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A Method to Calculate Frenet Apparatus of the Curves in Euclidean-5 Space

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***Abstract*—In this paper, a method to calculate Frenet Apparatus of the curves in five dimensional Euclidean space is presented.**

***Keywords*—Classical Differential Geometry, Euclidean-5 space, Frenet Apparatus.**

I. INTRODUCTION

IT is safe to report that to the many important results in the theory of the curves in E^3 were initiated by G. Monge; and G. Darboux pioneered the moving frame idea. Thereafter, Frenet defined his moving frame and his special equations which play important role in mechanics and kinematics as well as in differential geometry (for more details see [2]). At the beginning of the twentieth century, A.Einstein's theory opened a door of use of new geometries. These geometries mostly have higher dimensions. In higher dimensional Euclidean space, researchers treated some of classical differential geometry topics [3], [4] and [6].

It is well-known that, if a curve differentiable in an open interval, at each point, a set of mutually orthogonal unit vectors can be constructed. And these vectors are called Frenet frame or moving frame vectors. The rates of these frame vectors along the curve define curvatures of the curves. The set, whose elements are frame vectors and curvatures of a curve, is called Frenet apparatus of the curves.

In [5], author presented a method to determine Frenet apparatus of the curves in E^4 in analogy with the method in E^3.

In this work, using vector product defined in [1], the method in E^5 is expressed. Thus, this classical topic is extended to the space E^5.

II. PRELIMINARIES

To meet the requirements in the next sections, here, the basic elements of the theory of curves in the space E^5 are briefly presented.(A more complete elementary treatment can be found in [4]).

Let $\vec{\alpha}: I \subset R \to E^5$ be an arbitrary curve in the Euclidean space E^5. Recall that the curve $\vec{\alpha}$ is said to be of unit speed (or parameterized by arclength function s) if $\langle \vec{\alpha}', \vec{\alpha}' \rangle = 1$, where $\langle .,. \rangle$ is the standard scalar (inner) product of E^5 given by

$$\langle \vec{a}, \vec{b} \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4 + a_5 b_5 \qquad (1)$$

for each $\vec{a} = (a_1, a_2, a_3, a_4, a_5)$, $\vec{b} = (b_1, b_2, b_3, b_4, b_5) \in E^5$. In particular, the norm of a vector $\vec{a} \in E^5$ is given by $\|\vec{a}\| = \sqrt{\langle \vec{a}, \vec{a} \rangle}$.

Denote by $\{\vec{V}_1(s), \vec{V}_2(s), \vec{V}_3(s), \vec{V}_4(s), \vec{V}_5(s)\}$ the moving Frenet frame along the unit speed curve $\vec{\alpha}$. Then the Frenet formulas are given by (see [4])

$$\begin{bmatrix} \vec{V}_1' \\ \vec{V}_2' \\ \vec{V}_3' \\ \vec{V}_4' \\ \vec{V}_5' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 & 0 \\ 0 & -k_2 & 0 & k_3 & 0 \\ 0 & 0 & -k_3 & 0 & k_4 \\ 0 & 0 & 0 & -k_4 & 0 \end{bmatrix} \begin{bmatrix} \vec{V}_1 \\ \vec{V}_2 \\ \vec{V}_3 \\ \vec{V}_4 \\ \vec{V}_5 \end{bmatrix}. \qquad (2)$$

The functions $k_1(s), k_2(s), k_3(s)$ and $k_4(s)$ are called, respectively, the first, the second, the third and the fourth curvature of the curve $\vec{\alpha}$. If $k_4(s) \neq 0$ for each $s \in I \subset R$, the curve $\vec{\alpha}$ lies fully in E^5. Recall that the unit sphere S^4 in E^5, centered at the origin, is the hypersurface defined by

$$S^4 = \{\vec{x} \in E^5 : \langle \vec{x}, \vec{x} \rangle = 1\}. \qquad (3)$$

In [1], with an analogous way in Euclidean 3-space, the author defines a vector product in E^5 with following definition.

A. Let $\vec{a} = (a_1, a_2, a_3, a_4, a_5)$, $\vec{b} = (b_1, b_2, b_3, b_4, b_5)$, $\vec{c} = (c_1, c_2, c_3, c_4, c_5)$ and $\vec{d} = (d_1, d_2, d_3, d_4, d_5)$ be vectors in E^5. The vector product of E^5 is defined with the determinat

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$$\vec{a} \wedge \vec{b} \wedge \vec{c} \wedge \vec{d} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 & \vec{e}_4 & \vec{e}_5 \\ a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \end{vmatrix},$$

where $\vec{e}_i$ for $1 \le i \le 5$ are coordinat direction (basis) vectors of E^5 which satisfies

$$\vec{e}_1 \wedge \vec{e}_2 \wedge \vec{e}_3 \wedge \vec{e}_4 = \vec{e}_5 \,,\ \vec{e}_2 \wedge \vec{e}_3 \wedge \vec{e}_4 \wedge \vec{e}_5 = \vec{e}_1 \,,$$
$$\vec{e}_3 \wedge \vec{e}_4 \wedge \vec{e}_5 \wedge \vec{e}_1 = \vec{e}_2 \,, \vec{e}_4 \wedge \vec{e}_5 \wedge \vec{e}_1 \wedge \vec{e}_2 = \vec{e}_3$$
$$\vec{e}_3 \wedge \vec{e}_2 \wedge \vec{e}_1 \wedge \vec{e}_5 = \vec{e}_4 \,.$$

Via this product it is safe to report that

$$\left\langle \vec{a}, \vec{a} \wedge \vec{b} \wedge \vec{c} \wedge \vec{d} \right\rangle = \left\langle \vec{b}, \vec{a} \wedge \vec{b} \wedge \vec{c} \wedge \vec{d} \right\rangle = \left\langle \vec{c}, \vec{a} \wedge \vec{b} \wedge \vec{c} \wedge \vec{d} \right\rangle$$
$$= \left\langle \vec{d}, \vec{a} \wedge \vec{b} \wedge \vec{c} \wedge \vec{d} \right\rangle = 0$$

III. The Method

Let $\vec{X} = \vec{X}(s)$ be an unit speed curve in E^5. Our aim is to determine formulas of the set $\left\{k_1(s), k_2(s), k_3(s), k_4(s), \vec{V}_1(s), \vec{V}_2(s), \vec{V}_3(s), \vec{V}_4(s), \vec{V}_5(s)\right\}$. To do this, first we write following derivatives. Here ‘ denotes derivative respect to s.

$$\vec{X}' = \vec{V}_1 \,. \tag{4}$$

$$\vec{X}'' = k_1 \vec{V}_2 \,. \tag{5}$$

$$\vec{X}''' = -k_1^2 \vec{V}_1 + k_1' \vec{V}_2 + k_1 k_2 \vec{V}_3 \,. \tag{6}$$

$$\begin{aligned} \vec{X}^{(IV)} &= -3k_1k_1'\vec{V}_1 + (k_1'' - k_1^3 - k_1k_2^2)\vec{V}_2 + (2k_1'k_2 + k_1k_2')\vec{V}_3 \\ &\quad + k_1k_2k_3\vec{V}_4. \end{aligned} \tag{7}$$

$$\vec{X}^{(V)} = (...)\vec{V}_1 + (...)\vec{V}_2 + (...)\vec{V}_3 + (...)\vec{V}_4 + k_1k_2k_3k_4\vec{V}_5 \,. \tag{8}$$

Taking the norm of both sides of (5), we have the first curvature as

$$\left\| \vec{X}'' \right\| = k_1(s) \,. \tag{9}$$

And therefore, we obtain $\vec{V}_2$

$$\vec{V}_2 = \frac{\vec{X}''}{k_1} \,. \tag{10}$$

Then, the inner product $\left\langle \vec{X}''', \vec{V}_3 \right\rangle$ gives us the second curvature as

$$k_2 = \frac{\left\langle \vec{X}''', \vec{V}_3 \right\rangle}{\left\| \vec{X}'' \right\|} \,. \tag{11}$$

To calculate $\vec{V}_3$, let us form following equation:

$$\left\| \vec{X}'' \right\|^2 .(\vec{X}''' + \left\| \vec{X}'' \right\|^2 .\vec{X}') - \left\langle \vec{X}'', \vec{X}''' \right\rangle .\vec{X}'' = k_1^3 k_2 \vec{V}_3 \,. \tag{12}$$

If we take the norm of both sides of (12), it yields that

$$\vec{V}_3 = \frac{\left\| \vec{X}'' \right\|^2 .(\vec{X}''' + \left\| \vec{X}'' \right\|^2 .\vec{X}') - \left\langle \vec{X}'', \vec{X}''' \right\rangle .\vec{X}''}{\left\| \left\| \vec{X}'' \right\|^2 .(\vec{X}''' + \left\| \vec{X}'' \right\|^2 .\vec{X}') - \left\langle \vec{X}'', \vec{X}''' \right\rangle .\vec{X}'' \right\|} \,. \tag{13}$$

Now, let us calculate vector product of $\vec{V}_1 \wedge \vec{V}_2 \wedge \vec{X}''' \wedge \vec{X}^{(IV)}$ according to frame $\left\{\vec{V}_1, \vec{V}_2, \vec{V}_3, \vec{V}_4, \vec{V}_5\right\}$. This expression follows that

$$\vec{V}_1 \wedge \vec{V}_2 \wedge \vec{X}''' \wedge \vec{X}^{(IV)} = k_1^2 k_2^2 k_3 \vec{V}_5 \,. \tag{14}$$

Using (14), we easily have $\vec{V}_5$ and third curvature of the curve $\vec{X} = \vec{X}(s)$, respectively,

$$\vec{V}_5 = \eta \frac{\vec{V}_1 \wedge \vec{V}_2 \wedge \vec{X}''' \wedge \vec{X}^{(IV)}}{\left\| \vec{V}_1 \wedge \vec{V}_2 \wedge \vec{X}''' \wedge \vec{X}^{(IV)} \right\|} \,, \tag{15}$$

$$k_3 = \frac{\left\| \vec{V}_1 \wedge \vec{V}_2 \wedge \vec{X}''' \wedge \vec{X}^{(IV)} \right\|}{\left[\left\langle \vec{X}''', \vec{V}_3 \right\rangle \right]^2} \,. \tag{16}$$

η in the expression (15) is taken ± 1 to make $+1$ determinant of $\left[\vec{V}_1, \vec{V}_2, \vec{V}_3, \vec{V}_4, \vec{V}_5\right]$ matrix. By this way, Frenet frame positively oriented. Using inner product of (8) and (15), we write fourth curvature of $\vec{X}(s)$ as

$$k_4 = \frac{\left\langle \vec{X}^{(V)}, \vec{V}_5 \right\rangle . \left[\left\langle \vec{X}''', \vec{V}_3 \right\rangle \right]^2}{\left\| \vec{V}_1 \wedge \vec{V}_2 \wedge \vec{X}''' \wedge \vec{X}^{(IV)} \right\|} \,. \tag{17}$$

And, finally, the vector product

$$\vec{V}_4 = \eta .\vec{V}_3 \wedge \vec{V}_2 \wedge \vec{V}_1 \wedge \vec{V}_5 \tag{18}$$

gives us fourth vector fileld of the Frenet frame. Thus, we calculated Frenet apparatus of the curve $\vec{X} = \vec{X}(s)$.

IV. Conclusion

Throughout the presented paper, one of classical topic in the theory of the curves in E^5 is treated. In the recent papers, although Frenet frame vectors and curvatures are defined, there was not a method to calculate all Frenet apparatus of a unit speed curve which lies fully in E^5. Here, using vector product, we give formulas of frame vectors (and therefore curvatures). We hope these results will be helpful to mathematicians who are specialized on mathematical modeling.

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