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A New Approach to Optimal Control Problem Constrained by Canonical Form

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Abstract—In this article, it is considered a class of optimal control problems constrained by differential and integral constraints are called canonical form. A modified measure theoretical approach is introduced to solve this class of optimal control problems.

Keywords—Optimal control problem, Canonical form, Measure theory.

I. Introduction

CONSIDER a process described by the system of nonlinear differential equations as follows:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}(t, \mathbf{x}, \mathbf{u}), \quad \forall t \in (0, T], \tag{1}$$

with the initial and final conditions given by

$$\mathbf{x}(0) = \mathbf{x}^0, \tag{2}$$

$$\mathbf{x}(T) = \mathbf{x}^T, \tag{3}$$

where $\mathbf{x}(t) = [x_1(t), ..., x_n(t)]^\top \in R^n$, $\mathbf{u}(t) = [u_1(t), ..., u_m(t)]^\top \in R^m$ are the state and control vectors, respectively, and $\mathbf{f}(t) = [f_1(t), ..., f_n(t)]^\top \in R^n$ is continuously differentiable with respect to its respective arguments. Vectors $\mathbf{x}^0 = [x_1^0, ..., x_n^0]^\top \in R^n$ and $\mathbf{x}^T = [x_1^T, ..., x_n^T]^\top \in R^n$ are given constants. Let

$$A_{lu} = \{\mathbf{y}(.) = [y_1(.), ..., y_n(.)]^\top \in R^n; \quad {y_i}^l \le y_i(.) \le {y_i}^u,\ i = 1, ..., n\}, \tag{4}$$

$$U_{lu} = \{\mathbf{v}(.) = [v_1(.), ..., v_m(.)]^\top \in R^m; \quad {v_j}^l \le v_j(.) \le {v_j}^u,\ j = 1, ..., m\}, \tag{5}$$

where lower and upper bounds are given real numbers. It is obvious U_{lu} is a compact subset of R^m. A Boral measurable function $\mathbf{u} : [0, T] \to R^m$ is called an *admissible control* if $\mathbf{u} \in U_{lu}$. Let U denote the class of all such admissible controls. For each admissible control $\mathbf{u} \in U_{lu}$, let $\mathbf{x}(., \mathbf{u})$ denote the corresponding solution of the system (1) and satisfy the initial and final conditions (2)-(3). Such this state vector is referred to as an *admissible solution* of system (1) and (2)-(3) corresponding to $\mathbf{u} \in U_{lu}$, if $\mathbf{x} \in A_{lu}$. Let A denote the class of all such admissible states.
The canonical optimal control(COC) problem is now formulated as the following.

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Problem COC: Subject to the dynamical system (1) and (2)-(3), find a control $\mathbf{u} \in \mathsf{U}$ such that the cost functional

$$G_0(\mathbf{u}) = \Phi_0(\mathbf{x}(T|\mathbf{u})) + \int_0^T \mathsf{L}_0(t, \mathbf{x}(t|\mathbf{u}(t)), \mathbf{u}(t))\ dt, \tag{6}$$

is minimized over U and subject to

$$G_i(\mathbf{u}) = \Phi_i(\mathbf{x}(T|\mathbf{u})) + \int_0^T \mathsf{L}_i(t, \mathbf{x}(t|\mathbf{u}(t)), \mathbf{u}(t))\ dt = 0, \quad i = 1, ..., N_c, \tag{7}$$

$$G_i(\mathbf{u}) = \Phi_i(\mathbf{x}(T|\mathbf{u})) + \int_0^T \mathsf{L}_i(t, \mathbf{x}(t|\mathbf{u}(t)), \mathbf{u}(t))\ dt \le 0, \quad i = N_c + 1, ..., N, \tag{8}$$

where Φ_i and L_i for $i = 0, 1, ..., N$, are given real-valued functions.

It is assumed that functions $\mathbf{f}$ and L_i, $i = 0, 1, ..., N$, together with their partial derivatives respecting to each of the components of $\mathbf{x}$ and $\mathbf{u}$ are piecewise constants on $[0, T]$ for each $(\mathbf{x}, \mathbf{u}) \in R^n \times R^m$ and continuous on $R^n \times R^m$ for each $t \in [0, T]$. Functions Φ_i, $i = 0, 1, ..., N$, are continuously differentiable respecting to $\mathbf{x}$.

II. Measure theory

Firstly, without lose of generality, it may be suppose that $\Phi_i = 0$ for $i = 0, 1, ..., N$. A pair $\mathsf{p} = [\mathbf{x}, \mathbf{u}]$ is said to be an *admissible pair* if $\mathbf{x} \in \mathsf{A}$ and $\mathbf{u} \in \mathsf{U}$. Let P_{ad} denote the class of all such admissible pairs. Measure theoretical approach developed in [3] deals with integral equations and then applied by others [1][2]. Hence, it makes clear, using this approach for solving COC problem needs the differential equations of dynamical system (1) to be equivalent to integral ones. For this purpose, let B be an open ball in R^{n+1} containing $\mathsf{J} \times \mathsf{A}$ where $\mathsf{J} = [0, T]$. Furthermore, $\mathsf{C}^1(\mathsf{B})$ contains all real-valued continuously differentiable functions on B. Suppose function $\varphi^{\mathbf{f}}$ is defined as follows:

$$\varphi^{\mathbf{f}}(t, \mathbf{x}(t), \mathbf{u}(t)) = \varphi_{\mathbf{x}}(t, \mathbf{x}(t))\mathbf{f}(t, \mathbf{x}(t), \mathbf{u}(t)) + \varphi_t(t, \mathbf{x}(t)), \quad \forall \varphi \in \mathsf{C}^1(\mathsf{B}), \tag{9}$$

for all $\mathsf{p}=[\mathbf{x},\mathbf{u}]\in\mathsf{P}_{ad}$ and $t\in\mathsf{J}$. Since p is an admissible pair, it implies that

$$\begin{aligned}\int_0^T \varphi^{\mathbf{f}}(t,\mathbf{x}(t),\mathbf{u}(t))\ dt &= \int_0^T \big(\varphi_{\mathbf{x}}(t,\mathbf{x}(t))\mathbf{f}(t,\mathbf{x}(t),\mathbf{u}(t)) \\ &\quad +\varphi_t(t,\mathbf{x}(t))\big)\ dt \\ &= \int_0^T \frac{d}{dt}\varphi(t,\mathbf{x}(t))\ dt \\ &= \varphi(T,\mathbf{x}(T))-\varphi(0,\mathbf{x}(0)) \\ &= \triangle\varphi. \end{aligned}\tag{10}$$

Obviously, if $\Omega=\mathsf{J}\times\mathsf{A}\times\mathsf{U}$ then, one may verify that integral equations (10) for any $(t,\mathbf{x}(t),\mathbf{u}(t))\in\Omega$ are equivalent to differential equations (1). Moreover, it is known that the more constraints are appended in an optimization problem, the better optimal solution if there exists, is attained. Let $\mathsf{D}(\mathsf{J}^\circ)$ be the space of all infinitely differentiable real-valued functions with compact support in open interval J°. Regarding to dynamical system (1) and $\mathsf{p}\in\mathsf{P}_{ad}$, function ϕ_i, $i=1,...,n$, is constructed such that:

$$\phi_i^{\psi}(t,\mathbf{x}(t),\mathbf{u}(t))=x_i(t)\frac{d\psi(t)}{dt}+f_i(t,\mathbf{x}(t),\mathbf{u}(t))\psi(t),\quad \forall\psi\in\mathsf{D}(\mathsf{J}^\circ).\tag{11}$$

As follows from integrating of the above definition, it admits

$$\begin{aligned}&\int_0^T \phi_i^{\psi}(t,\mathbf{x}(t),\mathbf{u}(t))\ dt = \\ &\int_0^T x_i(t)\frac{d\psi(t)}{dt}\ dt+\int_0^T f_i(t,\mathbf{x}(t),\mathbf{u}(t))\psi(t)\ dt = \\ &x_i(t)\psi(t)\big|_0^T-\int_0^T\Big(\frac{dx_i(t)}{dt}- \\ &f_i(t,\mathbf{x}(t),\mathbf{u}(t))\Big)\psi(t)\ dt.\end{aligned}\tag{12}$$

Since $\psi\in\mathsf{D}(\mathsf{J}^\circ)$ has a compact support in J°, indeed, $\psi(0)=\psi(T)=0$, and the dynamical system (1) is satisfied, the right-hand side of (12) becomes zero. So, the following appendant constraints to COC problem can be given by

$$\int_0^T \Phi_\psi(t,\mathbf{x}(t),\mathbf{u}(t))\ dt=\mathbf{0},\tag{13}$$

where $\Phi_\psi=[\phi_1^\psi,...,\phi_n^\psi]^\top$. Furthermore, choosing functions depended only on the variable $t\in\mathsf{J}$ leads to

$$\int_0^T h(t,\mathbf{x}(t),\mathbf{u}(t))\ dt=a_h,\quad \forall h\in\mathsf{C}_1(\Omega),\tag{14}$$

where $\mathsf{C}_1(\Omega)$ as a subspace of $\mathsf{C}(\Omega)$, contains all continuous functions on Ω depending only on $t\in\mathsf{J}$. From the above assumptions it follows that the COC problem in integral form can be shown as the following:

Problem COCI: Minimize

$$G_0(\mathbf{u})=\int_0^T \mathsf{L}_0(t,\mathbf{x}(t),\mathbf{u}(t))\ dt,\tag{15}$$

subject to

$$G_i(\mathbf{u})=\int_0^T \mathsf{L}_i(t,\mathbf{x}(t),\mathbf{u}(t))\ dt = 0,\quad i=1,...,N_c,\tag{16}$$

$$G_i(\mathbf{u})=\int_0^T \mathsf{L}_i(t,\mathbf{x}(t),\mathbf{u}(t))\ dt \le 0,\quad i=N_c+1,...,N,\tag{17}$$

$$\int_0^T \varphi^{\mathbf{f}}(t,\mathbf{x}(t),\mathbf{u}(t))\ dt = \triangle\varphi,\ \forall\varphi\in\mathsf{C}^1(\mathsf{B}),\tag{18}$$

$$\int_0^T \Phi_\psi(t,\mathbf{x}(t),\mathbf{u}(t))\ dt = \mathbf{0},\ \forall\psi\in\mathsf{D}(\mathsf{J}^\circ),\tag{19}$$

$$\int_0^T h(t,\mathbf{x}(t),\mathbf{u}(t))\ dt = a_h,\ \forall h\in\mathsf{C}_1(\Omega).\tag{20}$$

The key to modified measure theoretical(MMT) approach lies in establishing the integral form of constraints and however cost functional. Note that the requirement for using MMT approach is justified so far. To begin with, respecting to $\mathsf{p}\in\mathsf{P}_{ad}$ the functional

$$\Lambda_{\mathsf{p}}:F\to\int_{\mathsf{J}}F(t,\mathbf{x}(t),\mathbf{u}(t))\ dt,\quad \forall F\in\mathsf{C}(\Omega),\tag{21}$$

defines a positive linear functional on $\mathsf{C}(\Omega)$, the space of all bounded continuous functions on Ω. Based on the proposed positive linear functional and the injective property of the mapping $\mathsf{p}\mapsto\Lambda_{\mathsf{p}}$ from P_{ad} into $\mathsf{C}^*(\Omega)$, Problem C may be considered on the dual space of $\mathsf{C}(\Omega)$, in words, $\mathsf{C}^*(\Omega)$ instead of P_{ad}.

Associated with functional (21), functional representation of problem COCI described by (16)-(20), becomes a new problem as follows.

Problem COCIF: Minimize

$$\Lambda_{\mathsf{p}}(\mathsf{L}_0),\tag{22}$$

subject to

$$\Lambda_{\mathsf{p}}(\mathsf{L}_i) = 0,\quad i=1,...,N_c,\tag{23}$$

$$\Lambda_{\mathsf{p}}(\mathsf{L}_i) \le 0,\quad i=N_c+1,...,N,\tag{24}$$

$$\Lambda_{\mathsf{p}}(\varphi^{\mathbf{f}}) = \triangle\varphi,\quad \forall\varphi\in\mathsf{C}^1(\mathsf{B}),\tag{25}$$

$$\Lambda_{\mathsf{p}}(\Phi_\psi) = \mathbf{0},\quad \forall\psi\in\mathsf{D}(\mathsf{J}^\circ),\tag{26}$$

$$\Lambda_{\mathsf{p}}(h) = a_h,\quad \forall h\in\mathsf{C}_1(\Omega).\tag{27}$$

Linear functional Λ_{p} can be uniquely defined in term of a positive Radon measure such that

$$\Lambda_{\mathsf{p}}(F)=\int_{\mathsf{J}}F\ dt=\int_\Omega F\ d\mu\equiv\mu(F),\quad \forall F\in\mathsf{C}(\Omega),\tag{28}$$

This result is a direct corollary of Riesz' representation theorem. In conjunction with positive Radon measure given by (28), problem COCIF is stated in the sense of measure exhibition.

Problem COCIM: Minimize

$$\mu(\mathsf{L}_0), \tag{29}$$

subject to

$$\begin{aligned} \mu(\mathsf{L}_i) &= 0, \quad i=1,...,N_c, && (30)\\ \mu(\mathsf{L}_i) &\leq 0, \quad i=N_c+1,...,N, && (31)\\ \mu(\varphi^{\mathbf{f}}) &= \triangle\varphi, \quad \forall\varphi\in \mathsf{C}^1(\mathsf{B}), && (32)\\ \mu(\Phi_\psi) &= \mathbf{0}, \quad \forall\psi\in \mathsf{D}(\mathsf{J}^\circ), && (33)\\ \mu(h) &= a_h, \quad \forall h\in \mathsf{C}_1(\Omega), && (34) \end{aligned}$$

where μ belongs to the space of all positive Radon measures on Ω, denoted by $\mathsf{M}^+(\Omega)$.
Let Q be a subset of $\mathsf{M}^+(\Omega)$ whose elements satisfy (30)-(34). If $\mathsf{M}^+(\Omega)$ is to be equipped by weak*-topology, as follows from Alaoghlu theorem one can prove that Q is a compact set. In the sense of this topology, the functional $\mathsf{I}:\mathsf{Q}\rightarrow R$ defined by $\mathsf{I}(\mu)=\mu(\mathsf{L}_0)$ is a linear continuous functional on the compact set Q. In fact, the functional I has at least a minimum on Q.

Problem COCIM is an infinite-dimensional linear programming(LP) problem. It is proceed with making up an approximate finite-dimensional LP problem whose optimal solution converges to minimizer of problem COCIM.
Let $\{\varphi_k,\ k=1,2,...\}$, $\{\psi_j,\ j=1,2,...\}$ and $\{h_s,\ s=1,2,...\}$ be sets of total functions in $\mathsf{C}^1(\mathsf{B})$, $\mathsf{D}(\mathsf{J}^\circ)$ and $\mathsf{C}_1(\Omega)$, respectively. If $\mathsf{Q}(M_1,M_2,L)$ denotes the subset of $\mathsf{M}^+(\Omega)$ containing of all measures which satisfy

$$\begin{aligned} \mu(\mathsf{L}_i) &= 0, \quad i=1,...,N_c, && (35)\\ \mu(\mathsf{L}_i) &\leq 0, \quad i=N_c+1,...,N, && (36)\\ \mu(\varphi_k^{\mathbf{f}}) &= \triangle\varphi_k, \quad k=1,2,...,M_1, && (37)\\ \mu(\Phi_{\psi_j}) &= \mathbf{0}, \quad j=1,2,...,M_2, && (38)\\ \mu(h_s) &= a_{h_s}, \quad s=1,2,...,L, && (39) \end{aligned}$$

Then, if M_1, M_2 and L tend to infinity

$$\{\eta_{(M_1,M_2,L)}=\inf_{\mathsf{Q}_{(M_1,M_2,L)}}\mu(\mathsf{L}_0)\}$$

converges to $\eta=\inf_{\mathsf{Q}}\mu(\mathsf{L}_0)$.

In what follows, purpose is to characterize optimal measure, say, μ^* in the space $\mathsf{Q}(M_1,M_2,L)$ at which $\mathsf{I}(\mu)=\mu(\mathsf{L}_0)$ taken minimum value.
As follows from Theorem A.5 of [3] and from [4], measure $\mu^*\in\mathsf{Q}(M_1,M_2,L)$ the minimizer of $\mathsf{I}(\mu)=\mu(\mathsf{L}_0)$ has the form

$$\mu^*=\sum_{r=1}^{M_1+M_2+L}\alpha_r^*\delta_{(z_r^*)}, \tag{40}$$

where $z_r^*\in\Omega$ and α_r^* for $r=1,2,...,M_1+M_2+L$, are non-negative coefficients. In the above formula $\delta_{(z_r^*)}$ is a unitary atomic measure defined by

$$\delta_{(z)}(F)=F(z), \quad \forall F\in\mathsf{C}(\Omega). \tag{41}$$

Consider the finite-dimensional optimization problem with objective functional $\mathsf{I}(\mu)=\mu(\mathsf{L}_0)$ and constraints (35)-(39). If μ in the latter optimization problem is substituted by μ^* defined by (40), then, the recent optimization problem is a non-linear problem because there exist unknown coefficients α_r^* and supports z_r^* for $r=1,2,...,M_1+M_2+L$. It is convenient to be focused on a LP problem taking into account only α_r^*, however, it will be done by taking fixed and determined points z_r approximating z_r^*, which z_r are chosen from a countable and dense subset of Ω.

There is a measure $\widehat{\mu}\in\mathsf{M}^+(\Omega)$ such that

$$|(\mu^*-\widehat{\mu})(\zeta_l)|<\epsilon, \quad l=0,1,...,N+1+M_1+M_2+L, \tag{42}$$

and $\widehat{\mu}$ has the form

$$\widehat{\mu}=\sum_{r=1}^{M_1+M_2+L}\alpha_r^*\delta_{(z_r)}, \tag{43}$$

where coefficients α_r^* are the same as in (40), $z_r\in\omega$ and $\{\zeta_l,\ l=0,...,1+N+M_1+M_2+L\}$ are $\{\mathsf{L}_i,\ i=0,...,N\}$, $\{\varphi_k^{\mathbf{f}},\ k=N+1,...,N+1+M_1\}$, $\{\Phi_{\psi_j},\ j=N+1+M_1,...,N+1+M_1+M_2\}$ and $\{h_s,\ s=N+1+M_1+M_2,...,N+1+M_1+M_2+L\}$.

Based on the concepts mentioned above, finite-dimensional LP problem may be constructed as follows:

Problem COCILP: Minimize

$$\sum_{r=1}^{\overline{N}}\alpha_r^*\mathsf{L}_0(z_r), \tag{44}$$

subject to

$$\begin{aligned} \sum_{r=1}^{\overline{N}}\alpha_r^*\mathsf{L}_i(z_r) &= 0, \quad i=1,...,N_c, && (45)\\ \sum_{r=1}^{\overline{N}}\alpha_r^*\mathsf{L}_i(z_r) &\leq 0, \quad i=N_c+1,...,N, && (46)\\ \sum_{r=1}^{\overline{N}}\alpha_r^*\varphi_k^{\mathbf{f}}(z_r) &= \triangle\varphi, \quad k=1,2,...,M_1, && (47)\\ \sum_{r=1}^{\overline{N}}\alpha_r^*\Phi_{\psi_j}(z_r) &= \mathbf{0}, \quad j=1,2,...,M_2, && (48)\\ \sum_{r=1}^{\overline{N}}\alpha_r^*h_s(z_r) &= a_h, \quad s=1,2,...,L, && (49)\\ \alpha_r^* &\geq 0, \quad r=1,2,...,\overline{N}, && (50) \end{aligned}$$

III. CONCLUSION

This paper proposes a new approach based on measure theory to solve a class of optimal control problems constrained by the canonical form of constraints. In this procedure the computations of the approximate optimal solution can be carried out easily by solving an LP problem which its optimal solution approximates the one of original optimal control problem.

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