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A Non-Standard Finite Difference Scheme for the Solution of Laplace Equation with Dirichlet Boundary Conditions

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Abstract**—**In this paper, we present a fast and accurate numerical scheme for the solution of a Laplace equation with Dirichlet boundary conditions. The non-standard finite difference scheme (NSFD) is applied to construct the numerical solutions of a Laplace equation with two different Dirichlet boundary conditions. The solutions obtained using NSFD are compared with the solutions obtained using the standard finite difference scheme (SFD). The NSFD scheme is demonstrated to be reliable and efficient.

Keywords**—**Standard finite difference schemes, non–standard schemes, Laplace equation, Dirichlet boundary conditions.

I. Introduction

LAPLACE equations have been used for many years in fluid mechanics, heat and mass transfer theory, elasticity, and electrostatics.

The Dirichlet boundary conditions for Laplace equation are defined as finding a solution of *u* on a domain *D*, and this domain has a boundary conditions defined as a given functions. As an interpretation and application of this problem, the heat equation with fix temperature on the boundaries of the domain and wait until the interior temperature does not change. The solution to the corresponding Dirichlet problem is the temperature distribution [1]. Recently, several methods are proposed to solve partial differential equations [2]-[4], the variational iteration method (VIM) has been applied for exact solutions of Laplace equations [5], Sadighi and Ganji [4] obtained the exact solution of Laplace equations with Dirichlet boundary conditions using the homotopy-perturbation method (HPM) and Adomian decomposition method (ADM), Mustafa [6] gives the numerical solution of the Laplace equations with Dirichlet boundary conditions using the homotopy analysis method (HAM).

One of the shortcomings of the SFD method is that the qualitative properties of the exact solution usually are not transferred to the numerical solution. Furthermore, are easily affected the stability properties of the standard approach. Also, in practice, using the standard method the limit of the step-size is not reached. What we obtain is the numerical solution for one or several values of step-size [7].

NSFD schemes have emerged as an alternative method for solving a wide range of problems whose mathematical models involve algebraic, differential and biological models as well as chaotic systems [8]. These techniques have many advantages over classical techniques and provide an efficient numerical solution. In fact,the NSFD method is an extension of the SFD method. Non-standard schemes as introduced by Mickens [9]-[12] are used to help resolve someof the issues related to numerical instabilities. In addition, Moaddy, Momani and Hashim used the NSFD for linear fractional partial differential equations in fluid mechanics [13], Dang and Hoang modeled a continuous-time predator–prey system with general functional response and recruitment for both species into a discrete-time by NSFD [14]. Furthermore, Mickens [9]-[12] introduced certain rules for obtaining the best difference equations.

In this paper we constructed a NSFD scheme to solve the Laplace equations with Dirichlet boundary conditions, the scheme depends on two different denominator functions ϕ_1 and ϕ_2. This technique has several advantages over the standard techniques, mainly in providing an efficient numerical solution with high accuracy. However, the NSFD scheme can be used to solve this problem effectively.

The rest of the paper is organized as follows. In the next section, we present the SFD scheme for solution of the Laplace equations with Dirichlet boundary conditions. Section III briefly describes the NSFD scheme to solve the Laplace equations with Dirichlet boundary conditions. In Section IV, we apply the NSFD scheme to solve Laplace equations with Dirichlet boundary conditions. Two different Dirichlet boundary conditions of the Laplace equation are considered as test examples, and we discuss numerical approximations to the solutions. In the last section, we summarize the conclusions.

II. SFD Scheme for Solution of the Laplace Equation

The two-dimensional Laplace equation has the following form:

$$\frac{\partial^2 u}{\partial^2 x}+\frac{\partial^2 u}{\partial^2 y} = 0 \qquad (1)$$

We present SFD scheme for (1). The discrete model is constructed using a central difference scheme for the second derivative.

$$\frac{\partial^2 u}{\partial^2 y} = \frac{u_{i,j+1}-2u_{i,j}+u_{i,j+1}}{(\Delta x)^2} \qquad (2)$$

$$\frac{\partial^2 u}{\partial^2 y} = \frac{u_{i,j+1}-2u_{i,j}+u_{i,j+1}}{(\Delta y)^2} \qquad (3)$$

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Substitution of these directives of (1) gives:

$$\frac{u_{i+1,j}-2u_{i.j}+u_{i-1,j}}{(\Delta x)^2}+\frac{u_{i+1,j}-2u_{i.j}+u_{i-1,j}}{(\Delta y)^2}=0 \quad (4)$$

where,

$$u_{i,j} = \frac{1}{2}\frac{\phi_1(\Delta x)(u_{i,j+1}+u_{i,j-1})+\phi_2(\Delta y)(u_{i+1,j}+u_{i-1,j})}{\phi_1(\Delta x)+\phi_2(\Delta y)} \quad (5)$$

$$\Delta x = \Delta y = h \quad (6)$$

$$x_i = ih, y_j = jh \quad (7)$$

from (5), (4) yields:

$$u_{i+1,j} - 4u_{i,j} + u_{i-1,j} + u_{i,j+1} + u_{i.j-1} = 0 \quad (8)$$

III. NSFD Scheme for Solution of the Laplace Equation

We will use (4) to simulate the solution of the Laplace equation by NSFD approximation. Based on the previous works and applying the NSFD scheme in [9] the following model is selected for Laplace equation:

$$\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{\phi_1(\Delta x)}+\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{\phi_2(\Delta y)} = 0 \quad (9)$$

where the denominator functions ϕ_1 and ϕ_2 satisfy the following condition:

$$\phi_m(h) = h^2 + o(h^4), m = 1,2 \quad (10)$$

Solving (9) for $u_{i,j}$ yields:

$$u_{i,j} = \frac{1}{2}\frac{\phi_1(\Delta x)(u_{i,j+1}+u_{i,j-1})+\phi_2(\Delta y)(u_{i+1,j}+u_{i-1,j})}{\phi_1(\Delta x)+\phi_2(\Delta y)}. \quad (11)$$

Note that this scheme has the following features:

1) The discrete model is explicit.
2) The denominator functions for the discrete second-derivative have non-standard form.
3) A central difference scheme replaces the second order space derivative.
4) If $\phi_1(\Delta x) = \phi_2(\Delta y)$ then, the exact analytical expression $\phi_m(h)$, for $\phi_m(h)$, $m = 1,2$ is not needed since, with the condition of (9), the denominator functions drop out of the calculation [9].
5) For both linear and nonlinear terms involving the dependent variable may require "nonlocal" discretizations; as in [12] for example.

$$u = 2u - u \to 2u_k - u_{k+1} \quad (12)$$

$$u^2 = u_k u_{k+1} \quad (13)$$

IV. Applications

We will apply the NSFD to two physical problems to illustrate the strength of the method and to establish numerical solutions for these problems.

Example I. Consider the two-dimensional Laplace equation

$$u_{xx} + u_{yy} = 0, x > 0, y < \pi$$

with boundary conditions:

$$u(0,y) = 0, u(\pi,y) = sinh\,\pi\,cos\,y \quad (14)$$

$$u(x,0) = sinh\,x, u(x,\pi) = -sinh\,x \quad (15)$$

The exact solution of this problem introduced by Mustafa [6] as follows:

$$u(x,y) = sinh\,x\,cos\,y \quad (16)$$

For NSFD scheme we write the denominator functions and ϕ_2 in the form:

$$\phi_1 = sinh^2\left(\frac{\Delta x}{2}\right), \phi_2 = sin^2\left(\frac{\Delta y}{2}\right) \quad (17)$$

Substitution the denominator functions ϕ_1 and ϕ_2 in (11) yields:

$$u_{i,j} = \frac{\sinh^2\left(\frac{\Delta x}{2}\right)(u_{i,j+1}+u_{i,j-1})+\sin^2\left(\frac{\Delta y}{2}\right)(u_{i+1,j}+u_{i-1,j})}{\sinh^2\left(\frac{\Delta x}{2}\right)+\sin^2\left(\frac{\Delta y}{2}\right)}. \quad (18)$$

Example II. We consider to solve the Laplace equations with Dirichlet boundary conditions

$$u_{xx} + u_{yy} = 0, x > 0, y < \pi$$

with boundary conditions:

$$u(0,y) = sin\,y, u(\pi,y) = cosh\,\pi\,sin\,y \quad (19)$$

$$u(x,0) = 0, u(x,\pi) = 0 \quad (20)$$

In this example we follow the same technique of the above example and using the denominator functions ϕ_1 and ϕ_2 in (17). The exact solution $u(x,y)$ of example II is given by Sadighi, and Ganji [1] as follows:

$$u(x,y) = cosh(x)\,sin(y) \quad (21)$$

V. Results and Discussion

In Maple, the number of variable digits controlling the number of significant digits is set to 18, In all the calculations done in this paper.

Tables I and II show the accuracy of the NSFD for solution of the Laplace equation in example I and example II, respectively. we present the absolute errors between NSFD and SFD solutions and the exact solutions at steps size ($h = \pi/5$), comparing NSFD results with the exact solution, we see that the maximum difference between the NSFD solution and the exact solution at time step ($h = \pi/5$) is of the order of magnitude of 10^{-09}.

TABLE I
COMPARISON OF SFD AND NSFD RESULTS FOR EXAMPLE I

$u(x,y)$	$\|Exact - SFD\|$	$\|Exact - NSFD\|$
$u(\pi/5,\pi/5)$	1.111E-02	4.616 E-10
$u(2\pi/5,\pi/5)$	2.449E-02	1.040E-09
$u(3\pi/5,\pi/5)$	4.018E-02	1.165E-09
$u(4\pi/5,\pi/5)$	4.815E-02	1.228E-09
$u(\pi/5,2\pi/5)$	5.849E-03	8.517E-11
$u(2\pi/5,2\pi/5)$	2.754E-02	2.218E-10
$u(3\pi/5,2\pi/5)$	2.046E-02	1.356E-09
$u(4\pi/5,2\pi/5)$	2.357E-02	2.013E-09
$u(\pi/5,3\pi/5)$	5.849E-03	6.851E-10
$u(2\pi/5,3\pi/5)$	1.275E-02	1.378E-09
$u(3\pi/5,3\pi/5)$	2.046E-02	2.643E-09
$u(4\pi/5,3\pi/5)$	2.357E-02	2.896E-09
$u(\pi/5,4\pi/5)$	1.110E-02	9.616E-10
$u(2\pi/5,4\pi/5)$	2.449E-02	2.040E-09
$u(3\pi/5,4\pi/5)$	4.018E-02	3.165E-09
$u(4\pi/5,4\pi/5)$	4.815E-02	4.228E-09

TABLE II
COMPARISON OF SFD AND NSFD RESULTS FOR EXAMPLE 2

$u(x,y)$	$\|Exact - SFD\|$	$\|Exact - NSFD\|$
$u(\pi/5,\pi/5)$	3.178E-02	6.152E-10
$u(2\pi/5,\pi/5)$	5.730E-02	1.572E-09
$u(3\pi/5,\pi/5)$	7.572E-02	2.235E-09
$u(4\pi/5,\pi/5)$	7.162E-02	3.057E-09
$u(\pi/5,2\pi/5)$	5.141E-02	9.974E-10
$u(2\pi/5,2\pi/5)$	9.272E-02	1.836-09
$u(3\pi/5,2\pi/5)$	0.123	3.387E-09
$u(4\pi/5,2\pi/5)$	0.116	5.507E-09
$u(\pi/5,3\pi/5)$	5.141E-02	9.974E-10
$u(2\pi/5,3\pi/5)$	9.272E-02	1.836E-09
$u(3\pi/5,3\pi/5)$	0.123	4.387E-09
$u(4\pi/5,3\pi/5)$	0.116	8.507E-09
$u(\pi/5,4\pi/5)$	3.1780E-02	7.152E-10
$u(2\pi/5,4\pi/5)$	5.730E-02	1.752E-09
$u(3\pi/5,4\pi/5)$	7.5728E-02	2.235E-09
$u(4\pi/5,4\pi/5)$	7.162E-02	4.057E-09

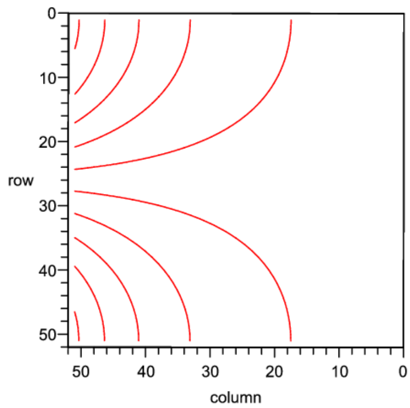

Fig. 1 The contour plot for the solution of example I using the NSFD where (y = π /50) and time step (h = π /50)

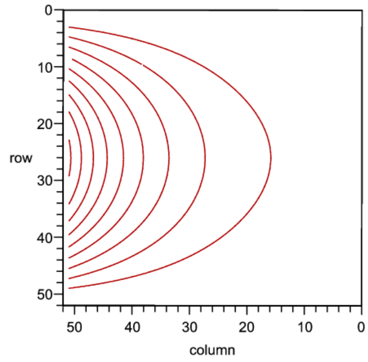

Fig. 2 The contour plot for the solution of example II using the NSFD where (y = π /50) and time step (h = π /50)

We note that if the denominator functions ϕ_1 and ϕ_2 are equal then the NSFD scheme become SFD scheme because the denominator functions will drop out of the calculation.

Figs. 2 and 3 show the counter plots of the Matrix's solutions of the Laplace equation using the NSFD scheme of example I and example II, respectively.

Figs. 3 and 4 show the solution of NSFD for the Laplace equation in example I and example II, respectively. We can conclude that the NSFD solutions for the time step ($h = \pi/50$) are sufficiently accurate.

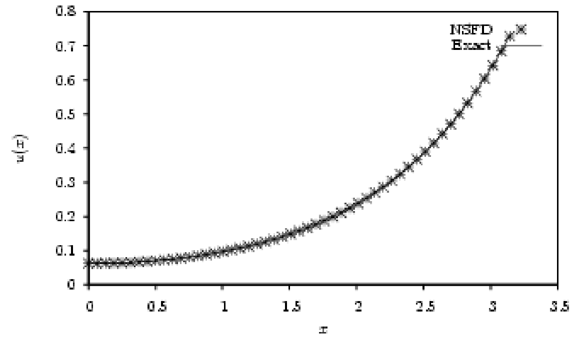

Fig. 3 Numerical solution for example 2 using NSFD where (y = π /50)

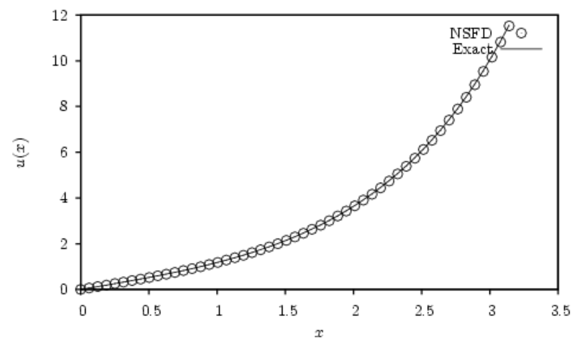

Fig. 4 Numerical solution for example I using NSFD where (y = π /50)

It was found that the NSFD solutions agree very well with the exact solutions and the maximum difference between the NSFD solution and the exact solution at time step ($h = \pi/50$) is of the order of magnitude of 10^{-16} and it was 10^{-5} for the SFD solution.

VI. Conclusion

In this paper, we construct a NSFD scheme for the Laplace equation. Solutions to the Laplace equation with Dirichlet boundary conditions were presented to demonstrate the efficiency of the NSFD scheme. Comparison of results showed that the NSFD scheme results in less error than did results using SFD.

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