

A Note on the Numerical Solution of Singular Integral Equations of Cauchy type

M. Abdulkawi, Z. K. Eshkuvatov, N. M. A. Nik Long

Abstract—This manuscript presents a method for the numerical solution of the Cauchy type singular integral equations of the first kind, over a finite segment which is bounded at the end points of the finite segment. The Chebyshev polynomials of the second kind with the corresponding weight function have been used to approximate the density function. The force function is approximated by using the Chebyshev polynomials of the first kind. It is shown that the numerical solution of characteristic singular integral equation is identical with the exact solution, when the force function is a cubic function. Moreover, it also shown that this numerical method gives exact solution for other singular integral equations with degenerate kernels.

Keywords—Singular integral equations, Cauchy kernel, Chebyshev polynomials, Interpolation.

I. INTRODUCTION

Let us consider the singular integral equation of the first kind on the finite segment

$$\int_{-1}^1 \frac{\varphi(t)}{t-x} dt + \int_{-1}^1 K(x,t) \varphi(t) dt = f(x), \quad -1 < x < 1, \quad (1)$$

where K and f are assumed to be real-valued functions belong to the class of Hölder continues functions on the sets $[-1, 1] \times [-1, 1]$ and $[-1, 1]$, respectively. φ is unknown function to be determined and the singular integral is interpreted as Cauchy principle value. Equation (1) is called Cauchy type singular integral equation of first kind which has many applications [2], [3]. The theory of this equation is well known and it is presented in the monographs [5], [7]. The characteristic singular integral equation of equation (1) is of the form

$$\int_{-1}^1 \frac{\varphi(t)}{t-x} dt = f(x), \quad -1 < x < 1. \quad (2)$$

The solution of equation (2) which is bounded at the end points $x = \pm 1$, is given by the following formula [1]

$$\varphi(x) = -\frac{1}{\pi^2} \sqrt{1-x^2} \int_{-1}^1 \frac{f(t)}{\sqrt{1-t^2}(t-x)} dt, \quad (3)$$

provided that

$$\int_{-1}^1 \frac{f(x)}{\sqrt{1-x^2}} dx = 0 \quad (4)$$

By solving equation (1) with respect to its characteristic part, we find that it is equivalent to the Fredholm equation type of

the second kind [2]

$$\left. \begin{aligned} \varphi(t) + \int_{-1}^1 K_0(t, \tau) \varphi(\tau) d\tau &= F(t), \\ K_0(t, \tau) &= -\frac{1}{\pi^2} \sqrt{1-t^2} \int_{-1}^1 \frac{K(x, \tau)}{\sqrt{1-x^2}(t-x)} dx, \\ F(t) &= -\frac{1}{\pi^2} \sqrt{1-t^2} \int_{-1}^1 \frac{f(x)}{\sqrt{1-x^2}(t-x)} dx. \end{aligned} \right\} \quad (5)$$

which one can apply the Fredholm's theorems to obtain the solution of this equation.

In addition, the condition

$$\int_{-1}^1 \frac{1}{\sqrt{1-x^2}} \left[f(x) - \int_{-1}^1 K(x, t) \varphi(t) dt \right] dx = 0 \quad (6)$$

must be implemented to obtain the bounded solution.

In this paper, we present a numerical method to solve equation (1) for which the solution is bounded at the end points $x \pm 1$. And we will show that the present numerical method gives exact solution for equation (2) when the force function f is a cubic function. Moreover, it will be shown that the numerical solution of equation (1) with some degenerate kernels is identical with the exact one.

II. THE APPROXIMATE TECHNIQUE

In this section, the numerical solution of equation (1) will be derived.

Approximating the known function $f(x)$ by using the first kind Chebyshev polynomial $f_n(x)$ of degree n

$$f_n(x) = \sum_{k=0}^n f_k T_k(x) \quad (7)$$

where

$$f_k = \frac{2}{\pi} \int_{-1}^1 \frac{1}{\sqrt{1-t^2}} f(t) T_k(t) dt, \quad (8)$$

and the Chebyshev polynomial of first kind T_j can be defined by the recurrence relation [6]

$$\left. \begin{aligned} T_0(x) &= 1, \quad T_1(x) = x, \\ T_n(x) &= 2x T_{n-1}(x) - T_{n-2}(x), \quad n \geq 2 \end{aligned} \right\} \quad (9)$$

The dash in \sum' denotes that the first term in the sum is to be halved.

Now, we will approximate the unknown function φ by the approximate function φ_n which is defined as follows

$$\varphi_n(x) = \sqrt{1-x^2} \sum_{j=0}^{n-1} \alpha_j U_j(x) \quad (10)$$

Corresponding author: Mohammad Abdulkawi, Department of Mathematics, Faculty of Science, University Putra Malaysia, 43400 Serdang, Selangor, Malaysia. Email: akawi@math.upm.edu.my, Phone: +6-012-3166114.

where U_j is the Chebyshev polynomial of second kind which is defined by the following recurrence relation

$$\left. \begin{aligned} U_0(x) &= 1, \quad U_1(x) = 2x, \\ U_n(x) &= 2x U_{n-1}(x) - U_{n-2}(x), \quad n \geq 2. \end{aligned} \right\} \quad (11)$$

Substituting (10) into (1) and using the following well-known relation

$$\int_{-1}^1 \frac{\sqrt{1-t^2} U_j(t)}{t-x} dt = -\pi T_{j+1}(x) \quad (12)$$

we obtain

$$-\pi \sum_{j=0}^{n-1} \alpha_j T_{j+1}(x) + \sum_{j=0}^{n-1} \alpha_j \eta_j(x) = f(x) \quad (13)$$

where

$$\eta_j(x) = \int_{-1}^1 \sqrt{1-t^2} K(x,t) U_j(t) dt \quad (14)$$

Approximating $\eta_j(x)$ by the orthogonal system $\{T_0, \dots, T_n\}$ as

$$\eta_j(x) = \sum_{k=0}^n \lambda_{j,k} T_k(x) \quad (15)$$

where

$$\lambda_{j,k} = \frac{2}{\pi} \int_{-1}^1 \frac{1}{\sqrt{1-t^2}} \eta_j(x) T_k(x) dx \quad (16)$$

Due to (15) we have

$$\sum_{j=0}^{n-1} \alpha_j \eta_j(x) = \sum_{k=0}^n \sum_{j=0}^{n-1} \alpha_j \lambda_{j,k}(x) T_k(x) \quad (17)$$

Now, the approximate solution of equation (1) is defined as the solution of equation

$$\int_{-1}^1 \frac{\varphi_n(t)}{t-x} dt + \int_{-1}^1 K(x,t) \varphi_n(t) dt = f_n(x) + \chi \quad (18)$$

where the parameter χ is chosen in such a way that the following condition is satisfied,

$$\int_{-1}^1 \frac{1}{\sqrt{1-t^2}} \left[F_n(t) - \int_{-1}^1 K(t,x) \varphi_n(x) dx \right] dt = 0, \quad (19)$$

where

$$F_n(t) = f_n(t) + \chi$$

Due to (7), (10) and (17) equation (18) becomes

$$-\pi \sum_{j=0}^{n-1} \alpha_j T_{j+1}(x) + \sum_{k=0}^n \sum_{j=0}^{n-1} \alpha_j \lambda_{j,k} T_k(x) = \sum_{k=0}^n f_k T_k(x) + \chi$$

Then the coefficients $\{\alpha_j\}_0^{n-1}$ and parameter χ are obtained by solving the following system of linear equations

$$\begin{aligned} 0 + \frac{1}{2} \sum_{j=0}^{n-1} \alpha_j \lambda_{j,0} &= \frac{1}{2} f_0 + \chi, \\ \alpha_0 - \frac{1}{\pi} \sum_{j=0}^{n-1} \alpha_j \lambda_{j,1} &= -\frac{1}{\pi} f_1, \\ \alpha_1 - \frac{1}{\pi} \sum_{j=0}^{n-1} \alpha_j \lambda_{j,2} &= -\frac{1}{\pi} f_2, \\ \dots &\dots \dots \\ \vdots &\vdots \vdots \\ \dots &\dots \dots \\ \alpha_{n-1} - \frac{1}{\pi} \sum_{j=0}^{n-1} \alpha_j \lambda_{j,n} &= -\frac{1}{\pi} f_n. \end{aligned} \quad (20)$$

The coefficients $\{f_k\}_1^n$ and $\{\lambda_{j,k}\}_0^{n-1,n}$ are given by (8) and (16), respectively, which can be computed analytically, or numerically using the following Gauss-Chebyshev formulas [4]

$$\left. \begin{aligned} f_j &\approx \frac{2}{n+1} \sum_{i=1}^{n+1} f(x_i) T_j(x_i), \\ x_i &= \cos \left[\frac{(2i-1)\pi}{2(n+1)} \right] \end{aligned} \right\}$$

and

$$\left. \begin{aligned} \lambda_{j,k} &\approx \frac{2}{n+2} \frac{\pi}{n+1} \sum_{i=1}^{n+1} \sum_{\ell=1}^{n+1} (1-t_\ell^2) K_{i,\ell} U_j(t_\ell) T_k(x_i), \\ K_{i,\ell} &= K(x_i, t_\ell), \quad t_\ell = \cos \left(\frac{\pi \ell}{n+2} \right) \end{aligned} \right\}$$

Note that, if system (20) is devoid of parameter χ then the number of equations in it exceeds the number of unknown variables so, in this case, the system has no solution. The parameter χ is considered as the variable to make system (20) well-posed and, therefore, will be called a regularizing variable.

The value of parameter χ can be also fined by solving the equation that is obtained by applying relations (7), (10) and (15) into condition (19) which will be equivalent to the first equation in system (20).

Lemma 1: If $f(x)$ in integral equation (2) is a cubic function, then the numerical solution (10) of equation (2) is exact.

Proof:

Let us consider the characteristic singular integral equation

$$\int_{-1}^1 \frac{\varphi(t)}{t-x} dt = f(x), \quad -1 < x < 1. \quad (21)$$

Let $f(x)$ be a cubic function, i.e.,

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3, \quad -1 < x < 1. \quad (22)$$

Substituting (22) into (8) yields

$$f_j = \frac{2}{\pi} \int_{-1}^1 \frac{a_0 + a_1 t + a_2 t^2 + a_3 t^3}{\sqrt{1-t^2}} T_j(t) dt \quad (23)$$

Using the Chebyshev recurrence relation of first kind (9) we obtain

$$\left. \begin{aligned} t^3 &= \frac{1}{4} [T_3(t) + 3T_1(t)] \\ t^2 &= \frac{1}{2} [T_2(t) + T_0(t)] \\ t &= T_1(t) \end{aligned} \right\} \quad (24)$$

Substituting (24) into (23) and using the following orthogonal property

$$\int_{-1}^1 \frac{1}{\sqrt{1-t^2}} T_i(t) T_j(t) dt = \begin{cases} 0, & i \neq j, \\ \pi, & i = j = 0, \\ \frac{\pi}{2}, & i = j \neq 0. \end{cases} \quad (25)$$

we find

$$\left\{ f_1 = a_1 + \frac{3a_3}{4} \quad f_2 = \frac{a_2}{2} \quad f_3 = \frac{a_3}{4} \right\} \quad (26)$$

Due to the system (20), when $K(x, t) = 0$, yields

$$\alpha_j = -\frac{1}{\pi} f_{j+1}, \quad j = 0, \dots, n-1. \quad (27)$$

Letting $n = 3$ in approximate solution (10) we get

$$\varphi_n(x) = -\frac{1}{\pi} \sqrt{1-x^2} \sum_{j=0}^2 f_{j+1} U_j(x) \quad (28)$$

Using (26-27) into (28) and taking into account (11), we obtain the numerical solution of equation (21)

$$\varphi_n(x) = -\frac{1}{\pi} \sqrt{1-x^2} \left[a_1 + \frac{a_3}{2} + a_2 x + a_3 x^2 \right] \quad (29)$$

Now, substituting (22) into the exact solution given by (3), yields

$$\varphi(x) = -\frac{1}{\pi^2} \sqrt{1-x^2} \int_{-1}^1 \frac{a_0 + a_1 t + a_2 t^2 + a_3 t^3}{\sqrt{1-t^2}(t-x)} dt \quad (30)$$

It is easy to see that

$$\left. \begin{aligned} \int_{-1}^1 \frac{1}{\sqrt{1-t^2}(t-x)} dt &= 0, \\ \int_{-1}^1 \frac{t}{\sqrt{1-t^2}(t-x)} dt &= \pi, \\ \int_{-1}^1 \frac{t^2}{\sqrt{1-t^2}(t-x)} dt &= \pi x \\ \int_{-1}^1 \frac{t^3}{\sqrt{1-t^2}(t-x)} dt &= \pi(x^2 + 0.5). \end{aligned} \right\} \quad (31)$$

Taking into account (31), the exact solution of equation (21) is

$$\varphi(x) = -\frac{1}{\pi} \sqrt{1-x^2} [a_1 + a_2 x + a_3(x^2 + 0.5)] \quad (32)$$

By comparing the approximate solution (29) with the exact solution (32) we complete the proof.

III. PARTICULAR RESULT

Let us consider the integral equation of the form (1) with degenerate kernel $K(x, t) = x^5 t^5$ and polynomial function $f(x) = x^5 + x^3 + x$ i.e.,

$$\int_{-1}^1 \frac{\varphi(t)}{t-x} dt + \int_{-1}^1 (x^5 t^5) \varphi(t) dt = x^5 + x^3 + x \quad (33)$$

Due to (8) we have

$$f_j = \frac{2}{\pi} \int_{-1}^1 \frac{1}{\sqrt{1-t^2}} (t^5 + t^3 + t) T_j(t) dt \quad (34)$$

It is easy to see that

$$t^5 = \frac{1}{16} [T_5(t) + 5T_3(t) + 10T_1(t)] \quad (35)$$

Using (24) and (35) into (34) we obtain

$$f_j = \frac{1}{8\pi} \int_{-1}^1 \frac{[T_5(t) + 9T_3(t) + 38T_1(t)] T_j(t)}{\sqrt{1-t^2}} dt \quad (36)$$

Applying the orthogonal property (25) into (36) we find

$$\left\{ f_1 = \frac{19}{8}, \quad f_2 = 0, \quad f_3 = \frac{9}{16}, \quad f_4 = 0, \quad f_5 = \frac{1}{16} \right\} \quad (37)$$

From (16) with knowing that $K(x, t) = x^5 t^5$ we have

$$\lambda_{j,k} = \frac{2}{\pi} \left[\int_{-1}^1 \frac{x^5 T_k(x)}{\sqrt{1-x^2}} dx \right] \left[\int_{-1}^1 \sqrt{1-t^2} t^5 U_j(t) dt \right] \quad (38)$$

Using recurrence relation (11), one can find that

$$t^5 = \frac{1}{32} [U_5(t) + 4U_3(t) + 5U_1(t)] \quad (39)$$

Due to (39) and applying the orthogonal property

$$\int_{-1}^1 \sqrt{1-t^2} U_i(t) U_j(t) dt = \begin{cases} 0, & i \neq j, \\ \frac{\pi}{2}, & i = j. \end{cases} \quad (40)$$

we obtain

$$\int_{-1}^1 \sqrt{1-t^2} t^5 U_j(t) dt = \begin{cases} 0, & j = 0, \\ \frac{5\pi}{64}, & j = 1, \\ 0, & j = 2, \\ \frac{\pi}{16}, & j = 3, \\ 0, & j = 4. \end{cases} \quad (41)$$

Similarly with helping of (9) and (25) yields

$$\int_{-1}^1 \frac{x^5}{\sqrt{1-x^2}} T_k(x) dx = \begin{cases} \frac{5\pi}{16}, & k = 1, \\ 0, & k = 2, \\ \frac{5\pi}{32}, & k = 3, \\ 0, & k = 4, \\ \frac{\pi}{32}, & k = 5. \end{cases} \quad (42)$$

From (41), (42) and (38) we obtain

$$\left. \begin{aligned} \lambda_{0,k} &= 0, \quad k = 1, \dots, 5, \quad \lambda_{1,1} = \frac{50\pi}{(32)^2}, \\ \lambda_{1,2} &= 0, \quad \lambda_{1,3} = \frac{25\pi}{(32)^2}, \quad \lambda_{1,4} = 0, \\ \lambda_{1,5} &= \frac{5\pi}{(32)^2}, \quad \lambda_{2,k} = 0, \quad k = 1, \dots, 5, \\ \lambda_{3,1} &= \frac{5\pi}{(2)^7}, \quad \lambda_{3,2} = 0, \quad \lambda_{3,3} = \frac{5\pi}{(16)^2}, \\ \lambda_{3,4} &= 0, \quad \lambda_{3,5} = \frac{\pi}{(16)^2}, \quad \lambda_{4,k} = 0, \quad k = 1, \dots, 5. \end{aligned} \right\} \quad (43)$$

Due to system (20), when $n = 5$, and taking into accounts (37) and (43), the coefficients α_j , $j = 0, \dots, 4$, will be obtained by solving the following system of linear equations

$$\begin{aligned} \alpha_0 - \frac{50}{(32)^2}\alpha_1 + 0 - \frac{10}{(16)^2}\alpha_3 + 0 &= -\frac{19}{8\pi} \\ 0 + \alpha_1 + 0 + 0 + 0 &= 0 \\ 0 - \frac{25}{(32)^2}\alpha_1 + \alpha_2 - \frac{5}{(16)^2}\alpha_3 + 0 &= -\frac{9}{16\pi} \\ 0 + 0 + 0 + \alpha_3 + 0 &= 0 \\ 0 - \frac{5}{(32)^2}\alpha_1 + 0 - \frac{1}{(16)^2}\alpha_3 + \alpha_4 &= -\frac{1}{16\pi} \end{aligned}$$

which are

$$\left\{ \alpha_0 = \frac{-19}{8\pi}, \alpha_1 = \alpha_3 = 0, \alpha_2 = \frac{-9}{16\pi}, \alpha_4 = \frac{-1}{16\pi} \right\} \quad (44)$$

Substituting the values of the coefficients $\{\alpha_j\}_0^4$ into approximate solution (10) yields

$$\varphi_n(x) = \sqrt{1-x^2} \sum_{j=0}^4 \alpha_j U_j(x) \quad (45)$$

Using (11) into (45) we obtain the numerical solution of equation (33)

$$\varphi_n(x) = -\frac{\sqrt{1-x^2}}{\pi} \left[x^4 + \frac{3}{2}x^2 + \frac{15}{8} \right] \quad (46)$$

which is exact.

IV. CONCLUSION

The known force function is approximated by using the Chebyshev polynomial of first kind while the unknown density function which is bounded at the end points of the finite segment $[-1, 1]$ is approximated using the Chebyshev polynomial of the second kind with corresponding weight function. Existence of the present numerical method is shown by Lemma 1 for characteristic singular integral equation when the force function is a cubic function. Particular result also shows that this numerical method gives the exact solution for other singular integral equations.

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