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A modification on Newton's method for solving systems of nonlinear equations

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***Abstract*—**In this paper, we are concerned with the further study for system of nonlinear equations. Since systems with inaccurate function values or problems with high computational cost arise frequently in science and engineering, recently such systems have attracted researcher's interest. In this work we present a new method which is independent of function evolutions and has a quadratic convergence. This method can be viewed as a extension of some recent methods for solving mentioned systems of nonlinear equations. Numerical results of applying this method to some test problems show the efficiently and reliability of method

I. Introduction

CONSIDER a system of nonlinear equations

$$\begin{cases} f_1(x_1, x_2, \ldots x_n) = 0, \\ f_2(x_1, x_2, \ldots x_n) = 0, \\ \vdots \\ f_n(x_1, x_2, \ldots x_n) = 0. \end{cases} \quad (1)$$

This system can be referred by $F(x) = 0$, where $F = (f_1, f_2, \ldots, f_n) : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable on an open neighborhood $D^* \subset D$ of a solution $x^* = (x_1^*, \ldots, x_n^*) \in D$ of the system (1). Each function f_i maps a vector $x = (x_1, x_2, \ldots, x_n)$ from the n-dimensional space $\mathbb{R}^n$ into $\mathbb{R}$. We assume that the system (1) admits a unique solution. The most known iterative method for solving systems of nonlinear equations is the classical Newton's method, given by

$$x^{p+1} = x^p - F'(x^p)^{-1} F(x^p), \; p \geq 0 \quad (2)$$

where $F'(x^p)$ denote the Jacobian matrix at the current approximation $x^p = (x_1^p, x_2^p, \ldots, x_2^p)$ and x^{p+1} is the next approximation. In general, there exists no method that yields an exact solution for such equations. In recent years, considerable interest in system of nonlinear equation has been stimulated due to their numerous applications in the areas of science and engineering [1]and many powerful methods have been presented [6–29]. For example by using essentially Taylor's polynomial [1,2], decomposition [3], quadrature formulas [4,5] and other techniques [6-10].

In real life applications, there exist many problems where the system is known with some precision only, e.g. when the function and derivative values depend on the results of numerical simulations [11] or the precision of the desired function is available at a prohibitive cost, for example where function value results from the sum of an infinite series (e.g. Bessel or Airy functions [12,13,14]). So, it is very important to obtain methods, which are function evaluations free. These methods are ideal for situations with unavailable accurate function values or high computational cost. In this direction, several methods have been proposed for example in[15] a method proposed which applied for polynomial only.

Also, there are some methods where, the function values in Newton's method are not directly evaluated from the corresponding component functions $f_i(x)$, but are approximated by using appropriate quantities, which called WFEN method [16] and IWFEN [17].

This paper is structured as follows. In Section 2, a brief outline of the WFEN, IWFEN methods for systems of nonlinear equations has discussed. In Section 3 we modify some presented notations, in [16-17] and by using a geometrical interpretation, a method which can be viewed as a new improved Newton's method without direct function evaluations is presented. Some numerical examples are stated in Section 4 and a comparison between proposed method and WFEN, IWFEN, Newton's methods on these examples is given. Finally, Conclusions are drawn in Section 5.

II. WFEN and IWFEN Methods

For $i = 1, 2, \ldots, n$, $p = 1, 2, \ldots$ and by using the point $x^p = (x_1^p, \ldots, x_{n-1}^p, x_n^p)$ the pivot points has been defined in [16-18], as

$$x_{Pivot}^{p,i} = (x_1^p, \ldots, x_{n-1}^p, x_n^{p,i}) \equiv (y^p, x_n^{p,i}) \quad (3)$$

Such points have the same $n-1$ components with the point $x^p = (x_1^p, \ldots, x_{n-1}^p, x_n^p)$ and differ only at the n-th component. These points have imposed lying on the solution surfaces of the corresponding functions $f_i(x)$, that is

$$f_i(x_{Pivot}^{p,i}) = 0 \quad (4)$$

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Hence, the unknown n -th component $x_n^{p,i}$ of pivot points can be found by solving each of the corresponding one-dimensional equations

$f_i(x_{Pivot}^{p,i}) = 0$

Implicit Function Theorem [2], guaranties the existence the unique mappings φ_i such that $x_n = \varphi_i(y), f_i(y;\varphi_i(y)) = 0$ and $x_n^{p,i} = \varphi_i(y^p)$.

Based on the definition of pivot points, the method namely WFEN (Without direct Function Evaluations Newton), is given by

$$x^{p+1} = x^p - F'(x^p)^{-1} W(x^p), \quad p \ge 0. \tag{5}$$

Where $w_i(x^p) = \partial_n f_i(x_{Pivot}^{p,i})(x_n^p - x_n^{p,i})$. (for more details refer to [16]).

The other iterative scheme namely IWFEN (Improved Without direct Function Evaluations Newton), As a modification of method (5) has presented, is given by

$$x^{p+1} = x^p - L'(x^p)^{-1} L(x^p), \quad p \ge 0. \tag{6}$$

Where

$$L(x^p) = \begin{pmatrix} x_n^p - \varphi_1(y^p) \\ x_n^p - \varphi_2(y^p) \\ \vdots \\ x_n^p - \varphi_n(y^p) \end{pmatrix},$$

$$L'(x^p) = \begin{pmatrix} \frac{\partial_1 f_1(x^p)}{\partial_n f_1(x^p)} & \frac{\partial_2 f_1(x^p)}{\partial_n f_1(x^p)} & \cdots & \frac{\partial_{n-1} f_1(x^p)}{\partial_{n-1} f_1(x^p)} & 1 \\ \frac{\partial_1 f_2(x^p)}{\partial_n f_2(x^p)} & \frac{\partial_2 f_2(x^p)}{\partial_n f_2(x^p)} & \cdots & \frac{\partial_{n-1} f_2(x^p)}{\partial_{n-1} f_2(x^p)} & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\partial f_n(x^p)}{\partial_n f_n(x^p)} & \frac{\partial_2 f_n(x^p)}{\partial_n f_n(x^p)} & \cdots & \frac{\partial_{n-1} f_n(x^p)}{\partial_{n-1} f_n(x^p)} & 1 \end{pmatrix}$$

also $\varphi_i(y^p)$'s are the mappings were mentioned above.(for more details refer to [17]).

III. Derivation of the New Method

A. Some new definitions

To develop a new method, let us have the following definitions.

Definition 1. For any $i, j(i) \in \{1,2,\ldots,n\}$ and $p = 1,\ldots$, we define functions $g_{j(i)}^p : \mathbb{R} \to \mathbb{R}$ as

$g_{j(i)}^p(t) = f_i(y_{j(i)}^p, t)$

Where $y_{j(i)}^p = (x_k^p \mid k \in \{\{1,\ldots,n\} - \{j(i)\}\})$ and $j(i) \in \{1,2,\ldots,n\}$ introduce as followed in next subsection.

Definition2. For any $i, j(i) \in \{1,2,\ldots,n\}$ and $p = 1,\ldots$, we extend the notion of pivot points(3) as the following form

$$x_{Pivot,j(i)}^{p,i} = (x_1^p, x_2^p, \ldots, x_{j(i)}^{p,i}, \ldots, x_{n-1}^p, x_n^p) \tag{7}$$

Form $n-1$ components of current point x^p. The $j(i)$ -th unknown component $x_{j(i)}^{p,i}$ of modified pivot points can be found by solving each of the corresponding one dimensional equation

$$g_{j(i)}^p(t) = 0. \tag{8}$$

According to Implicit Function Theorem there exist unique mappings φ_i such that $x_{j(i)} = \varphi_i(y)$, $x_{j(i)} = \varphi_i(y), f_i(y;\varphi_i(y)) = 0$ and therefore

$x_{j(i)}^{p,i} = \varphi_i(y^p)$

In this paper, we use Newton's method using the initial guess $x_{j(i)}^p$ for solving each of the corresponding one-dimensional equations(8)(For simplicity, the Maple command *NewtonsMethod* can be used).

It is clear that the solution of (8) is depending on the expression of the components f_i and the current approach x^p. That is, if any of the Eq. (8) has no zeros, we are not able to apply our proposed method on a system of equations. Here, similar to what brought in [17], we can adopted some techniques to guarantee the existence of pivot points. For example choosing a different component for solving (8) as stated in [19], or applying either a reordering technique like in [20] or a linear combination between the components f_i like in [21]). For the needs of this work we consider that we are always able to find the zeros of (8) is possible.

B. Illustration of new method

The key idea in this paper is to define new quantities to approximate function values in Newton's method(2).

At the first, we use the first order's Taylor expansion of $g_{j(i)}^p(t)$ around the point $t = x_{j(i)}^p$ as

$$\begin{aligned} g_{j(i)}^p(t) &\approx g_{j(i)}^p(x_{j(i)}^p) \\ &+ \frac{dg_{j(i)}^p}{dt}(x_{j(i)}^p)(t - x_{j(i)}^p) \end{aligned} \tag{9}$$

Setting $t = x_{j(i)}^{p,i}$ at (9), based on the definitions of $g_{j(i)}^p(t)$, we have

$$\begin{aligned} g_{j(i)}^p(x_{j(i)}^{p,i}) &\approx f_i(x^p) \\ &+ \partial_{j(i)} f_i(x^p)(x_{j(i)}^{p,i} - x_{j(i)}^p) \end{aligned} \tag{10}$$

Due to the definition of pivot points and (7) and $g_{j(i)}^p(x_{j(i)}^{p,i}) = 0$, the above relation (10) becomes

$$f_i(x^p) \approx \partial_{j(i)} f_i(x^p)(x_{j(i)}^p - x_{j(i)}^{p,i}) \tag{11}$$

In Fig. 1, for any $j(i) \in \{1,2,\ldots,n\}$ we can see the behavior of function $g_{j(i)}^p(x)$ around the point $A = (x_{j(i)}^p, 0)$. Also we have the If we bring from the pivot point $B = (x_{Pivot,j(i)}^{p,i}, 0)$, the parallel line to the tangent of the function at the point $P = (x_{j(i)}^p, f_i(x^p))$, this line cuts the segment AP at the point $Q = (x_{j(i)}^p, \partial_{j(i)} f_i(x^p)(x_{j(i)}^{i,p} - x_{j(i)}^p))$.

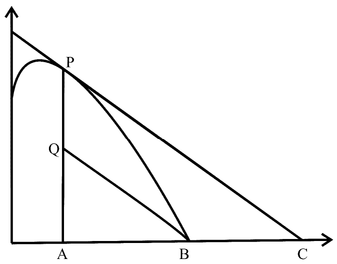

Fig.1 The behavior of function $g^{p}_{j(i)}(x)$

Also it can be easily verified that the coordinate of point C is $(x^{p}_{j(i)} - \frac{f_i(x^p)}{\partial_{j(i)} f_i(x^p)}, 0)$. From the similar triangles, the function value $f_i(x^p)$, denoted by the segment AP, can be approximated by the quantity $\partial_{j(i)} f_i(x^p)(x^{p,i}_{j(i)} - x^{p}_{j(i)})$, denoted by the segment AQ.suitable direction of $j(i)$.

Using similarity in triangles, it can be verified that whenever segment BC has a fewer length, the approximation $\partial_{j(i)} f_i(x^p)(x^{p,i}_{j(i)} - x^{p}_{j(i)})$ instead of $f_i(x^p)$ is more valid. So , we should choose that direction $j(i)$ which minimizes the expression.

$$|BC| = \left| x^{p,i}_{j(i)} - x^{p}_{j(i)} + \frac{f_i(x^p)}{\partial_{j(i)} f_i(x^p)} \right|$$

From triangular inequality, we have

$$|BC| = \left| x^{p,i}_{j(i)} - x^{p}_{j(i)} + \frac{f_i(x^p)}{\partial_{j(i)} f_i(x^p)} \right| \le \left| x^{p,i}_{j(i)} - x^{p}_{j(i)} \right| + \left| \frac{f_i(x^p)}{\partial_{j(i)} f_i(x^p)} \right|$$

It is clear that, the expression $\left| \frac{f_i(x^p)}{\partial_{j(i)} f_i(x^p)} \right|$ minimizing whenever the numerator expression achieves its maximum value. Hence, in this paper we set $j(i) = J$ when $\left|\partial_J f_i(x^p)\right| \ge \left|\partial_k f_i(x^p)\right|$, for any $k \in \{1,\ldots,n\}$, i.e. that direction which has to steepest slope of the gradient vector at the point x^p.

Now, using (11) in Newton method (2), we have

$$\widehat{V}(x^p)\widehat{L}(x^p) + F'(x^p)(x - x^p) = 0. \qquad (12)$$

Where

$$\widehat{V}(x^p) = \begin{pmatrix} \partial_{j(1)} f_1(x^p) & 0 & \ldots & 0 \\ 0 & \partial_{j(2)} f_2(x^p) & \ldots & 0 \\ \vdots & \ddots & \ldots & 0 \\ 0 & 0 & \ldots & \partial_{j(n)} f_n(x^p) \end{pmatrix},$$

$$\widehat{L}(x^p) = \begin{pmatrix} x^{p}_{j(1)} - \varphi_1(y^p) \\ x^{p}_{j(2)} - \varphi_2(y^p) \\ \vdots \\ x^{p}_{j(n)} - \varphi_n(y^p) \end{pmatrix}$$

Under the assumptions of Implicit Function Theorem the diagonal matrix $\widehat{V}(x^p)$ is invertible and (12) becomes

$$\widehat{V}(x^p)^{-1} F'(x^p)(x - x^p) = -\widehat{L}(x^p). \qquad (13)$$

Now, we consider the function

$$\widehat{L}(x) = \left(x_{j(1)} - \varphi_1(y), \ldots, x_{j(n)} - \varphi_n(y) \right)^t \qquad (14)$$

Utilizing again the Implicit Function Theorem to derive $\partial_{j(i)} \varphi_i(x)$ we get

$$(\widehat{L'}(x)^{-1})_{k,m} = \frac{\partial_m f_k(x)}{\partial_{j(k)} f_k(x)} \qquad (15)$$

Eqs. (14) and (15) introduce iterative method given by

$$x^{p+1} = x^p - \widehat{L'}(x^p)^{-1} \widehat{L}(x^p), \quad p \ge 0. \qquad (16)$$

We will refer to this iteration iterative scheme (16) as BGM method, as the modified Newton method to solve systems of nonlinear equations.

we have a following convergence theorem that shows the order of convergence of the proposed method(16).

Theorem 1. Suppose that $F = (f_1, f_2,, f_n) : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is twice continuously differentiable on an open neighborhood $D \subset D^*$ of a point $x^* = (x^*_1, x^*_2, \ldots, x^*_n) \in D$ for which $F(x^*_n) = 0$ and $F'(x^*_n)$ is nonsingular. Then the iterations $x^p, p = 1, 2, \ldots$ of the new method, given by (16) will converge to x^* provided the initial guess $x_\circ$ is sufficiently close to x^*. Moreover the order of convergence will be two.

Proof. Refer to[17].

.

IV. Numerical Examples

In this section, some examples are presented to illustrate the efficiency of proposed iterative family. In order to compare the results, we take the same examples were presented in [16-18].

In Tables 1-4 we present the results obtained, for various initial points, by Newton's method and the schemes (5) , (6) and (16).

Example 1. The first system has two roots $r_1 = (0.1, 0.1, 0.1)$ and $r_2 = (-0.1, -0.1, -0.1)$.. It is given by

$$f_1(x_1, x_2, x_3) = x_1^3 - x_1 x_2 x_3 = 0$$
$$f_2(x_1, x_2, x_3) = x_2^2 - x_1 x_3 = 0$$
$$f_2(x_1, x_2, x_3) = 10 x_1 x_3 + x_2 - x_1 - 0.1 = 0$$

Example 2. The second example is

$$f_1(x_1, x_2, x_3) = x_1^3 - x_1 x_2 x_3 = 0$$
$$f_2(x_1, x_2, x_3) = x_2^2 - x_1 x_3 = 0$$
$$f_2(x_1, x_2, x_3) = 10 x_1 x_3 + x_2 - x_1 - 0.1 = 0$$

with the solution

$$r = (-0.9999001 \times 10^{-4}, -0.9999001 \times 10^{-4}, 0.9999001 \times 10^{-4})$$

Due to the definition of pivot points(3), It is clear that so called “WFEN” and “IWFEN” methods are sensitive to the order of place of unknowns in component functions $f_i(x)$. This is a clear shortcoming of this methods in some systems. This sensitivity are shown in the followings examples.

Example 3. Rewrite Example 1. by changing the rule of x_3, x_2, as

$$f_1(x_1,x_2,x_3) = x_1^3 - x_1x_2x_3 = 0$$

$$f_2(x_1,x_2,x_3) = x_3^2 - x_1x_2 = 0$$

$$f_2(x_1,x_2,x_3) = 10x_1x_2 + x_3 - x_1 - 0.1 = 0$$

According to example1., this system has the $r_1 = (0.1, 0.1, 0.1)$ and $r_2 = (-0.1, -0.1, -0.1)$ roots.

Example 4. Reconsider Example 2. by changing the rule of x_3, x_1, given by

$$f_1(x_1,x_2,x_3) = x_1x_3 - x_1e^{x_3^2} + 10^{-4} = 0$$

$$f_2(x_1,x_2,x_3) = x_3(x_3^2 + x_2^2) + x_2^2(x_1 - x_2) = 0$$

$$f_2(x_1,x_2,x_3) = x_2^3 + x_1^3 = 0$$

According to example2., this system has the same root $r = (0.9999001\times10^{-4}, -0.9999001\times10^{-4}, -0.9999001\times10^{-4})$.

In Tables 1 and 4, `IT' indicates the number of the iterations, `FE' the number of the function evaluations (including derivatives). Results were obtained by using Maple software via 30 digit floating point arithmetic (Digits:=30) and following stopping criteria

$$\|x^{k+1} - x^k\| + \|f(x^k)\| \le 10^{-14}$$

V. Conclusion

In this paper, a modification of some existing method for solving system of nonlinear equations are presented. This method is independent of function evaluation and can be used in some systems that function calculations are quite costly or can’t be done precisely. As seen in tables[1-4], the numerical results of proposed method are quite satisfactory and admit the geometrical explanations. in some cases the results of our are very acceptable and there is a sufficient reduction on the number of iterations and hence the proposed method look be a reliable refinement for Newton’s method. It can be viewed as an improvement and refinement of the Newton’s methods and some resent methods.

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TABLE I
COMPARISON DIFFERENT METHODS FOR EXAMPLE 1

x_1^0	x_1^0	x_1^0	Newton		WFEN		IMFEN		BGM	
			IT	FE	IT	FE	IT	FE	IT	FE
0.4	0.5	0.5	53	636	20	240	20	180	16	144
-4	-2	1	33	396	33	396	33	297	23	207
-1	-2	0.6	51	612	51	612	51	459	10	90
-1	-2	1	29	384	29	384	29	261	19	171
0.5	2	1	54	648	54	648	54	486	18	162
5	-2	-2	38	456	38	456	38	342	18	162
10	-2	-2	39	468	39	468	39	351	33	297

TABLE II
COMPARISON DIFFERENT METHODS FOR EXAMPLE 2

x_1^0	x_2^0	x_3^0	Newton		WFEN		IMFEN		BGM	
			IT	FE	IT	FE	IT	FE	IT	FE
2	2	2	42	504	38	456	38	342	12	108
-2	-2	-2	27	324	27	324	27	243	7	63
3	3	5	92	1104	43	516	18	162	24	216
4	4	4	73	876	26	312	26	234	27	243
0.5	0.5	0.5	46	552	32	384	32	288	7	63
1	1	5	37	444	37	444	37	333	11	99
-4	-1	-2	28	336	26	312	26	234	17	153

TABLE III

COMPARISON DIFFERENT METHODS FOR EXAMPLE 3

x_1^0	x_2^0	x_3^0	Newton		WFEN		IMFEN		BGM	
			IT	FE	IT	FE	IT	FE	IT	FE
0.4	0.5	0.5	53	636	Div	-	Div	-	16	144
-4	1.5	5	38	456	Div	-	Div	-	13	208
-1	6	-2	15	180	Div	-	Div	-	13	208
1	-2	-2	11	132	Div	-	Div	-	10	90
1	2	0.5	54	648	Div	-	Div	-	18	288
5	2	2	15	180	Div	-	Div	-	10	90
-2	-2	10	39	468	Div	-	Div	-	33	528

Div= Divergent

TABLE IV

COMPARISON DIFFERENT METHODS FOR EXAMPLE 3

x_1^0	x_2^0	x_3^0	Newton		WFEN		IMFEN		BGM	
			IT	FE	IT	FE	IT	FE	IT	FE
2	2	2	42	504	Div	-	Div	-	12	108
-2	-2	-2	27	324	46	552	46	414	7	63
3	5	3	92	1104	Div	-	Div	-	24	216
4	4	4	73	876	51	612	43	387	27	243
.5	.5	.5	46	552	Div	-	Div	-	7	63
1	1	5	37	444	Div	-	Div	-	11	99
-2	-1	-4	28	336	Div	-	Div	-	17	153

Div= Divergent