

# A Method for Modeling Flexible Manipulators: Transfer Matrix Method with Finite Segments

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**Abstract**—This paper presents a computationally efficient method for the modeling of robot manipulators with flexible links and joints. This approach combines the Discrete Time Transfer Matrix Method with the Finite Segment Method, in which the flexible links are discretized by a number of rigid segments connected by torsion springs; and the flexibility of joints are modeled by torsion springs. The proposed method avoids the global dynamics and has the advantage of modeling non-uniform manipulators. Experiments and simulations of a single-link flexible manipulator are conducted for verifying the proposed methodologies. The simulations of a three-link robot arm with links and joints flexibility are also performed.

**Keywords**—Flexible manipulator, transfer matrix method, linearization, finite segment method.

## I. INTRODUCTION

IN the last decades, significant efforts have been devoted to studying the flexible manipulators due to their higher operational speed and higher payload-to-weight ratio compared with conventional rigid manipulators. However, flexible manipulators produce considerable deformations and oscillations when operating at high speed. The flexible robots are continuous dynamic systems with an infinite number of degrees of freedom. Their dynamics are governed by nonlinear coupled, ordinary and partial differential equations. The exact solution of such systems does not exist. Therefore, the complex dynamic equations are truncated to some finite dimensional models. The modeling approaches in the literature are mainly classified into several categories [1]: Assumed Mode Method (AMM) [2], Finite Element Method (FEM) [3], [4], Lumped Mass Method (LMM) [5] and Finite Segment Method(FSM) [6].

In this paper, a computationally efficient method for modeling flexible robots is presented. This method is based on the Discrete Time Transfer Matrix Method (DT-TMM) incorporated with Finite Segment Method (FSM). The FSM was proposed in [7], assuming that a flexible beam is composed by a certain number of rigid segments connected by adjacent springs and dampers. Compared with FEM and AMM, the FSM has the simplest mathematical formulation. It has the advantage of modeling flexible multibody systems with geometrical nonlinearity and material nonlinearity [7]. Compared with LMM, it offers more accurate results considering the inertia of a body.

The Transfer Matrix Method was first proposed in the 1920s [8] used to solve one-dimension linear systems. Later in 1950, TMM was used on more general vibration studies

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[9]. Nevertheless, the classical Transfer Matrix Method could only deal with the elastic structure mechanics problems for one-dimensional linear systems. The vibration characteristics of linear multi-rigid-flexible systems and dynamics of general multi-body systems are beyond its applicability. To handle the problems, in the last decade, DT-TMM was developed, including linear TMM of multi-body system [10], DT-TMM of multi-body system [11] and TMM of controlled multi-body system [12]. DT-TMM was combined with AMM for applying on flexible systems in [13].

DT-TMM has several advantages that make it attractive regarding the multi-body rigid and flexible systems. First of all, the establishment of global dynamic equations for modeling a system is not needed. DT-TMM describes systems by multiplying corresponding transfer matrices of components. This property makes the characterization of a system simple, concise and straightforward. Another significant strength is that the orders of the matrices involve in the calculation always remain small regardless of the number of elements in the model, which significantly increases the computational speed.

In this work, a method combining DT-TMM with FSM is presented. This method combines the discretization modeling strategy of the Finite Segment Method and takes advantages of the computational efficiency of DT-TMM. Compared with AMM based DT-TMM, the FSM-DT-TMM is superior in modeling variable cross-section flexible links as well as in modeling multi-flexible manipulator with joint flexibility. In addition, the corresponding transfer matrices have more concise formulations, and fewer transfer matrices are needed to be defined. To validate this method, an experiment of a single flexible link is performed. The experimental results well agree with the numerical simulations. Moreover, this approach is extended to model a three-link manipulator with links and joints flexibility.

## II. DYNAMIC MODELING OF A MANIPULATOR WITH MULTIPLE FLEXIBLE LINKS AND FLEXIBLE JOINTS

The schematic of the structure of a planar manipulator with  $n$ -flexible links and flexible revolute joints is presented in Fig. 1. This serial manipulator is actuated by individual motors. No floating frame and no rotational matrix are needed in the proposed method.

The following assumptions are considered in developing the model of the  $n$ -link manipulator:

- The motion of the manipulator is assumed to be in the horizontal plane. Therefore, gravitational body force is not included, and deformations are only considered in the horizontal direction.



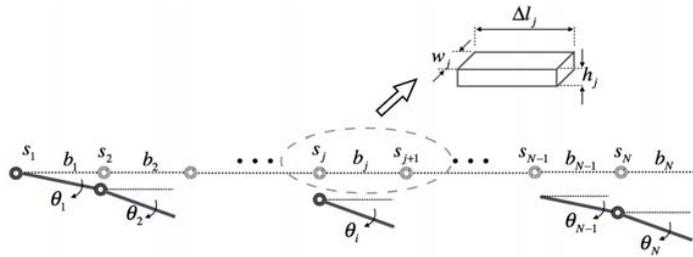


Fig. 2 Finite segment model of a flexible link

$A(t_{i-1}), B(t_{i-1}), C(t_{i-1})$  and  $D(t_{i-1})$  are known functions of other variables ( $r, \dot{r}, \ddot{r}$ ) at time  $t_{i-1}$ . Next, the Newmark- $\beta$  integration scheme [15] is applied to define  $A, B, C$  and  $D$ , as listed in Table I.

TABLE I  
COEFFICIENTS OF THE NEWMARK- $\beta$  INTEGRATION SCHEME

Constant	Linearized expression
$A$	$\frac{1}{\beta\Delta T^2}$
$B$	$\frac{1}{\beta\Delta T^2}[-r(t_{i-1}) - \dot{r}(t_{i-1})\Delta T - (\frac{1}{2} - \beta)\ddot{r}(t_{i-1})\Delta T^2]$
$C$	$\frac{\gamma}{\beta\Delta T}$
$D$	$\dot{r}(t_{i-1}) + (1 - \gamma)\ddot{r}(t_{i-1})\Delta T + \gamma B(t_i)\Delta T$

where  $\Delta T = t_i - t_{i-1}$  is the time interval,  $\beta$  and  $\gamma$  are the weighing parameters on the Newmark- $\beta$  integration and play a key role in the stability and convergence of analysis.

Generally, the geometric positions of a component in dynamic systems contain the trigonometric terms. Thus, the linearization of trigonometric terms are described using Taylor series expansion with truncation error of  $\Delta T^2$ , that is

$$\begin{aligned} \cos \theta(t_i) &= -\sin \theta(t_{i-1})\theta(t_i) + G_1 + o(\Delta T^2) \\ \sin \theta(t_i) &= \cos \theta(t_{i-1})\theta(t_i) + G_2 + o(\Delta T^2) \end{aligned} \quad (6)$$

where

$$\begin{aligned} G_1 &= \cos \theta(t_{i-1}) \left( 1 - \frac{1}{2} (\dot{\theta}(t_{i-1})\Delta T)^2 \right) \\ &\quad + \theta(t_{i-1}) \sin \theta(t_{i-1}) \\ G_2 &= \sin \theta(t_{i-1}) \left( 1 - \frac{1}{2} (\dot{\theta}(t_{i-1})\Delta T)^2 \right) \\ &\quad - \theta(t_{i-1}) \cos \theta(t_{i-1}) \end{aligned}$$

Thus,  $A, B, C, D, G_1$  and  $G_2$  are defined as functions of quantities from previous time step  $r(t_{i-1}), \dot{r}(t_{i-1}), \ddot{r}(t_{i-1})$ .

### B. Transfer Matrices of Elements Moving in Plane

The dynamics equations of the  $j$ th element can be linearized using the numerical integration, and then be assembled into a single transfer equation as expressed as (3). The moment momentum balance and mass center motion equation for a planar rigid body is given as in (7).

$$\begin{aligned} q_{x,I} - q_{x,O} + f_{x,C} &= \ddot{x}_C m \\ q_{y,I} - q_{y,O} + f_{y,C} &= \ddot{y}_C m \\ J_I \ddot{\theta}_I + m x_{IC} \ddot{y} - m y_{IC} \ddot{x} &= -M_I + M_O + M_C \\ + q_{x,O} y_{IO} - q_{y,O} x_{IO} - f_{x,C} y_{IC} + f_{y,C} x_{IC} \end{aligned} \quad (7)$$

where  $q_{x,I}, q_{y,I}$  are internal forces on inboard.  $q_{x,O}, q_{y,O}$  are internal forces on outboard.  $f_{x,C}, f_{y,C}$  are assumed external forces acting on the mass center.  $m$  is the mass of the rigid body and  $(x_C, y_C)$  is the mass center coordinate.  $q_{x,I}, q_{y,I}$  are internal forces acting on the inboard and  $q_{x,O}, q_{y,O}$  are internal forces acting on the outboard.  $M_I, M_O$  and  $M_C$  are the moments acting on the inboard, outboard and mass center, respectively.  $J_I$  is the rotation inertia about its inboard. Linearize the dynamic equations and the transfer equation of a planar rigid body can be written as

$$\vec{z}_O = U_{rigid} \vec{z}_I \quad (8)$$

where the  $\vec{z}_I$  and  $\vec{z}_O$  are its inboard and outboard state vectors. The transfer matrix  $U_{rigid}$  is expressed as

$$U_{Rigid} = \begin{bmatrix} 1 & 0 & u_{1,3} & 0 & 0 & 0 & u_{1,7} \\ 0 & 1 & u_{2,3} & 0 & 0 & 0 & u_{2,7} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ u_{4,1} & u_{4,2} & u_{4,3} & u_{4,5} & u_{4,6} & u_{4,7} & \\ u_{5,1} & 0 & u_{5,3} & 0 & 1 & 0 & u_{5,7} \\ 0 & u_{6,2} & u_{6,3} & 0 & 0 & 1 & u_{6,7} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (9)$$

The components of the transfer matrix are listed in [16].

The transfer matrix for a linear torsion spring is

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{K_t} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (10)$$

where  $K_t$  is the coefficient of the elasticity of the torsion spring.

The transfer matrix for a active joint is

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \theta_j^{joint} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (11)$$

where  $\theta_j^{joint}$  is the angle change provided by the joint.

As illustrated in Fig. ??, describing the system by assembling the transfer matrices of corresponding components from inboard to the outboard of the system, we obtain the overall transfer equation as

$$\vec{z}_O = U_{Link,n} U_{Joint,n} \cdots U_{Link,2} U_{Joint,2} U_{Link,1} U_{Joint,1} \vec{z}_I \quad (12)$$

where  $U_{Link}$  is transfer matrix for the flexible link, and  $U_{Joint}$  is transfer matrix for the flexible or rigid joint.

Combine the DT-TMM with FSM; a flexible link is modeled as serial multi-rigid links connecting by torsion springs. Hence, the corresponding transfer equation for a flexible link can be expressed as

$$U_{Link} = U_{b_N} U_{s_N} \cdots U_{b_1} \quad (13)$$

where transfer matrices of segment links  $U_b$  follow (9) and spring coefficient of segment torsion springs  $U_s$  are determined from (1).

The algorithmic solving procedure of FSM-DT-TMM is illustrated in a flow chart in Fig. 3. The entire process can be repeated until it reaches the desired computational time.

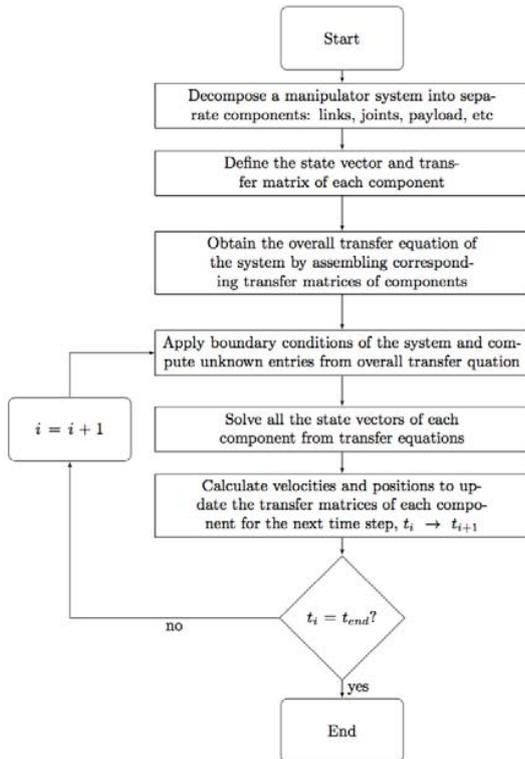


Fig. 3 Flow chart of the algorithms of solution procedures

#### IV. NUMERICAL SIMULATIONS AND EXPERIMENTS

In this section, numerical simulations and experiment tests are conducted to assess the validity of the proposed method. A uniform cross-section area manipulator is considered. Simulation and testing results from this case are compared and discussed. Furthermore, FSM-DT-TMM is extended to

model a three-link flexible manipulator. The influence of the joint flexibility and the manipulator vibration behavior is then examined.

##### A. Simulation and Testing of a Single Flexible Uniform Link Manipulator with Rigid Joint

The experimental set-up of a single flexible manipulator is shown in Fig. 4. The system was built up for experimental verification of the FMS-DT-TMM considering the single-link flexible manipulator with uniform cross-section area moves in the horizontal plane. This system was developed using NI Labview real-time module and CompactRIO for real-time control of the DC motor. This manipulator is composed of a thin aluminum beam with parameters listed in Table II and a DC motor. A strain gauge is mounted on the beam to find the strain and thus the deflection of the manipulator.

If flexibility in the shaft is not considered, the system could be modeled using two elements: An active rigid joint which provides the required rotational motion and a flexible link which can be divided into a certain number of rigid beams connected by torsion springs.

TABLE II  
PARAMETERS OF THE SINGLE-FLEXIBLE MANIPULATOR

Symbol	Value	Unit	Parameters
$L$	0.252	[m]	Length of beam
$h$	0.00048	[m]	Height of beam
$w$	0.028	[m]	Width of the beam
$E$	$7 \cdot 10^9$	[N/m <sup>2</sup> ]	Young's modulus
$\rho$	2766	[kg/m <sup>3</sup> ]	Density

As seen in Fig. 2, following (12) for one flexible link manipulator with a rigid joint, the overall transfer equation is expressed as

$$\vec{z}_O = U_{Link} U_{Joint} \vec{z}_I \quad (14)$$

$$U_{Link} = U_{b_N} U_{s_N} \cdots U_{b_1}$$

where  $U_{Link}$  is the transfer matrix of the flexible beam.  $U_s$  is the transfer matrix of the equivalent torsion spring, where its stiffness can be determined from (1).  $U_b$  is the transfer matrix of a segment rigid link and it can be obtained from (9). According to the definition of the state vectors and boundary conditions of the system, the inboard and outboard state vectors are

$$\vec{z}_I(t_i) = [0 \ 0 \ 0 \ m(t_i) \ q_x(t_i) \ q_y(t_i) \ 1]^T \quad (15)$$

$$\vec{z}_O(t_i) = [x(t_i) \ y(t_i) \ \theta(t_i) \ 0 \ 0 \ 0 \ 1]^T$$

where at the inboard, the position and orientation are known and set to be 0. At the outboard boundary, no external forces or moments acting on the tip of the manipulator. Hence the last three terms in  $\vec{z}_O$  are 0.

The dynamic model is simulated with integral parameters  $\beta = 0.7$  and  $\gamma = 0.8$  which add algorithmic damping into the simulation [17] for modeling the physical structure damping in the system. The time step is  $\Delta T = 0.002[s]$  which is the data sample time step of the DC motor. The flexible link is modeled by 60 segment rigid links.

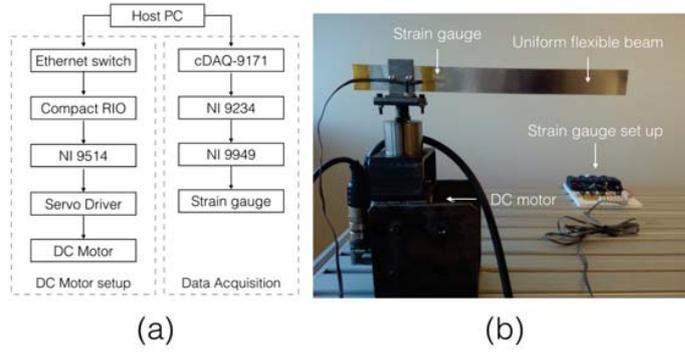


Fig. 4 Experimental setup of single flexible manipulator: (a)schematic diagram; (b)uniform cross-section area

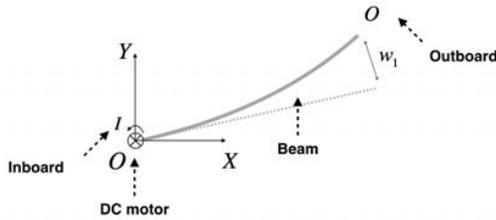


Fig. 5 Schematic of the single link flexible robot with uniform cross-section area, as shown in Fig. 4 (b)

Fig. 6 gives the comparison of results between simulation and experiment. The angular motion of the single flexible link manipulator is illustrated in Fig. 6 (a) and this angular motion is used as the signal input for the simulation. Fig. 6 (b) compares simulation and experimental results of the manipulator deflection responses. Fig. 6 (c) shows the comparison of the frequency response of the tip deflection using Fast Fourier Transform. The vibration behaviors of the system are clearly exposed in this plot, in which the main peaks for experiment and simulation are  $6.05[Hz]$  and  $6.16[Hz]$ , respectively. The frequency response results show that both the simulation and experiment are consistent in the theoretical natural frequency which is  $6.10[Hz]$ .

From the comparison of the results, one can note that the vibration behaviors in Fig. 6 (b) and the tip deformations in Fig. 6 (c) show good agreement. Furthermore, as seen in Fig. 6 (c), it indicates that the dynamics of the manipulator system are characterized by several vibration modes. During the rotating, the beam is subjected to the driven torques and considered to be forced vibration. After the motor stops, the residual vibration of the link is deemed to be free vibration with clamped-free boundary condition. With the given rotating angle, the numerical simulation accurately performs the vibration characterizations under different boundary conditions. It should be mentioned that the model of FMS-DT-TMM of this single flexible link manipulator is purely linear. The internal damping in the beam and joint friction are not included in the simulation while  $\beta = 0.8$  and  $\gamma = 0.7$  are used to introduce algorithmic damping. To sum up, the simulation results regarding the tip deformations and frequency responses are efficient and

accurate, and hence, it can be concluded that the proposed method is validated in modeling a single flexible manipulator.

### B. Simulation of a 3-Link Manipulator with Flexible Links and Joints Moving in Plane

In this case, we extended the applicability of the FMS-DT-TMM to a 3-flexible manipulator with flexible joints moves in the horizontal plane, as illustrated in Fig. 7.

The overall transfer equation for this system is expressed as

$$\begin{aligned} \tilde{z}_O &= U_{Link3}U_{Joint3}U_{Link2}U_{Joint2}U_{Link1}U_{Joint1}\tilde{z}_I \\ U_{Link,j} &= U_{bN,j}U_{sN,j} \cdots U_{b1,j}U_{m1,j} \\ U_{Joint,j} &= U_{K,j}U_{Motor,j} \end{aligned} \quad (16)$$

where  $U_K$  denote the flexibility of the joints. The driven angles for each joint are

$$\theta_{1,2,3} = \begin{cases} \frac{\pi}{12} \left( 1 - \cos\left(\frac{2\pi t}{t_0}\right) \right), & 0 \leq t \leq \frac{t_0}{2} \\ \frac{\pi}{12}, & \frac{t_0}{2} \leq t \end{cases} \quad (17)$$

where in this case,  $t_0 = 0.8s$  is the period of oscillation. Parameters for the simulation are tabulated in Table III.

Symbol	Value	Unit	Parameters
$L$	0.254	$[m]$	Length of each link
$h$	0.00508	$[m]$	Height of beam
$w$	0.00508	$[m]$	Width of beam
$I$	$wh^3/12$	$[m^4]$	Area moment of inertia
$E$	$71 \cdot 10^9$	$[N/m^2]$	Young's modulus
$\rho$	2710	$[kg/m^3]$	Density
$K_m$	100	$[N \cdot m/rad]$	Motor elasticity
$M_m$	0.04	$[kg]$	Mass of each joint

The effect of inertial damping in links have been neglected in this study, and the joints connected between links are considered frictionless.  $\gamma = 0.51$ ,  $\beta = 0.7$  and time step  $\Delta T = 0.001s$  are used in the simulation. The value of  $\gamma = 0.51$  introduces a relatively small algorithmic damping to make the simulation more stable with negligible influence on response amplitude.

The comparison results of the flexible manipulator with rigid and with flexible joints are shown in Figs. 8 and 9. The end point error in  $X$  and  $Y$  direction are illustrated in

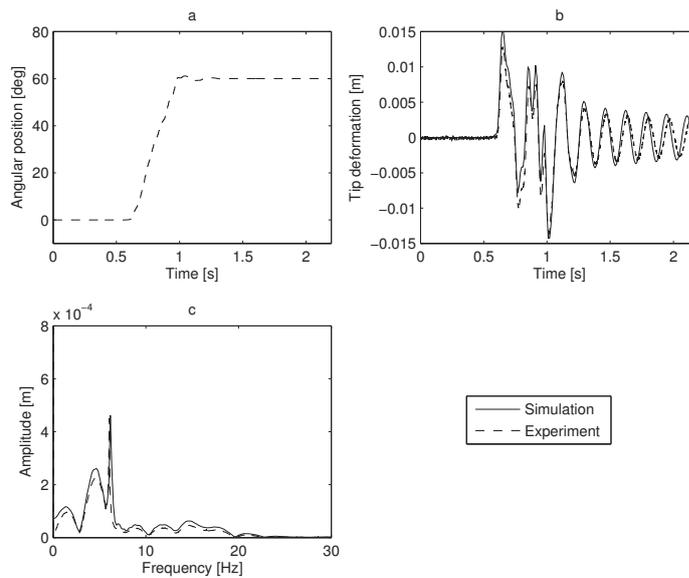


Fig. 6 Response of the manipulator: (a) angular position response of the manipulator; (b) end point deflection; (c) frequency response of the deflection

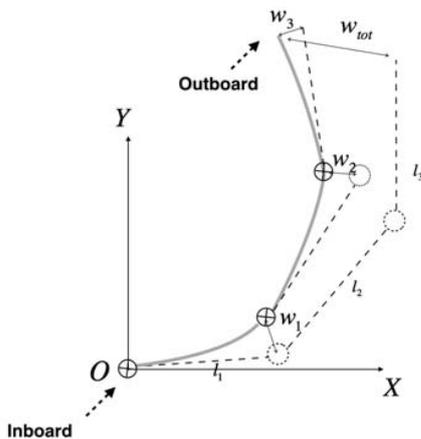


Fig. 7 The schematic of a three-link manipulator with link and joint flexibility

Fig. 8. It can be noted that the manipulator with flexible joints performs considerable larger end point error than that with rigid joints. In addition, the manipulator frequency response with flexible joints shows lower natural frequency. Consequently, one can see from Fig. 8 that, due to the joint flexibility, the manipulators exhibit more oscillatory behavior than those with rigid joints. In Figs. 9 (a)-(c), the deformations of the three links are compared with/without joint elasticity and in Fig. 9 (d), the links deformations are compared with each other with the existence of flexible joint.

The simulations present that the results considering and not considering the joint flexibility are quite different considering the endpoint error and link deflections. The results indicate that joint flexibility plays a important role in the dynamic behavior of manipulators with elasticity. Furthermore, the notable link

and joint deformation differences among one manipulator could provide the insights in design and controlling for manipulator mechanisms.

## V. CONCLUSIONS

In this paper, a computationally efficient method has been demonstrated to obtain the dynamic equations of motion for a generalized framework of an n-flexible link manipulator with joints elasticity. The proposed method is based on the Discrete Time Transfer Matrix Method and the Finite Segment Method, in which the flexible linkage is discretized by a certain number of rigid links connecting by adjacent torsion springs. This approach avoids the establishment of global dynamic equations by multiplying corresponding component transfer matrices. A single flexible link manipulator experiment was performed, and simulations of a three flexible link robot were presented. The sound agreement of results between experiments and simulations approved the applicability and accuracy of the proposed method.

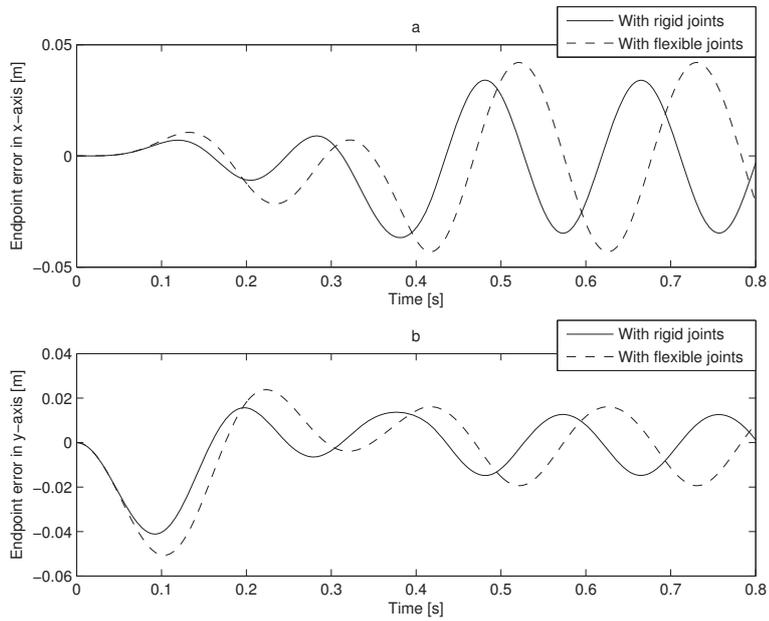


Fig. 8 The endpoint error in X direction (a) and Y direction (b)

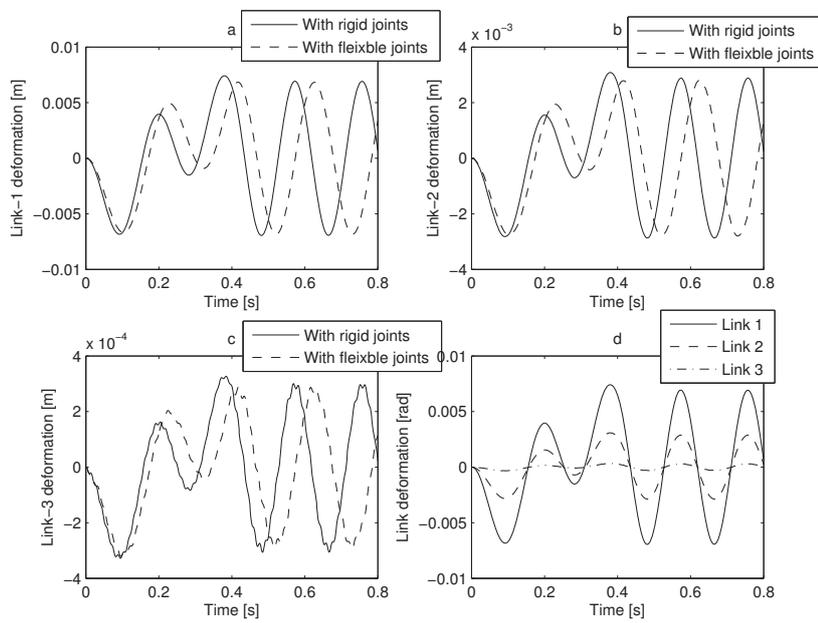


Fig. 9 Responses of the manipulator: (a) link-1 deformation; (b) link-2 deformation; (c)link-3 deformation; (d)joints deformations

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