A Robust Controller for Output Variance Reduction and Minimum Variance with Application on a Permanent Field DC-Motor

Mahmood M. Al-Imam and M. Mustafa,

Abstract—In this paper, we present an experimental testing for a new algorithm that determines an optimal controller's coefficients for output variance reduction related to Linear Time Invariant (LTI) Systems. The algorithm features simplicity in calculation, generalization to minimal and non-minimal phase systems, and could be configured to achieve reference tracking as well as variance reduction after compromising with the output variance. An experiment of DC-motor velocity control demonstrates the application of this new algorithm in designing the controller. The results show that the controller achieves minimum variance and reference tracking for a preset velocity reference relying on an identified model of the motor.

Keywords—Output variance, minimum variance, over-parameterization, DC-Motor.

I. INTRODUCTION

THE external disturbances is the pivotal factor that deteriorates performance of a feedback control Linear Time Invariant (LTI) system [1], [2]. Therefore, optimal disturbance rejection was the focus of researchers to design addresses optimal disturbance rejection [3].

Minimum Variance (MV) appears as a result of optimal disturbance rejection, which was early studied in the valuable work of [4], [2]. MV regulation only applies to plants having a discrete-time model with stable zeros. Generalized Minimum Variance (GMV) came to the scene in treating plants with unstable zeros. It is an extension to the existence MV strategy [2].Literatures of Generalized variance are well investigated in early work of [5], [6], [2]. Recent literatures can be found in the work of [7]

Linear Quadratic Gaussian (LQG), in other hand, was introduced to overcome problems encounters in GMV account [8], [9]. However, LQG control scheme exhibits heavy calculation load [8], [4] in addition to no guaranty to the conventional desired gain and phase margin [10]

The objective of this paper is to design an algorithm that calculates the coefficients of a minimum variance controller that overcomes limitation of MV and GMV, features fast calculated coefficients comparing with LQG preparatory setting, in addition to robustness against system pole locations. The controller also can track a reference set-point. The concept is based on over-parameterization

Mahmood is with department of Electrical and Computer Engineering, Georgia Institute of Technology, Atlanta, GA, 30332 USA e-mail: (see http://www.michaelshell.org/contact.html).
M. Marzuki with Anonymous University.
Manuscript received April 19, 2005; revised January 11, 2007.

of polynomials using *Diophantine Equation* and *Sylvester Theorem*. Early adoption of the same concept is found in [11], which consists of developing a single structure controller that track reference signal of LTI system in addition achieving variance reduction on the output. [11] assumed always a unity characteristic equation and assumes heuristically that the developed algorithm reduces the output variance after a careful tuning to a weighting matrix. In this paper, we overcome the assumption of unity characteristics equation by working with any prescribed characteristics equation given by the user. [12] proves analytically that an over-extending of parameter of its proposed controller to infinity, the variance reduction tends to a variance of an LQG controller.

The approach of the work is to adopt, as mentioned earlier, *Diophantine equation* in order to calculate the over-parametrized coefficients for controller of an ARMAX model. The minimum variance for a general system output that may includes unstable poles is analytically derived. The result is truncated which yields to a dual Diophantine Equations problem, that both can be over-parameterized. This step is essential in overcoming complexity of calculating the H_2 norm of the minimum variance. The calculated polynomial that minimizes the H_2 norm of the first over-parameterized polynomial minimizes also H_2 norm of the second, which yields to the solution. The proof is realized by introducing a sub-optimal solution and analytically proving that its over-parameterized H_2 norm converges to zero. This causes to the optimal solution to converge to minimal as well, as the optimal H_2 norm is sandwiched by the sub-optimal norm from above.

In order to achieve a dual objective (i.e. variance reduction and reference tracking), an incorporating of an integrator to the controller raises slightly the variance, but establishes a reference tracking with zero steady-state error. Such advantages compromises the variance as it will be shown in the results that variance increase is not significantly comparing with the minimum variance in addition taking into account the simplicity of the algorithm that develops the controller. Recent similar work can be found in [3]

The results are obtained from applying the controller on a permanent field DC-motor speed control. A certain reference point of speed was applied to the system. The results are divided into two parts; reference point tracking and variance reduction. The controller achieves minimum variance after extending the controller with a higher order. The results also shows the incorporation of the integrator inside the controller

raises the variance slightly while maintaining the steady state error to zero.

II. OUTPUT VARIANCE AND EXTENDED CONTROLLERS

A linear single-input single-output system represented by an autoregressive moving average with auxiliary input ARMAX model is represented in figure (1). $A(z^{-1})$ and $B(z^{-1})$ are polynomials, which represent the plant's transfer function numerator and denominator respectively. The noise model is represented with $C(z^{-1})$ and $A(z^{-1})$ as $e(t)$ is a source of white noise. The signals are expressed in discrete time t. [4] A servo-controller described with $G(z^{-1})$ and $F(z^{-1})$ are the controller's transfer function numerator and denominator respectively. $H(z^{-1})$ is the feed-forward term and it is normally chosen to give a good set-point tracking. In this paper, $H(z^{-1})$ is normally chosen so that the overall DC gain for the closed loop system is unity.
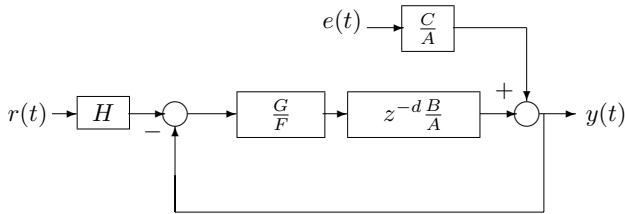

Fig. 1: ARMAX model

The characteristics equation of the system (i.e. the denominator of the closed loop transfer function) is denoted by $T(z^{-1})$ and is described in (1)

$$T(z^{-1}) = A(z^{-1})F(z^{-1}) + z^{-d}B(z^{-1})G(z^{-1}) \quad (1)$$

equation (1) has a unique solution if

$$n_f = n_d + k - 1 \text{ and } n_g = n_g - 1 \quad (2)$$

$F_0(z^{-1})$ and $G_0(z^{-1})$ are the minimal order for the controller satisfying equations (1) and (2). The set of all controllers that satisfy equation (1) are given by

$$F = F_0 + z^{-d}BP_{n_P} \quad (3)$$
$$G = G_0 - AP_{n_P} \quad (4)$$

The operator z^{-1} has been omitted for simplicity. Polynomial P_{n_P} in equations (3) and (4) denotes a polynomial of degree n_P that is used to extend the original solution of $F(z^{-1})$ and $G(z^{-1})$ [12]
Using equations (3), (4), (1) and the model of ARMAX model, it can be shown that

$$y(t) = \frac{z^{-d}BGH}{T}r(t) + \left(\frac{F_0 + z^{-d}BP_{n_P}}{T}\right)Ce(t) \quad (5)$$

From equation (5) the output variance is given by

$$var_y = E\left(\frac{F_0 + z^{-d}BP_{n_P}}{T}Ce(t)\right)^2 = \left\|\frac{F_0 + z^{-d}BP_{n_P}}{T}C\right\|_{L_2}^2 \quad (6)$$

The objective is to minimize the cost function that defines the output variance, described in equation (6), which is a function of the coefficients of P_{n_P}.

A. Minimum variance for Minimal and Non-Minimal Phase Systems

Let polynomial $B = B_+B_-$, where B_+ and B_- denote the stable and unstable factors of the polynomial respectively. Polynomial B_-^+ is the normalized reciprocal polynomial of B_- (i.e $B_-^+ = q^{-n_b+1}B_*^*/b_0$). It is already proven in () that

$$P_{n_P} = -\frac{Q}{B_+B_-^+C} \quad (7)$$

where polynomial Q is obtained from the Diophantine equation

$$RT + z^{-d}QB_- = F_0B_-^+C \quad (8)$$

where,
$R(0) = 1, deg(R) = deg(B_-)+d-1, deg(Q) = deg(T)-1$
Expanding the expectation function in equation (6) into two terms and decomposing the first term of the result into stable and causal, and unstable and non-causal terms. After arranging, It can be proven that the minimum output variance is a function of R. According to [4], the polynomials Q and R are the solution for the General Minimum Variance problem GMV. The closed loop system after the application of high order pole-placement controller becomes

$$var_y = \left\|\frac{R}{B_-^+}\right\|_{L_2}^2 \quad (9)$$

III. ANALYSIS OF REDUCED VARIANCE CONTROLLER WITH SUB-OPTIMAL SOLUTION

In order to simplify the calculation, [11] neglects the effect of the characteristics equation, T, in equation (1) and minimized the cost function

$$J = \left\|(F_0 + z^{-d}BP_{n_P})C\right\|_{L2}^2 \quad (10)$$

Equation (10) can be described as a sum of coefficients squares of $(F_0 + z^{-d}BP_{n_P})C$. [11] argued heuristically that the proposed method reduces also the output variance. However, by neglecting the effect of polynomial T, the method works only for restricted cases when $T = 1$. The difficulty for other choice of T is that the formulation of equation (6) into a set of linear equations in term of unknown coefficients of polynomial as proved by [12] is computationally cumbersome. The developed method here retains simplicity comparing with the algorithm proposed by [11] and overcomes its limitation. According to section two, the output variance in (6) is minimized when it equals to (9)

$$\left(\frac{F_0 + z^{-d}BP_{n_P}}{T}\right)C = \frac{R}{B_-^+} \simeq \zeta_{n_\zeta} \quad (11)$$

It is important to recall that B_-^+ is the normalized reciprocal of the unstable part of polynomial B. Because the derivation

of the proposed algorithm is relying on equation (11). this leads to the proposed controller is applicable to non-minimal phase systems. Polynomial ζ_{n_ζ} is an approximation of the long division of equation (11). It is a polynomial with order n_ζ with $\zeta(0) = 1$. The output variance can be computed as the sum of square of coefficients $\zeta_{n_\zeta}(z^{-1})$. Obviously, it is desired to minimize these coefficients, because it results minimum output variance. Specifically, finding the coefficients of P_{n_P} that minimizes equation (6) leads also to minimize the norm $\left\| \zeta_{n_\zeta}(z^{-1}) \right\|_{L2}^2$. Arranging (11) to get another Diophantine equation

$$T\zeta_{n_\zeta} - z^{-d}BP_{n_P}C = F_0C \qquad (12)$$

Equation (12) can be reconfigured using *Sylvester's Theorem* into a set of linear equations and with a unique solution. It is remarkable that once n_P is selected, n_ζ can be selected from $n_\zeta = n_P + n_b + n_C - n_T$. This extra freedom can be exploited to reduce $\left\| \zeta_{n_\zeta}(z^{-1}) \right\|_{L2^2}$. The design problem can be formulated as a problem to choose n_P greater than its solution so that P_{n_P} minimizes (6). A proposed algorithm for synthesizing the controller is introduced.

Let P_0 and ζ_0 are the minimal order solution to equation (12). Therefore, the general solution using Diophantine equations is defined by

$$P_{n_P} = P_0 + T\phi \qquad (13)$$
$$\zeta_{n_\zeta} = \zeta_0 + B'C\phi \qquad (14)$$

Polynomial B' represents $z^{-d}B$ and polynomial ϕ is a polynomial with arbitrary coefficients. According to equations (11), choosing ϕ that minimizes the square of L_2 norm (14) leads also to minimize square of L_2 norm of (13).

Let ϕ^0 denotes the solution that minimizes the norm in L_2 of (14) and $\zeta_{n_\zeta}^0(z^{-1})$ is the polynomial $\zeta_{n_\zeta}(z^{-1})$ after substituting ϕ^0 in (14). Therefore, $\zeta_{n_\zeta}^0(z^{-1})$ is considered the minimal square norm in L_2 space and the following bound is valid.

$$\left\| \zeta_{n_\zeta}^0 \right\|_{L_2}^2 \leq \left\| \zeta_{n_\zeta}^1 \right\|_{L_2}^2 \qquad (15)$$

Equation (15) is interpreted as the norm of any suboptimal solution is greater or equal to the norm of the optimal solution for any order of n_ζ. The importance of equation (15) will be clarified in the next section.

A. Convergence of Algorithm for Variance Reduction

Defining the fraction R/B_-^+ from equation (11) as a summation series denoted

$$J_{min} = \left\| \frac{R}{B_-^+} \right\|_{L_2}^2 = \left\| \sum_{n=1}^{\infty} \alpha_i \right\|_{L_2}^2 \qquad (16)$$

where $\alpha \in \mathbf{R}$ and $\alpha_0 = 1$ therefore, one concludes that the minimum variance that could be is always one.

The objective is to develop polynomial $\zeta_{n_\zeta}(z^{-1})$, which is defined in (14) that has its square norm L_2 approaches the

optimal norm, $\zeta_{n_\zeta}^0$ of equation (15). Adopting an approach similar to [12], we define a suboptimal solution of the problem.

$$\frac{\zeta_0(z^{-1})}{B'(z^{-1})C(z^{-1})} = \sum_{i=0}^{\infty} w_i z^{-i} \qquad (17)$$

Denote $B'(z^{-1})C(z^{-1}) = D(z^{-1})$. The infinity series in equation (17) can be represented by a finite and infinity series respectively.

$$\zeta_0 = DW_{n-1} + z^{-n}V_n \qquad (18)$$

where

$$\frac{V_n}{D} = \sum_{i=n}^{\infty} w_i z^{n-i} \text{ and } W_{n-1} = \sum_{i=0}^{n-1} w_i z^{-i}$$

From equation (18), we can relate the term $\left| V_n(z^{-1})/D(z^{-1}) \right|$ as a function of $\zeta_0(z^{-1})$ and $W_{n-1}(z^{-1})$. Substituting equations (18) into (14)) with a suboptimal solution $P_{n_P}^1(z^{-1}) = -W_{n_P}$ yields

$$\zeta_{n_\zeta^1}^1(z^{-1}) = z^{-(n_P+1)}V_{n_P+1} \qquad (19)$$

$$= z^{-(n_P+1)}D(z^{-1}) \sum_{i=n_P+1}^{\infty} w_i z^{-(i-n_P-1)}$$

The term, n_ζ^1, is the order of the polynomial $\zeta_{n_\zeta^1}^1$. It is important to highlight that ζ_0 is a stable polynomial. From equation (18), it is clear that the norm L_2 of $\zeta_{n_\zeta^1}^1$ is a function in n_P. Defining

$$\gamma = max\{|z| : z^{n_D}D(z^{-1}) = 0\} \qquad (20)$$

Let $n = n_P + 1$ and evaluating the term $\left| V_n(z^{-1})/D(z^{-1}) \right|$ for $|z| = 1$ using *Cauchy integral* of radius γ' centered on the origin, satisfying $1 > \gamma' > \gamma$.

$$\left| \frac{V_n(z^{-1})}{D(z^{-1})} \right| = \left| \frac{1}{2\pi j} \oint_{|z|=\gamma'} \frac{\sum_{i=n}^{\infty} w_i z^{-(i-n)}}{z-1} dz \right| \qquad (21)$$

Recalling that $\left| \oint f(z)dz \right| \leq max_{z \in \mathbf{C}}(|f(z)|)L_C$, where L_C is the circumference of the unity circle and using (18)

$$\left| \frac{V_n(z^{-1})}{D(z^{-1})} \right| \leq \frac{(\gamma')^{n+1}}{\gamma'-1} \, max_{|z|=\gamma'} \left\{ \frac{\zeta_0(z^{-1})}{D(z^{-1})} \right\} \to 0 \quad n \to \infty \qquad (22)$$

Because the term $W_{n-1}(z^{-1})$ is a finite summation and its Cauchy integral equals to zero. Taking the norm in L_2 of equation (22) and substituting it in equation (19)

$$\left\| \zeta_{n_\zeta^1}^1(z_0^{-1}) \right\|_{L_2}^2 \leq \frac{(\gamma')^{2(n+1)}}{(\gamma'-1)^2} \, max_{|z|=\gamma'} \left| \frac{\zeta_0(z^{-1})}{D(z^{-1})} \right|^2 D^2(z^{-1}) \to 0 \qquad (23)$$

$$n \to \infty$$

Equation (23) proves that the norm of the sub-optimal solution tends to zero as the order of n tends to infinity. Recalling the lower bound in equation (15), the norm of the optimal solution tends to zero also as n tends to infinity. It can be interpreted alternatively if the suboptimal solution tends to zero for the same n_P that goes to infinity. Consequently, the

optimal solution tends to zero also for the same n_P. A suitable algorithm that is used to design the higher order controller is presented.

a) Algorithm:

Step 1 Select T to have a desired servo behavior and solve (1) to obtain the minimal order solution F_0 and G_0.

Step 2 Solve for the minimal P_0 and ζ_0 using equation (12)

Step 3 Select the desired n_ϕ that will give $n_\phi = n_P - n_T$

Step 4 Find ϕ that minimizes $\|\zeta_0 + B'C\phi\|_{L_2}^2$

Step 5 obtain P_{n_P} using equation (13)

Step 6 Form the high order controller using equation (5)

IV. Controller Testing on a DC Motor

A permanent magnet DC servo motor with its driver is utilized in this experiment. The input of the motor is an analog voltage in the range -10 V to +10 V supplied to the driver. An encoder, attached to the motor, transmits the speed pulses to a *Frequency to Voltage Converter* circuit, in which the latter produces an analog voltage in the range from 0 to 10 V and represents the feedback signal. An illustration of the experiment setting is introduced in figure 3 and the real setting of the experiment is in figure 2.
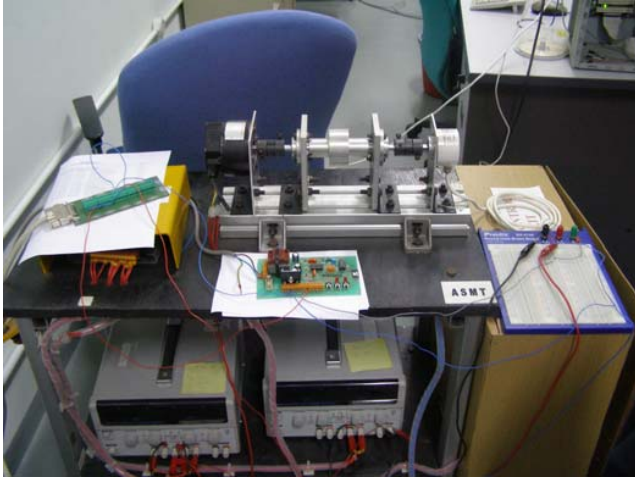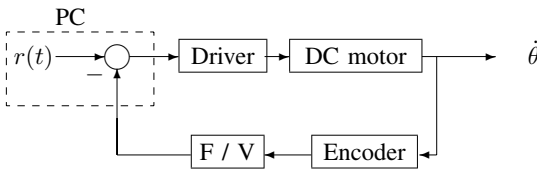

Fig. 2: DC motor

Fig. 3: Diagram of experimental setting

A. ARMAX system Identification

The system identification was achieved using a recursive algorithm in Matlab software. The experiments was carried using *XPC Target Box* device that was programmed with a sampling rate 60 m sec after careful choice to prevent the

poles to shift to edge of the unity circle in the z-plan.

A parametric model identification type ARMAX model is according to figure 1. Using AIC, MDL, and Best fit criteria, the order of the system was estimated. The selection of the order n_C was decided from the best fit analysis to the validating data set. The best fit was the ARMAX model that was supported after conducting a residual analysis for the cross and auto-correlation. The identified model is described using the shift operator q, that is $q^{-1}f(t) = f(t-1)$.

$$A(q)y(t) = B(q)u(t) + C(q)e(t) \quad (24)$$
$$A(q^{-1}) = 1 - 0.8251(\pm 0.01178)q^{-1}$$
$$B(q^{-1}) = 0.243(\pm 0.01587)q^{-1}$$
$$C(q^{-1}) = 1 - 0.6824(\pm 0.04435)q^{-1}$$

The choice of our identified ARMAX model that will be used in the experiment will be with the absolute coefficients (i.e $A(q^{-1}) = 1 - 0.8251q^{-1}$, $B(q^{-1}) = 0.243q^{-1}$, $C(q^{-1}) = 1 - 0.6824q^{-1}$). The candidates controller planned to be used is

$$G_c(z^{-1}) = \frac{z^{-1} - 0.8z^{-2}}{1 - 0.4z^{-1} - 0.92z^{-2} - 0.48z^{-3}} \quad (25)$$

The choice of this controller lies on an apparent step response with zero steady state error. The transfer function of the controller in equation (25) represents the minimal order solution of $F(q^{-1})$. The poles of this controller are 1.0, -0.8, -0.6 with a $\|F_0\|_{L_2}^2 = 2.2368$

V. Results and Discussion

The implementation of the controller of (25) with the identified ARMAX model, leads to set two experiments. The first, is to show the performance for tracking step reference signal for the candidate minimal order controller, extended order controller, and extended order controller with integrator form controller. We apply a reference signal of 2.5V to the driver of the motor. The results of tracking are illustrated in figures (4a, 4b, 4c) In this experiment, the minimal order solution tracks the reference, while the extended controller achieves the tracking with a steady state error equals to 0.8V. The extended controller with integrator resets the steady state error back to zero. The compromise of the extended controller with integrator instead of extended controller alone is simply to achieve a reference tracking without a significant loss in the output variance reduction. figures (5a, 5b, 5c) show the zero-response with minimal order controller, extended controller without integral, and extended controller with a forced integral respectively. The calculated variance of the three cases are given in table I. The minimal order solution is 1.5402 of variance. Noticing that the extended order achieves the minimal variance (i.e. unity variance). The extended controller with integral raises-up the variance up to 1.2041.
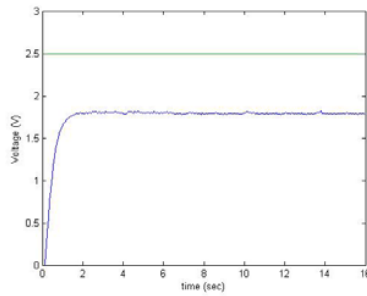
VI. Conclusion

A fast calculated controller for output reduce variance of minimum and non-minimum phase system was introduced
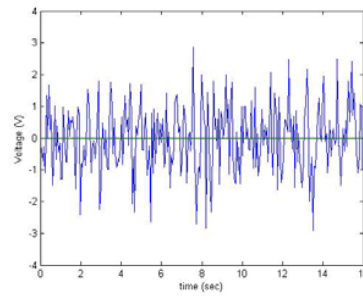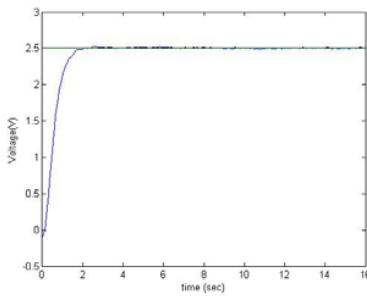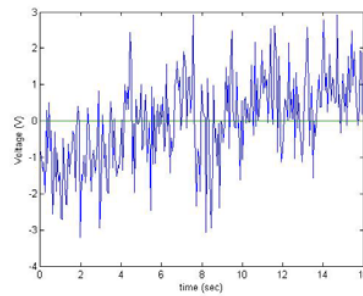
(a) Minimal orer

(b) Extended order

(c) Extended order with forced integral
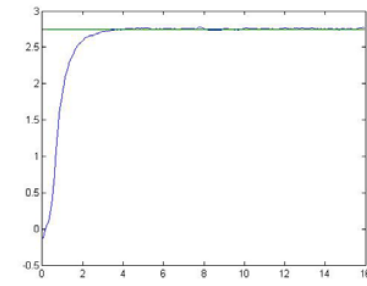
Fig. 4: Step response with minimal and extended form of low order controller
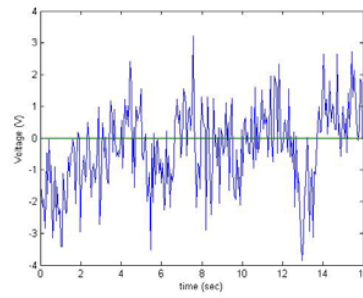
(a) Minimal order

(b) Extended order

(c) Extended order with forced integral

Fig. 5: Zero response with minimal and extended form of controller

TABLE I: Measured variance

controller's type	$F(z^{-1})$	$var(y)$
Minimal order $F_0(z^{-1})$	$1 + 0.4z^{-1} - 0.92z^{-2}$ $-0.48z^{-3}$	1.7124
Extended order $F_{n_F}(z^{-1})$	$1 - 0.425z^{-1} - z^{-2}$ $+0.084z^{-3} + 0.396z^{-4}$	1.0818
Extended order with integral $F_{n_F} = F_0 + (1 - z^{-1}) BP_{n_P}$	$1 - 0.425z^{-1}$ $-0.182z^{-2} + 0.1716z^{-3}$ $-0.1686z^{-4} - 0.396z^{-5}$	1.43

features robustness against the identification parameters of the system. The calculated over-parameterized controller, based on the new algorithm, works on the result of identification and achieves minimum variance. This over-parameterized controller was, initially, based on a minimal-order controller, which achieves certain level of output variance in the experiment. in order to ensure reference tracking for the over-parameterized controller, an integrator was incorporated in the controller. The effect of incorporating the integrator raises the output variance not significantly comparing with the minimum variance, while realizing reference tracking.

in this paper. It was proven in the paper that the introduced algorithm converges to minimum variance with over-parameterizing the controller's coefficients.

The paper introduces the application of the new controller on a real physical system. It was shown that the new controller

REFERENCES

[1] H. Kwakernaak, "Minimax frequency domain performance and robustness optimization of linear feedback systems," *IEEE Transactions on Automatic Control*, vol. AC-30, no. 10, pp. 994–1004, 1985.

[2] I. D. Landau and G. Zito, *Digital Control Systems Design,Identification, and Implementation*, ser. Communications and Control Engineering. Springer London, 2006, ch. Design of Digital Controllers in the Presence of Random Disturbances, pp. 169–199.

[3] H.-W. Gao, G.-Y. Tang, and C. Li, "Optimal disturbance rejection with zero steady-state error for time delay systems," in *Proceedings of the 6th World Congress on Intelligent Control and Automation*. Dalian, China: IEEE, June 2006, pp. 511–515.

[4] K. J. Astrom and B. Wittenmark, *Computer-controlled systems: Theory and Design*, 3rd ed. Prentice Hall Information and Sstem Sciences Series, November 1996.

[5] D. W. Clarke and P. J. Gawthrop, "A self-tuning controller," *IEEE Procedinng*, vol. 122, 1975.

[6] ——, "A self-tuning controller," *IEEE Procedinng*, vol. 126, 1979.

[7] S. Ertunc, B. Akay, H. Boyacioglu, and H. Hapoglu, "Self-tuning control of dissolved oxygen concentration in a batch bioreactor," *Food and Bioproducts Processing*, vol. 87, pp. 46–55, 2009.

[8] M. Sternad and A. Ahlen, *Polynomial methods in optimal Control and filtering*, ser. IEE Control Engineering. Peter Peregrinus Ltd., United Kingdom, 1993, no. 49, ch. LQ control design and self-tuning.

[9] A. Krishnan and M. Das, "Minimum variance & lqg control for active noise cancellation - a comparison," in *Proc. 43rd IEEE Midwest Symp. on Circuits and Systems*, 2000, pp. 1358–1361.

[10] A. J. Krener, "A brief tutorial on linear and nonlinear control theory," Department of Mathematics, University of California, Tech. Rep., n.d.

[11] M. E. Halpern, "Modified pole-assignment controller for plant models with exact or near pole-zero cancellation," *IEE Proceeding*, vol. 135, no. 3, pp. 189–195, 1988.

[12] R. Davies and M. B. Zarrop, "On reduced variance overparameterized pole assignment control," *International Journal of Control*, vol. 69, no. 1, pp. 131–144, 1998.