

A new sufficient conditions of stability for discrete time non-autonomous delayed Hopfield neural networks

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Abstract—In this paper, we consider the uniform asymptotic stability, global asymptotic stability and global exponential stability of the equilibrium point of discrete Hopfield neural networks with delays. Some new stability criteria for system are derived by using the Lyapunov functional method and the linear matrix inequality approach, for estimating the upper bound of Lyapunov functional derivative.

Keywords—Hopfield Neural Networks, Uniform asymptotic stability, Global asymptotic stability, Exponential stability.

I. INTRODUCTION

DURING the past several years, the stability of a unique equilibrium point of continuous-time delayed Hopfield neural networks has received much attention due to its importance in many applications such as associative memories, pattern recognition, automatic control, static image processing, combinatorial optimizations problems and other areas. Stability results that impose constraint conditions on the network parameters will be dependent of the intended applications in investigating the stability properties of neural networks. So far, many researchers have investigated the global asymptotic stability and global exponential stability of continuous Hopfield neural networks and obtained various results, we refer the reader to [1-8]. In conducting numerical simulation of continuous neural network. Hence, stability for discrete Hopfield neural networks has also received considerable attention from many researchers, (see [9-14]).

In this paper, we consider a class of discrete-time HNN with delays. Some new sufficient conditions for uniform asymptotic stability, global asymptotic stability and global exponential stability of the equilibrium point for such system are obtained by means of using a Lyapunov functional and linear matrix inequality. The conditions on global exponential stability are simpler and less restrictive versions of some recent results.

This paper is organized as follows: In section 2, a discrete Hopfield neural networks is described. In addition, we present some basic definition and lemma. New stability criteria for discrete-time non-autonomous delayed Hopfield neural networks are derived in section 3. An example is presented to illustrate the efficiency of the results in section 4. Finally, some conclusions are drawn in section 5.

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II. PRELIMINARIES

The dynamic behavior of discrete Hopfield neural networks can be described as follows

$$\begin{cases} x_i(n+1) = a_i(n)x_i(n) + \sum_{j=1}^m b_{ij}(n)f_j(x_j(n-\kappa)), & n \neq n_k \\ \Delta x_i \setminus n=n_k = x_i(n_k) - x_i(n_k-1) = d_k^{(i)}(x_i(n_k-1) - \bar{x}_i) \end{cases} \quad (1)$$

where $d_k^{(i)} \in \mathbb{R}$ for $i \in \{1, 2, \dots, m\}$, $n \in \{0, 1, 2, \dots\}$, m corresponds to the number of units in a neural network; $x(n) = [x_1(n), \dots, x_m(n)]^T \in \mathbb{R}^m$ corresponds to the state vector; $f(x(n)) = [f_1(x_1(n)), \dots, f_m(x_m(n))]^T \in \mathbb{R}^m$ denotes the activation function of the neurons; $f(x(n-\kappa)) = [f_1(x_1(n-\kappa)), \dots, f_m(x_m(n-\kappa))]^T \in \mathbb{R}^m$; $A(n) = \text{diag}(a_i(n))$ ($a_i(n) \in]0, 1[$) represents the rate with which the i th unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs. $B(n) = \{b_{ij}(n)\}$ represents the delayed feedback matrix, κ is a positive integer and denotes the transmission delay along the axon of the j th unit.

The initial conditions associated with system (3) are of the form

$$y_i(l) = \varphi_i(l), \quad i \in \{1, 2, \dots, m\}, \quad (2)$$

where l is an integer with $l \in [-\kappa, 0]$.

Since \bar{x} is an equilibrium point of system (1), one can derive from system (1) that the transformation $y_i = x_i - \bar{x}_i$ transforms such system into the following system:

$$\begin{cases} y_i(n+1) = a_i(n)y_i(n) + \sum_{j=1}^m b_{ij}(n)g_j(y_j(n-\kappa)), & n \neq n_k \\ y_i(n_k) = (1 + d_k^{(i)}) \cdot y_i(n_k-1), \end{cases} \quad (3)$$

where $g_j(y_j(n-\kappa)) = f_j(\bar{x}_j + y_j(n-\kappa)) - f_j(\bar{x}_j)$.

In this paper, we will assume that the activation functions g_i , $i = 1, 2, \dots, m$ satisfy the following conditions

$$|g_i(\xi_1) - g_i(\xi_2)| \leq L_i |\xi_1 - \xi_2|, \quad \forall \xi_1, \xi_2 \in \mathbb{R} \quad (4)$$

$$g_i(0) = 0$$

and $D_k = \text{diag}(1 + d_k^{(1)}, 1 + d_k^{(2)}, \dots, 1 + d_k^{(n)})$.

Now, we define various types of stability for system (3) at its equilibrium point, and we introduce lemmas used in our work. Some definitions and lemmas of stability for system (3) at its equilibrium point are introduced as follows:

A. Definitions [1]

(i) If, for any $k_0 \geq 0$ and $\epsilon > 0$, there exists a $\delta_1 = \delta(k_0, \epsilon) > 0$ such as

$$\|y(k_0)\| < \delta(k_0, \epsilon) \implies \|y(k)\| < \epsilon, \forall k \geq k_0$$

then system (3) is stable (in the Lyapunov sense) at the equilibrium point.

(ii) If the system (3) is stable at the equilibrium point, and if there exists a $\delta_2 = \delta(k_0) > 0$ such as

$$\|y(k_0)\| < \delta(k_0) \implies \lim_{k \rightarrow \infty} y(k) = 0$$

then system (3) is asymptotically stable at the equilibrium point.

(iii) The solution of system (3) is exponentially stable if for all solution $y_i(n, \varphi_i)$ with initial condition $y_i(l) = \varphi_i(l)$, $\forall l \in [-\kappa, 0]$ there exist two constants $\epsilon \in]0, 1]$ and $M \geq 1$ such as : $\|y(n)\| \leq M\|\phi\|\epsilon^n$, $\forall n > 0$, where $\|\phi\| = \max_{l \in [-\kappa, 0]} \{\|\phi(l)\|\}$.

(iv) If δ_1 in (i) (or δ_2 in (ii)) can be chosen independently of k_0 , then the system is uniformly stable (or uniformly asymptotically stable) at the equilibrium point.

(v) If δ_2 in (ii) (or δ_3 in (iii)) can be an arbitrarily large, finite number, then the system is globally asymptotically stable (or globally exponentially stable) at the equilibrium point.

We now need the following basic lemmas used in our work.

B. Lemma [17]

Let $X \in \mathbb{R}^{n \times n}$, then

$$\lambda_{\min}(X)a^T a \leq a^T X a \leq \lambda_{\max}(X)a^T a$$

for any $a \in \mathbb{R}^n$, if X is a symmetric matrix, where $\lambda_{\min}(X)$ and $\lambda_{\max}(X)$ are respectively the smallest and biggest eigenvalue of the matrix X .

C. Lemma [16]

Let $M(x) = M^T(x)$, $P(x) = P^T(x) > 0$ and $Q(x)$ depend affinely on x . Then

$$\begin{pmatrix} Q(x) & M(x) \\ M^T(x) & -P(x) \end{pmatrix} < 0$$

is equivalent to

$$Q(x) + M(x)P^{-1}(x)M^T(x) < 0$$

III. MAIN RESULTS

Now, we shall establish some theorems which provide sufficient conditions for global exponential stability, uniform asymptotic stability and global asymptotic stability of system (3). At first, we consider the global exponential stability.

A. Theorem

System (3) is asymptotically stable and globally exponentially stable if there exist two positive definite matrices P, Q and $\epsilon \in]0, 1]$ such as $E =$

$$\begin{pmatrix} A(n)PA(n) - P + \lambda_{\max}(Q)L^2 & A(n)PB(n) \\ B^T(n)PA(n) & B^T(n)PB(n) - Q \end{pmatrix} < 0$$

$$\prod_{n_0 \leq n_k \leq n} \max\left\{\frac{\xi_k}{\lambda_{\min}(P)}, 1\right\} \leq \epsilon^{2n}, \quad \forall n > 0,$$

where $L = \text{diag}(L_i)$, ξ_k and $\lambda_{\max}(Q)$ are respectively the largest eigenvalues of the matrix D_kPD_k , $k \in \mathbb{Z}_+^*$ and the matrix Q .

If $Q = I$, $P = 2I$ and $L = I$ in Theorem III-A, we can easily obtain this Corollary :

B. Corollary

System (3) is asymptotically stable and globally exponentially stable if there exist $\epsilon \in]0, 1]$ such as

(i)

$$E = \begin{pmatrix} 2A(n)A(n) - I & 2A(n)B(n) \\ 2B^T(n)A(n) & 2B^T(n)B(n) - I \end{pmatrix} < 0$$

(ii)

$$\prod_{n_0 \leq n_k \leq n} \max\left\{\frac{\xi_k}{2}, 1\right\} \leq \epsilon^{2n}, \quad \forall n > 0,$$

where ξ_k is the largest eigenvalues of the matrix $2D_k^2$, $k \in \mathbb{Z}_+^*$.

Remark: Based on Lemma II-C, if $2B^T(n)B(n) - I < 0$, then condition (i) in Corollary III-B can be rewritten as

$$E' = 2A(n)A(n) - I - 4A(n)B(n)(2B^T(n)B(n) - I)^{-1}B^T(n)A(n) < 0$$

Next we can establish a theorem which provides sufficient conditions for uniform stability, uniform asymptotic stability and global asymptotic stability of system (3).

C. Theorem

System (3) is uniformly stable if it exist $\epsilon^* \in]0, 1]$ and two positive definite matrix P, Q such as:

(i)

$$E = \begin{pmatrix} A(n)PA(n) - a(P + \frac{1}{1-\epsilon^*}\lambda_{\max}(Q)L^2) & A(n)PB(n) \\ B^T(n)PA(n) & B^T(n)PB(n) - \frac{b}{1-\epsilon^*}Q \end{pmatrix}$$

< 0

with $a = \frac{1+\epsilon^*(n-n_0)^2}{1+\epsilon^*(n+1-n_0)^2}$ and $b = \frac{1+\epsilon^*(n-\kappa-n_0)^2}{1+\epsilon^*(n+1-n_0)^2}$

(ii) $\frac{\prod_{0 \leq n_k \leq n} \max\{\frac{\xi_k}{\lambda_{\min}(P)}, 1\}}{1+\epsilon^*(n-n_0)^2} < \infty$, $n \rightarrow \infty$, with ξ_k is the largest eigenvalue of matrix D_kPD_k , $k \in \mathbb{Z}_+^*$.

If besides this condition is verified

(iii) $\frac{\prod_{0 \leq n_k \leq n} \max\{\frac{\xi_k}{\lambda_{\min}(P)}, 1\}}{1+\epsilon^*(n-n_0)^2} \rightarrow 0$, $n \rightarrow \infty$, the system (3) is uniformly asymptotically stable and globally asymptotically stable.

If $P = 2I$ and $Q = I$ in Theorem III-C, we can get the following Corollary :

D. Corollary

System (3) is uniformly stable if it exist $\epsilon^* \in]0, 1]$ such as:

(i)

$$E = \begin{pmatrix} 2A(n)A(n) - a(2I + \frac{1}{1-\rho}I) & 2A(n)B(n) \\ 2B^T(n)A(n) & 2B^T(n)B(n) - \frac{b}{1-\rho}I \end{pmatrix} < 0$$

with $a = \frac{1+\epsilon^*(n-n_0)^2}{1+\epsilon^*(n+1-n_0)^2}$ and $b = \frac{1+\epsilon^*(n-\kappa-n_0)^2}{1+\epsilon^*(n+1-n_0)^2}$

(ii) $\frac{\prod_{0 \leq n_k < n} \max\{\frac{\xi_k}{2}, 1\}}{1+\epsilon^*(n-n_0)^2} < \infty$, $n \rightarrow \infty$, with ξ_k is the largest eigenvalue of matrix $2D_k^2$, $k \in \mathbb{Z}_+^*$

If besides this condition is verified

(iii) $\frac{\prod_{0 \leq n_k < n} \max\{\frac{\xi_k}{2}, 1\}}{1+\epsilon^*(n-n_0)^2} \rightarrow 0$, $n \rightarrow \infty$ the system (3) is uniformly asymptotically stable and globally asymptotically stable.

Remark: Based on Lemma II-C, if $2B^T(n)B(n) - \frac{b}{1-\rho}I < 0$, then condition (i) in Corollary III-D can be rewritten as:

$$E'' = 2A(n)A(n) - a(2 - \frac{1}{1-\rho})I - 4A(n)B(n)(2B^T(n)B(n) - \frac{b}{1-\rho}I)^{-1}B^T(n)A(n) < 0$$

IV. NUMERICAL APPLICATIONS

In this section, we will give an example to show the validity of the results given in this paper.

A. Example

Consider the following discrete Hopfield neural network

$$\begin{cases} x(n+1) = A(n)x(n) + B(n)g(x(n-\kappa)), & n \neq n_k \\ \Delta x_i|_{n=n_k} = x_i(n_k) - x_i(n_k-1) \\ = d_k^{(i)}(x_i(n_k-1) - x_i^*) & i = 1, 2, k = 1, 2, \dots \end{cases} \quad (5)$$

Where $d_k^{(1)} = \sqrt{1 + \frac{1}{5k^2}} - 1$, $d_k^{(2)} = \sqrt{1 + \frac{1}{6k^2}} - 1$, $n_k \in \mathbb{Z}_+^*$ with the matrices

$$A(n) = \begin{pmatrix} \sqrt{\frac{20}{91}} & 0 \\ 0 & 0.05e^{-n} \end{pmatrix} \text{ and } B(n) = \begin{pmatrix} 0.2 & 0.4 \\ 0.2 & -0.4 \end{pmatrix}$$

and the nonlinear input-output function is chosen as $f(x) = \tanh(x)$. It can be verified that this function satisfies assumption(4) with $L_1 = L_2 = 1$. Furthermore, we have

$$2B^T(n)B(n) - I = \begin{pmatrix} -0.6 & 0 \\ 0 & -0.36 \end{pmatrix} < 0$$

and one can easily check that

$$E' = \begin{pmatrix} -\frac{1}{9} & -\frac{68}{45}\sqrt{\frac{20}{91}}0.05e^{-n} \\ -\frac{68}{45}\sqrt{\frac{20}{91}}0.05e^{-n} & \frac{91}{9000}e^{-2n} - 1 \end{pmatrix}$$

The determinants of the principal submatrices of E' are $-\frac{1}{9} < 0$ and $\frac{1}{9} - 0.0024e^{-2n} > 0$. Based on Hurwitz's Theorem, we can conclude that $E' < 0$. Therefore, from Remark III-B, it follows that system (5) is asymptotically stable.

$$2D_k^2 = \begin{pmatrix} 4 + \frac{2}{5k^2} - 4\sqrt{1 + \frac{1}{5k^2}} & 0 \\ 0 & 4 + \frac{2}{6k^2} - 4\sqrt{1 + \frac{1}{6k^2}} \end{pmatrix}$$

Therefore,

$$\prod_{n_0 \leq n_k \leq n} \max\{\frac{\xi_k}{2}, 1\} = 1 \leq \epsilon^n, \text{ with } \epsilon = 1.$$

So, system (5) is globally exponentially stable.

While using the Corollary III-D, we can even show that system (5) is uniformly stable .

We choose $\rho = 0$: Therefore, the condition (i) in Corollary III-D is verified:

$$\lim_{\epsilon^* \rightarrow 0} E'' = 2A(n)A(n) - 3I - 4A(n)B(n)(2B^T(n)B(n) - I)^{-1}B^T(n)A(n) < 0$$

Besides

$$\frac{\prod_{0 \leq n_k < n} \max\{\frac{\xi_k}{2}, 1\}}{1 + \epsilon^*(n - n_0)^2} = \frac{1}{1 + \epsilon^*(n - n_0)^2} < \infty$$

Hence, system (5) is uniformly stable.

Remark: In [16], the authors proved that system (5) is asymptotically stable. In this work, we showed that system (5) is also globally exponentially stable and uniformly stable.

V. CONCLUSION

In this paper, a class of discrete HNN with delay is considered. We obtain some new sufficient criteria ensuring uniform asymptotic stability, global stability and global exponential stability of the equilibrium point for system (3) by using the Lyapunov method and linear matrix inequality. Our results show effect of delay on the stability of HNN. The results here are discussed from the point of view of its more generality than earlier results. In order to validate our results, an example is given to illustrate their feasibility and efficiency. Has the continuation of this work us can look for new criterias of stability for high-order Hopfield type neural networks by refined and generalized our results.

APPENDIX A
PROOF OF THE THEOREM III-A

Consider the following Lyapunov function:

$$V(n) = y^T(n)Py(n) + \sum_{i=n-\kappa}^{n-1} g^T(y(i))Qg(y(i)) \quad (6)$$

We have

$$\begin{aligned} V(n+1) &= y^T(n+1)Py(n+1) + \sum_{i=n+1-\kappa}^n g^T(y(i))Qg(y(i)) \\ &= y^T(n)A(n)PA(n)y(n) \\ &\quad + 2y^T(n)A(n)PB(n)g(y(n-\kappa)) \\ &\quad + g^T(y(n-\kappa))B^T(n)PB(n)g(y(n-\kappa)) \\ &\quad + \sum_{i=n+1-\kappa}^{n-1} g^T(y(i))Qg(y(i)) + g^T(y(n))Qg(y(n)) \end{aligned}$$

Therefore,

$$\begin{aligned} \Delta V(n) &= V(n+1) - V(n) \\ &= y^T(n)A(n)PA(n)y(n) + 2y^T(n)A(n)PB(n)g(y(n-\kappa)) \\ &\quad + g^T(y(n-\kappa))B^T(n)PB(n)g(y(n-\kappa)) \\ &\quad + \sum_{i=n+1-\kappa}^{n-1} g^T(y(i))Qg(y(i)) \\ &\quad + g^T(y(n))Qg(y(n)) \\ &\quad - y^T(n)Py(n) - \sum_{i=n-\kappa}^{n-1} g^T(y(i))Qg(y(i)) \\ &\leq y^T(n)[A(n)PA(n) - P + \lambda_{\max}(Q)L^2]y(n) \\ &\quad + 2y^T(n)A(n)PB(n)g(y(n-\kappa)) \\ &\quad + g^T(y(n-\kappa))[B^T(n)PB(n) - Q]g(y(n-\kappa)) \\ &= \begin{pmatrix} y(n) \\ g(y(n-\kappa)) \end{pmatrix}^T E \begin{pmatrix} y(n) \\ g(y(n-\kappa)) \end{pmatrix} < 0 \end{aligned}$$

It is clear that

$$V(n) \leq \lambda_{\max}(P)\|y(n)\|^2 + \lambda_{\max}(Q)\|y(n)\|^2 L^2 \kappa$$

Therefore,

$$V(n) \leq [\lambda_{\max}(P) + \lambda_{\max}(Q)L^2\kappa]\|y(n)\|^2$$

So while using Lemma II-B, we will have the following result

$$\begin{aligned} \lambda_{\min}(P)\|y(n)\|^2 &\leq V(n) \\ &\leq (\lambda_{\max}(P) + \lambda_{\max}(Q)L^2\kappa)\|y(n)\|^2 \end{aligned}$$

However we have

$$V(n_k) \leq \max\left\{\frac{\xi_k}{\lambda_{\min}(P)}, 1\right\}V(n_k - 1)$$

Therefore,

$$\begin{aligned} \lambda_{\min}(P)\|y(n)\|^2 &\leq V(n) \\ &\leq V(n_0) \prod_{n_0 \leq n_k < n} \max\left\{\frac{\xi_k}{\lambda_{\min}(P)}, 1\right\} \end{aligned}$$

Therefore,

$$\begin{aligned} \|y(n)\|^2 &\leq \left[\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} + \frac{\lambda_{\max}(Q)L^2\kappa}{\lambda_{\min}(P)}\right]\|\varphi\|^2 \\ &\quad \prod_{n_0 \leq n_k < n} \max\left\{\frac{\xi_k}{\lambda_{\min}(P)}, 1\right\} \end{aligned}$$

So

$$\|y(n)\|^2 \leq \left[\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} + \frac{\lambda_{\max}(Q)L^2\kappa}{\lambda_{\min}(P)}\right]\|\varphi\|^2 \epsilon^{2n}$$

Then,

$$\|y(n)\| \leq M\|\varphi\|\epsilon^n, \forall n \geq 0$$

With

$$M = \sqrt{\left[\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} + \frac{\lambda_{\max}(Q)L^2\kappa}{\lambda_{\min}(P)}\right]} \geq 1 \text{ Which completes the proof.}$$

APPENDIX B
PROOF OF THE THEOREM III-C

We start with showing that system (3) is uniformly stable. According with the condition (iii), it exists a constant $M^* > 0$ as:

$$\frac{\prod_{0 \leq n_k < n} \max\left\{\frac{\xi_k}{\lambda_{\min}(P)}, 1\right\}}{1 + \epsilon^*(n - n_0)^2} \leq M^*, n \geq n_0$$

$\forall n \geq n_0$ if $y(n_0, \varphi)$ is a solution of (3), $\forall \epsilon > 0$ we choose:

$$\delta = \sqrt{\frac{\lambda_{\min}(P)}{(\lambda_{\max}(P) + \frac{\lambda_{\max}(Q)}{1-\rho}N(\kappa+1))M^*}} \cdot \epsilon$$

We considered the Lyapunov function:

$$\begin{aligned} V(n) &= (1 + \epsilon^*(n - n_0)^2)y^T(n)Py(n) \\ &\quad + \frac{1}{1-\rho} \sum_{i=n-\kappa}^{n-1} (1 + \epsilon^*(i - n_0)^2)g^T(y(i))Qg(y(i)) \end{aligned}$$

Therefore,

$$\begin{aligned} V(n+1) &= (1 + \epsilon^*(n+1 - n_0)^2)y^T(n+1)Py(n+1) \\ &\quad + \frac{1}{1-\rho} \sum_{i=n+1-\kappa}^n (1 + \epsilon^*(i - n_0)^2)g^T(y(i))Qg(y(i)) \\ &= [1 + \epsilon^*(n+1 - n_0)^2][A(n)y(n) \\ &\quad + B(n)g(y(n-\kappa))]^T P [A(n)y(n) + B(n)g(y(n-\kappa))] \\ &\quad + \frac{1}{1-\rho} \sum_{i=n+1-\kappa}^{n-1} (1 + \epsilon^*(i - n_0)^2)g^T(y(i))Qg(y(i)) \\ &\quad + \frac{1}{1-\rho} (1 + \epsilon^*(n - n_0)^2)g^T(y(n))Qg(y(n)) \\ &= [1 + \epsilon^*(n+1 - n_0)^2][y(n)^T A(n)P \\ &\quad + g^T(y(n-\kappa))B^T(n)P][A(n)y(n) + B(n)g(y(n-\kappa))] \\ &\quad + \frac{1}{1-\rho} \sum_{i=n+1-\kappa}^{n-1} (1 + \epsilon^*(i - n_0)^2)g^T(y(i))Qg(y(i)) \\ &\quad + \frac{1}{1-\rho} (1 + \epsilon^*(n - n_0)^2)g^T(y(n))Qg(y(n)) \end{aligned}$$

$$\begin{aligned}
&= [1 + \epsilon^*(n+1-n_0)^2][y^T(n)A(n)PA(n)y(n) \\
&+ 2y^T(n)A(n)PB(n)g(y(n-\kappa)) \\
&+ g^T(y(n-\kappa))B^T(n)PB(n)g(y(n-\kappa))] \\
&+ \frac{1}{1-\rho} \sum_{i=n+1-\kappa}^{n-1} (1 + \epsilon^*(i-n_0)^2)g^T(y(i))Qg(y(i)) \\
&+ \frac{1}{1-\rho} (1 + \epsilon^*(n-n_0)^2)g^T(y(n))Qg(y(n))
\end{aligned}$$

Now, calculate the difference $\Delta V(n) = V(n+1) - V(n)$

$$\begin{aligned}
\Delta V(n) &= [1 + \epsilon^*(n+1-n_0)^2][y^T(n)[A(n)PA(n) \\
&- \frac{1+\epsilon^*(n-n_0)^2}{1+\epsilon^*(n+1-n_0)^2}P]y(n) \\
&+ 2y^T(n)A(n)PB(n)g(y(n-\kappa)) \\
&+ g^T(y(n-\kappa))B^T(n)PB(n)g(y(n-\kappa))] \\
&+ \frac{1}{1-\rho} (1 + \epsilon^*(n-n_0)^2)g^T(y(n))Qg(y(n)) \\
&- \frac{1}{1-\rho} (1 + \epsilon^*(n-\kappa-n_0)^2)g^T(y(n-\kappa))Qg(y(n-\kappa))
\end{aligned}$$

$$\begin{aligned}
&\leq [1 + \epsilon^*(n+1-n_0)^2][y^T(n)[A(n)PA(n) \\
&- \frac{1+\epsilon^*(n-n_0)^2}{1+\epsilon^*(n+1-n_0)^2}P]y(n) \\
&+ 2y^T(n)A(n)PB(n)g(y(n-\kappa)) \\
&+ g^T(y(n-\kappa))B^T(n)PB(n)g(y(n-\kappa))] \\
&- \frac{1}{1-\rho} \cdot \frac{1+\epsilon^*(n-\kappa-n_0)^2}{1+\epsilon^*(n+1-n_0)^2}Q]g(y(n-\kappa)) \\
&= [1 + \epsilon^*(n+1-n_0)^2] \begin{pmatrix} y(n) \\ g(y(n-\kappa)) \end{pmatrix}^T E \begin{pmatrix} y(n) \\ g(y(n-\kappa)) \end{pmatrix} \\
&\leq [1 + (n+1-n_0)^2] \begin{pmatrix} y(n) \\ g(y(n-\kappa)) \end{pmatrix}^T E \begin{pmatrix} y(n) \\ g(y(n-\kappa)) \end{pmatrix}
\end{aligned}$$

Therefore,

$$\Delta V(n) < 0 \quad (7)$$

$$\begin{aligned}
V(n_k) &= (1 + \epsilon^*(n_k - n_0)^2)y^T(n_k)Py(n_k) \\
&+ \frac{1}{1-\rho} \sum_{i=n_k-\kappa}^{n_k} (1 + \epsilon^*(i-n_0)^2)g^T(y(i))Qg(y(i)) \\
&= (1 + \epsilon^*(n_k - n_0)^2)y^T(n_k-1)D_kPD_ky(n_k-1) \\
&+ \frac{1}{1-\rho} \sum_{i=n_k-1-\kappa}^{n_k-1} (1 + \epsilon^*(i-n_0)^2)g^T(y(i))Qg(y(i)) \\
&\leq (1 + \epsilon^*(n_k - n_0)^2)\xi_k y^T(n_k-1)y(n_k-1) \\
&+ \frac{1}{1-\rho} \sum_{i=n_k-1-\kappa}^{n_k-1} (1 + \epsilon^*(i-n_0)^2)g^T(y(i))Qg(y(i)) \\
&\leq (1 + \epsilon^*(n_k - n_0)^2)\frac{\xi_k}{\lambda_{\min}(P)}y^T(n_k-1)Py(n_k-1) \\
&+ \frac{1}{1-\rho} \sum_{i=n_k-1-\kappa}^{n_k-1} (1 + \epsilon^*(i-n_0)^2)g^T(y(i))Qg(y(i))
\end{aligned}$$

Therefore,

$$V(n_k) \leq \max\left\{\frac{\xi_k}{\lambda_{\min}(P)}, 1\right\}V(n_k-1) \quad (8)$$

We have from Lemma II-B

$$\begin{aligned}
&\lambda_{\min}(P)(1 + \epsilon^*(n-n_0)^2)\|y(n)\|^2 \leq V(n) \\
&\leq \lambda_{\max}(P)(1 + \epsilon^*(n-n_0)^2)\|y(n)\|^2
\end{aligned}$$

$$\begin{aligned}
&+ \frac{\lambda_{\max}(Q)}{1-\rho}N\left(\sum_{n=\kappa}^{n-1} 1 + \sum_{i=n-\kappa}^{n-1} \epsilon^*(i-n_0)^2\right)\|y(n)\|^2 \\
&\leq \lambda_{\max}(P)(1 + \epsilon^*(n-n_0)^2)\|y(n)\|^2 \\
&+ \frac{\lambda_{\max}(Q)}{1-\rho}N\kappa(1 + \epsilon^*(n-n_0)^2)\|y(n)\|^2 \\
&\leq [\lambda_{\max}(P) + \frac{\lambda_{\max}(Q)}{1-\rho}N\kappa](1 + \epsilon^*(n-n_0)^2)\|y(n)\|^2
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\lambda_{\min}(P)(1 + \epsilon^*(n-n_0)^2)\|y(n)\|^2 \leq V(n) \\
&\leq [\lambda_{\max}(P) + \frac{\lambda_{\max}(Q)}{1-\rho}N\kappa](1 + \epsilon^*(n-n_0)^2)\|y(n)\|^2 \quad (9)
\end{aligned}$$

Collecting (7), (8) et (9), then we get:

$$\begin{aligned}
&\lambda_{\min}(P)(1 + \epsilon^*(n-n_0)^2)\|y(n)\|^2 \leq V(n) \\
&\leq V(n_0) \prod_{0 \leq n_k \leq n} \max\left\{\frac{\xi_k}{\lambda_{\min}(P)}, 1\right\} \quad (10)
\end{aligned}$$

Of (9) we can raise $V(n_0)$:

$$V(n_0) \leq [\lambda_{\max}(P) + \frac{\lambda_{\max}(Q)}{1-\rho}N\kappa]\|\varphi\|^2$$

by following:

$$\begin{aligned}
&\|y(n)\|^2 \leq [\lambda_{\max}(P) + \frac{\lambda_{\max}(Q)}{1-\rho}N\kappa] \frac{\|\varphi\|^2}{\lambda_{\min}(P)} \prod_{0 \leq n_k \leq n} \max\left\{\frac{\xi_k}{\lambda_{\min}(P)}, 1\right\} \\
&\frac{\lambda_{\max}(Q)}{1-\rho}N\kappa] \frac{\|\varphi\|^2}{\lambda_{\min}(P)} \frac{1}{1+\epsilon^*(n-n_0)^2}
\end{aligned}$$

From where

$$\begin{aligned}
&\|y(n)\|^2 \leq [\lambda_{\max}(P) + \frac{\lambda_{\max}(Q)}{1-\rho}N\kappa] \frac{\|\varphi\|^2}{\lambda_{\min}(P)} M^* \\
&\leq [\lambda_{\max}(P) + \frac{\lambda_{\max}(Q)}{1-\rho}N\kappa] \frac{\delta^2}{\lambda_{\min}(P)} M^* \leq \epsilon^2
\end{aligned}$$

What implies that

$$\|y(n)\|^2 \leq \epsilon^2, \quad n \geq n_0$$

Therefore,

$$\|y(n)\| \leq \epsilon, \quad n \geq n_0$$

Then the solution of (3) is uniformly stable.

It is clear that if the condition(iii) is verified then: $\limsup_{n \rightarrow +\infty} \|y(n)\|^2 = 0$, and for this cas, (3) is also uniformly asymptotically stable and globally asymptotically stable.

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