.001n} - ne^{-.001} + n - 1\right)}{.000000999}$$

$$r_2 = \frac{\left(e^{-0.1n} - ne^{-0.1} + n - 1\right)}{.009055917}$$

$$s_1 = \frac{\left(e^{-.001n} - 1\right)}{-.0009995}$$

$$s_2 = \frac{\left(e^{-0.1n} - 1\right)}{-.095162582}$$

We now substitute for r_i and s_i in i=1(1)2 in the matrix equation. Then, computing and simplifying on component by component basis, we have

$$y_{1n} = \frac{2}{3}e^{-.001n} - \frac{1}{3}e^{-0.1n} + \frac{1}{9}\times\frac{.000005997}{.000000999}\left(e^{-.001n} - ne^{-.001} + n - 1\right) + \frac{1}{9}\times\frac{.0059999995}{-.0009995}\left(e^{-.001n} - 1\right)$$

$$y_{2n} = -\frac{1}{3}e^{-.001n} + \frac{2}{3}e^{-0.1n} + \frac{1}{9}\times\frac{-.0000029985}{.000000999}\left(e^{-.001n} - ne^{-.001} + n - 1\right) + \frac{1}{9}\times\frac{-.00299999975}{-.0009995}\left(e^{-.001n} - 1\right)$$

Taking note during computation (as defined in Section III) that any number a x 10^{-r} = 0, provided |a| < 10 and r ≥ 6, we obtain after simplification;

$$y_{1n} = \frac{2}{3}e^{-.001n} - \frac{1}{3}e^{-0.1n} + \left(6.666666\times10^{-4}\right)n$$

$$y_{2n} = -\frac{1}{3}e^{-.001n} + \frac{2}{3}e^{-0.1n} - \left(3.333333056\times10^{-4}\right)n$$

The NSDL numerical solutions (given to 8 decimal places or more) are obtained for random points of t in the interval $0 \le t \le 1$ and compared with the theoretical solutions in Table I.

TABLE I
COMPARING THE NSDL AND THEORETICAL SOLUTIONS OF EXAMPLE I: $T_0 = 0$, H = .001

t	$n=\frac{t-t_0}{.001}$	y(t)	y_n(NSDL)
.001	1	0.3650545272 0.2698914454	0.3650545272 0.2698914454
0.1	100	0.66987648 -0.33491554	0.66987647 -0.33491554
0.5	500	0.73768710 -0.36884355	0.73768710 -0.36884354
1.0	1000	0.91191960 -0.45595980	0.91191980 -0.45595980

From Table I, we see that the NSDL and theoretical solutions coincide at all random points chosen in the interval of integration. This shows the efficiency of the NSDL in stiff systems.

Example 2

$$\frac{dx}{dt} = 4x - 5y + 4t - 1$$

$$\frac{dy}{dt} = x - 2y + t$$

$$x(0) = y(0) = 0, \qquad 0 \le t \le 1$$

This is a nonstiff system from Krasnov et al. [5]. The theoretical solutions are

$$x(t) = -t$$

$$y(t) = 0.$$

The RHS is linear i.e.

$$b(t) = at + c, \text{ where } a = \begin{pmatrix}4\\1\end{pmatrix} \text{ and } c = \begin{pmatrix}-1\\0\end{pmatrix}$$

We shall let $x = y_1$ and $y = y_2$. The eigenvalues of the Jacobian are $\lambda_1 = 3\ \lambda_2 = -1$.

From (14), $u = a(t_0 + \frac{h}{2}) + c$. Therefore, taking h = 0.001, we find $u = \begin{pmatrix}-0.998\\0.0005\end{pmatrix}$.

By the procedures in Section III, with h = .001, we observe that the first part of (15) i.e. $\overline{y_n}$ vanishes since $y_1(0) = y_2(0) = 0$.

The integration formula then reduces to

$$y_n = \frac{1}{4}\begin{pmatrix}.000015022523 & .0000009995\\ .000003004504 & .0000009995\end{pmatrix}\begin{pmatrix}r_1\\r_2\end{pmatrix} + \frac{1}{4}\begin{pmatrix}-.0049999549 & .000999991652\\ -.00099999098 & .000999991652\end{pmatrix}\begin{pmatrix}s_1\\s_2\end{pmatrix}$$

where

$$r_1 = \frac{\left(e^{.003n} - ne^{.003} + n - 1\right)}{.000009027047}$$

$$r_2 = \frac{\left(e^{-.001n} - ne^{-.001} + n - 1\right)}{.000000999}$$

$$s_1 = \frac{(e^{.003n} - 1)}{.0030045045\ \ 03}$$

$$s_2 = \frac{(e^{-.001n} - 1)}{-.0009995001\ \ 67}$$

As in Example I, we substitute for r_i and s_i i=1(1)2 and reduce to the component by component basis to obtain;

$$y_{1n} = \frac{1}{4}\times\frac{.000015022\ 523}{.000009027\ 047}\left(e^{-.003n} - ne^{-.003} + n - 1\right) + \frac{1}{4}\times\frac{.000000999\ 5}{.000000999}\left(e^{-.001n} - ne^{-.001} + n - 1\right)$$
$$+ \frac{1}{4}\times\frac{-.004999954\ 9}{.003004500\ 4503}\left(e^{.003n} - 1\right) + \frac{1}{4}\times\frac{.000999991\ 652}{-.000999500\ 167}\left(e^{-.001n} - 1\right)$$

$$y_{2n} = \frac{1}{4}\times\frac{.000003004504}{.000009027047}\left(e^{-.003n} - ne^{.003} + n - 1\right) + \frac{1}{4}\times\frac{.0000009995}{.000000999}\left(e^{-.001n} - ne^{-.001} + n - 1\right)$$

$$+\frac{1}{4}\times\frac{-.00099999098}{.0030045004503}\left(e^{.003n} - 1\right) + \frac{1}{4}\times\frac{.000999991652}{-.000999500167}\left(e^{-.001n} - 1\right)$$

Setting numbers or resultant numbers of the type $a \times 10^{-r} = 0$, where $|a| < 10$ and $r \geq 6$, we obtain after simplification

$$y_{1n} = -0.001n$$

$$y_{2n} = 0$$

The NSDL solutions are written down only to 4 decimal places for the random points of t, in the interval $0 \leq t \leq 1$ and compared with the theoretical solutions in Table II.

TABLE II
COMPARING THE NSDL AND THEORETICAL SOLUTIONS OF EXAMPLE 2, t_0=0, H = .001

t	$n=\frac{t-t_0}{.001}$	y(t)	y_n(NSDL)
0.001	1	-0.0010 0.0000	-0.0010 0.0000
0.5	500	-0.5000 0.0000	-0.5000 0.0000
0.625	625	-0.6250 0.0000	-0.6250 0.0000
1.0	1000	-1.0000 0.0000	-1.0000 0.0000

We find from Table II that the NSDL and the theoretical solutions coincide in the interval of integration. It demonstrates that NSDL is also efficient in the integration of nonstiff systems with linear RHSs.

Example 3

$$y_1^1 = -0.1y_1 - 199.9y_2$$

$$y_2^1 = -200y_2$$

$$y_1(0) = 2, \quad y_2(0) = 1$$

This is a stiff system with zero RHS, which means that the system is homogeneous. The theoretical solutions are

$$y_1(t) = e^{-0.1t} + e^{-200t}$$

$$y_2(t) = e^{-200t}$$

It was first obtained from Fatunla [4], but has been solved accurately in Jibunoh [1]. The original application of the nonstiff method of Adams-Moulton, Shampine and Gordon and the application of stiff Backward Differentiation Codes of Gear did not yield accurate results [4].

NSDL reduces to NSD in homogeneous systems, and only the first part of (12) survives. Therefore, we merely reproduce the NSD procedure in Jibunoh [1]. The eigenvalues of the Jacobian are $\lambda_1 = -0.1$, $\lambda_2 = -200$.

By (11) we obtain

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -0.1 & -200 \end{pmatrix}^{-1}\begin{pmatrix} I \\ A \end{pmatrix}$$

where A is the constant Jacobian. Thus,

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \frac{1}{199.9}\begin{pmatrix} 200 & 1 \\ -0.1 & -1 \end{pmatrix}\begin{pmatrix} I \\ A \end{pmatrix}$$

Then,

$$A_1 = \frac{1}{199.9}\left[\begin{pmatrix} 200 & 0 \\ 0 & 200 \end{pmatrix} + \begin{pmatrix} -0.1 & -199.9 \\ 0 & -200 \end{pmatrix}\right]$$

i.e.

$$A_1 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

Now, $y(0) = (2, \quad 1)^T$

By (19),

$$y_*(t) = \left[A_1 y_0, \quad A_2 y_0\right]\begin{pmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \end{pmatrix}$$

$$\text{i.e.} \quad y_*(t) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^{-0.1t} \\ e^{-200t} \end{pmatrix}$$

= y(t), the exact theoretical solution.

Example 4

$$\frac{dy_1}{dt} = 4y_2 + y_3$$

$$\frac{dy_2}{dt} = y_3$$

$$\frac{dy_3}{dt} = 4y_2$$

$$y_1(0) = 5,\ y_2(0) = 0,\ y_3(0) = 4$$

This is a nonstiff linear system from Krasnov et al. [5]. The RHS = 0, which implies that the system is homogeneous.

The eigenvalues of the Jacobian are $\lambda_1 = 0,\ \lambda_2 = 2,\ \lambda_3 = -2$. The theoretical solutions are given by

$$y_1(t) = 1 + 3e^{2t} + e^{-2t}$$

$$y_2(t) = e^{2t} - e^{-2t}$$

$$y_3(t) = 2e^{2t} + 2e^{-2t}$$

By the spectral decomposition method in (11),

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 4 \end{pmatrix}^{-1} \begin{pmatrix} I \\ A \\ A^2 \end{pmatrix}$$

where A, as usual, is the constant Jacobian of the system. Eventually, this yields

$$A_1 = \frac{1}{4}\begin{pmatrix} 4 & -4 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A_2 = \frac{1}{4}\begin{pmatrix} 0 & 6 & 3 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{pmatrix}$$

$$A_3 = \frac{1}{4}\begin{pmatrix} 0 & -2 & 1 \\ 0 & 2 & -1 \\ 0 & -4 & 2 \end{pmatrix}$$

Now $y(0) = (5,\ 0,\ 4)^T$.

So by (19),

$$y_*(t) = \begin{bmatrix} A_1 y_0, A_2 y_0, \ldots A_3 y_0 \end{bmatrix} \begin{pmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \\ e^{\lambda_3 t} \end{pmatrix}$$

i.e.

$$y_*(t) = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & -1 \\ 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ e^{2t} \\ e^{-2t} \end{pmatrix}$$

$= y_*(t)$, the exact theoretical solution.

Example 5

$$\frac{dy_1}{dt} = 2y_1 - 4y_2 + 1$$

$$\frac{dy_2}{dt} = -y_1 + 5y_2$$

$$y_1(0) = 3, \quad y_2(0) = 1$$

This is a nonstiff system with constant RHS, $b_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, adapted from Krasnov et al. [5]. The theoretical solutions are

$$y_1(t) = 4e^t - \frac{1}{6}(e^{6t} + 5)$$

$$y_2(t) = e^t + \frac{1}{6}(e^{6t} - 1)$$

The matrix $A = \begin{pmatrix} 2 & -4 \\ -1 & 5 \end{pmatrix}$ is the constant Jacobian with eigenvalues $\lambda_1 = 1,\ \lambda_2 = 6$. We take h = 0.001. Then, decomposing e^{hA} we have

$$A_1 = \frac{1}{5}\begin{pmatrix} 4 & 4 \\ 1 & 1 \end{pmatrix} \text{ and } A_2 = \frac{1}{5}\begin{pmatrix} 1 & -4 \\ -1 & 4 \end{pmatrix}.$$

To apply the integration formula (26), we first obtain

$$A^{-1}b_0 = \frac{1}{6}\begin{pmatrix} 5 \\ 1 \end{pmatrix} \text{ and } y_{0*} = y_0 + A^{-1}b_0 = \frac{1}{6}\begin{pmatrix} 23 \\ 7 \end{pmatrix}$$

Therefore, following (26), we have

$$y_*(t) = \begin{bmatrix} A_1 y_{0*}, & A_2 y_{0*} \end{bmatrix} \begin{pmatrix} e^t \\ e^{6t} \end{pmatrix} - \frac{1}{6}\begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

as the integration formula for the system. i.e.

$$y_*(t) = \frac{1}{6}\begin{pmatrix} 24 & -1 \\ 6 & 1 \end{pmatrix}\begin{pmatrix} e^t \\ e^{6t} \end{pmatrix} - \frac{1}{6}\begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 4e^t - \frac{1}{6}e^{6t} - \frac{5}{6} \\ e^t + \frac{1}{6}e^{6t} - \frac{1}{6} \end{pmatrix}$$

= y(t), the exact theoretical solution.

V. Conclusion

The spectral decomposition method developed in this paper (i.e. the NSDL) is tailored to integrate directly stiff and nonstiff ODE systems with linear or constant RHSs. The method is an offshoot of the Jibunoh Spectral decomposition for the general linear systems developed in [1].

This method for the aforementioned class of systems obtained solutions at any random points of t by direct evaluation, and the computations for the implementation of the

NSDL may be handled manually or by use of a computer. In the latter case, a computer program needs to be developed by an interested researcher.

All the solutions obtained for the special ODE systems coincide with the theoretical solutions. In particular, solutions of homogeneous systems are exemplary as the numerical solutions turn out to be exactly equal to the theoretical (analytical) solutions of the systems, in terms of t. This recommends the present method as an efficient technique for homogeneous linear systems.

It must be noted that the exact or nearly exact eigenvalues of the Jacobians are needed for the implementation of the NSDL, if we wish to maintain the high accuracy of the integrations.

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