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A Method for Identifying Physical Parameters with Linear Fractional Transformation

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***Abstract*—**This paper proposes a new parameter identification method based on Linear Fractional Transformation (LFT). It is assumed that the target linear system includes unknown parameters. The parameter deviations are separated from a nominal system via LFT, and identified by organizing I/O signals around the separated deviations of the real system. The purpose of this paper is to apply LFT to simultaneously identify the parameter deviations in systems with fewer outputs than unknown parameters. As a fundamental example, this method is implemented to one degree of freedom vibratory system. Via LFT, all physical parameters were simultaneously identified in this system. Then, numerical simulations were conducted for this system to verify the results. This study shows that all the physical parameters of a system with fewer outputs than unknown parameters can be effectively identified simultaneously using LFT.

***Keywords*—**Identification, Linear Fractional Transformation, Right inverse system

I. Introduction

AMONG model-based technologies such as control system design and state estimation, it is extremely important to acquire an accurate dynamics (model) of the target system. This research field is called "System Identification" and has been widely researched [1], [2]. Among them, there exist the cases which cannot be obtained accurate model of the target system because some or all parameters which specify the system property have uncertainties or are unknown; nevertheless the model structure of the target system is known beforehand. As focusing on these parametric uncertainties and unknowns, the method which identifies the unknown parameters directly is known as "Parameter Identification" [3], [4]. This paper deals with a parameter identification method which focuses on physical parameters such as masses, damping coefficients, and spring coefficients in dynamics.

These "structure known but physical parameters unknown" cases have been researched by Dasgupta and Anderson [5]. They reveal the parameterization of unknown physical components on structured systems and the method for identifying these unknown parameters. Demourant and Ferreres performed additional research of the identification for such the cases [6].

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They express the system to be identified as Linear Fractional Transformation (LFT) form. They also develop the scheme to identify the unknown parameters by pulling them out via LFT. This identification method is effective to the system with sufficient outputs, but not to the system with fewer outputs. Therefore, although the method performed in this paper is also based on LFT, we focus on the case which has fewer outputs than that of unknown parameters. The properties of this kind of system are more common.

Identification of unknown physical parameters takes important roles in large number of fields such as preventive healthcare, productive industries, fault diagnoses, and so on. For instance, on the predictive healthcare, capturing a physical condition (e.g., stiffness and viscosity) of affected area helps doctors to evaluate its severity as well as existence of tumors. Also in the productive industries, production costs can be reduced by the system which eliminates defective products by watching mass errors via the parameter identification method.

This paper addresses aforementioned physical parameter identification in terms of extraction of parameter deviations by means of LFT. Note that LFT is useful to the extraction for both of these two types of system; "all" the parameters are unknown, and "partial" parameters are unknown but the others are accurately known. This paper mainly discusses the farmer case. The main subject is the case that the number of outputs of the target Multi Input Multi Output (MIMO) system is less than the number of unknown parameters. This paper proposes a new method to identify all the physical parameters simultaneously in such cases. We emphasize that the method is superior to such things as follows:

1) We deal with the fewer outputs than unknown parameters.

2) We acquire the identified values in real time or little time delay with a time window.

3) The method effectively performs even if the target system has large parameter deviations; therefore we can roughly choose the nominal values.

In order to verify the effectiveness of this method, it is implemented to one degree of freedom vibratory system. Then, numerical simulations with MATLAB have been conducted for the system. The results show that the physical parameters of the system can be effectively identified simultaneously using LFT.

II. LFT Expression for Target System

A. System Description with Unknown Parameters via LFT

In this paper, suppose that the target system to be identified is limited to a class of Linear Time Invariant (LTI) system.

Therefore the target system obeys linear state equation below;

$$\begin{cases} \dot{x}_s(t) = Ax_s(t) + Bu(t) \\ y(t) = Cx_s(t) + Du(t) \end{cases} \tag{1}$$

where $x_s(t)\in R^n$ is a state variable vector, $u(t)\in R^m$ is an input vector, and $y(t)\in R^r$ is an output vector of the target system. The transfer function matrix therefore holds

$$G(s) = D + (sI - A)^{-1}B \equiv \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]. \tag{2}$$

Now we assume that the coefficient matrices (A, B, C, D) include unknown parameters p_i (i=1,…,k) which respectively consist of the nominal value p_{0i} and the parameter deviation δp_i. k represents the number of unknown parameters. Thus, these are assumed to be given by

$$p_i = p_{0i} + \delta p_i\,. \tag{3}$$

Here, we define the nominal system $G_0(s)$ as if all the parameter deviations are 0 in the system $G(s)$, i.e., δp_i =0. The nominal system is therefore given by

$$G_0(s) = \left[\begin{array}{c|c} A_0 & B_0 \\ \hline C_0 & D_0 \end{array}\right]. \tag{4}$$

Note that the transfer function matrix of the target system is assumed to be proper and asymptotically stable. Without loss of generality, we can choose nominal values of parameters such that the nominal system is proper and asymptotically stable.

To acquire further description of the system $G(s)$, Linear Fractional Transformation (LFT) is convenient [7]. In this paper, we utilize the lower LFT in order to pull out an influence of parameter deviations δp_i from the prior known information of the nominal system $G_0(s)$. Figure 1 exhibits a system representation via LFT. In Fig. 1, $M(s)$ denotes a generalized controlled object corresponding to $G(s)$ and $G_0(s)$. $M(s)$ is the known transfer function matrix which does not include any δp_i. Δ is a parameter deviation matrix which consists of δp_i. It is often assumed that Δ is diagonal. Therefore,

$$\Delta = \mathrm{diag}[\delta p], \delta p = [\delta p_1 \ \dots \ \delta p_k]^{\mathrm{T}}. \tag{5}$$

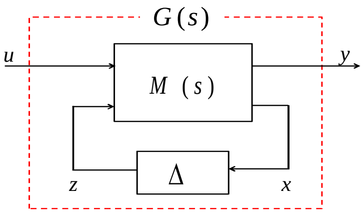

Fig. 1 LFT based system representation

In this paper, we assume that all the parameter deviations do not vary with time, hence have constant values. We also define the input and output (I/O) relationship around Δ as $x(s)$ and $z(s)$ in frequency domain, and $x(t)$ and $z(t)$ in the time domain. Because Δ is a diagonal matrix, x and z have k-dimensional elements. By these definitions, the generalized system is formulated as the following form:

$$\begin{cases} \begin{bmatrix} y(s) \\ x(s) \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} u(s) \\ z(s) \end{bmatrix} \\ z(s) = \Delta x(s) \end{cases} \tag{6}$$

Eliminating $x(s)$ and $z(s)$ from first equation of (6) by substituting second equation of (6) gives the relationship between $y(s)$ and $u(s)$; that is to say $G(s)$. Therefore $G(s)$ satisfies

$$G(s) = M_{11} + M_{12}\Delta(I - M_{22}\Delta)^{-1}M_{21}. \tag{7}$$

Equation (7) gives the definition of LFT with respect to the system $G(s)$. In (7), let all the parameter deviations be neglected, i.e., Δ=0, the nominal system $G_0(s)$ is given as the other form:

$$G_0(s) = M_{11} \tag{8}$$

Second term of (7) therefore represents a mismatch between the nominal system $G_0(s)$ and the real system $G(s)$. We next introduce the state space expression for the target system with pulling out parameter deviations via LFT, as shown in Fig. 1. By regarding the inputs for the target system as $[u(t)^{\mathrm{T}}\ z(t)^{\mathrm{T}}]^{\mathrm{T}}$, and the outputs as $[y(t)^{\mathrm{T}}\ x(t)^{\mathrm{T}}]^{\mathrm{T}}$ in (1), the state equation is given by considering (6), (8), and (4) as follows:

$$\begin{cases} \dot{x}_s(t) = A_0x_s(t) + B_0u(t) + B_1z(t) \\ y(t) = C_0x_s(t) + D_0u(t) + D_1z(t) \\ x(t) = C_1x_s(t) + D_2u(t) + D_3z(t) \\ z(t) = \Delta x(t) \end{cases} \tag{9}$$

Equation (9) also provides the state space expressions of each element of $M(s)$ in (6) such as;

$$\begin{cases} M_{11}(s) = \left[\begin{array}{c|c} A_0 & B_0 \\ \hline C_0 & D_0 \end{array}\right], M_{12}(s) = \left[\begin{array}{c|c} A_0 & B_1 \\ \hline C_0 & D_1 \end{array}\right] \\ M_{21}(s) = \left[\begin{array}{c|c} A_0 & B_0 \\ \hline C_1 & D_2 \end{array}\right], M_{22}(s) = \left[\begin{array}{c|c} A_0 & B_1 \\ \hline C_1 & D_3 \end{array}\right] \end{cases}. \tag{10}$$

B. Example of LFT Expression

This subsection discusses an example of the LFT expression of the target system. The example discussed here is a one degree of freedom (1-DOF) vibratory system as shown in Fig. 2. In the system of Fig. 2, the left side actuator which controls its position

x_1 and its velocity $\dot{x}_1$ excites the right side vibrator. Because the integral of $\dot{x}_1$ generates x_1, the input of the system is given as $u = \dot{x}_1$. We define the displacement of the vibrator as x_2, and its mass, damping coefficient, and spring coefficient as m, d, and k, respectively. For the outputs of the system, we select a force measured at the right edge of the actuator $F = d(\dot{x}_2 - \dot{x}_1) + k(x_2 - x_1)$ and the acceleration of the vibrator $\ddot{x}_2$. Therefore, the output vector is given as $y = \begin{bmatrix} F^{\mathrm{T}} & \ddot{x}_2^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$. The physical parameters m, d, and k respectively have the nominal values and the parameter deviations. Therefore, they satisfy the following condition and their deviations gives the parameter deviation vector corresponding to Δ:

$$\begin{aligned} & m = m_0 + \delta m, d = d_0 + \delta d, k = k_0 + \delta k \\ & \qquad \Rightarrow \delta p = \begin{bmatrix} \delta k & \delta d & \delta m \end{bmatrix}^{\mathrm{T}} \end{aligned} \tag{11}$$

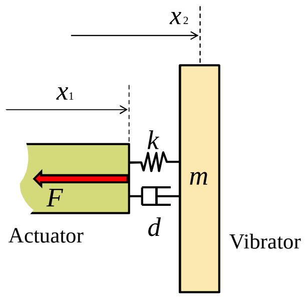

Fig. 2 One degree of freedom vibratory system

And now, we derive the transfer function matrix of generalized controlled object $M(s)$ for this system by extracting the parameter deviations Δ=diag[δk δd δm] via LFT. First of all, an equation of motion of the vibrator is calculated as

$$m\ddot{x}_2 + d(\dot{x}_2 - \dot{x}_1) + k(x_2 - x_1) = 0 . \tag{12}$$

Taking the state variable vector as $x_s = \begin{bmatrix} x_2 & \dot{x}_2 & x_1 \end{bmatrix}^{\mathrm{T}}$ provides the transfer function matrix of the target system $G(s)$ with the state space expression as

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} \begin{bmatrix} 0 & 1 & 0 \\ -\dfrac{k}{m} & -\dfrac{d}{m} & \dfrac{k}{m} \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ \dfrac{d}{m} \\ 1 \end{bmatrix} \\ \hline \begin{bmatrix} k & d & -k \\ -\dfrac{k}{m} & -\dfrac{d}{m} & \dfrac{k}{m} \end{bmatrix} & \begin{bmatrix} -d \\ \dfrac{d}{m} \end{bmatrix} \end{array}\right] . \tag{13}$$

Next procedure is to pull out the parameter deviations from $G(s)$. So as to complete it intuitively, we employ a block diagram for clarifying the deviations. Figure 3 shows the block diagram of the target system (13) with taking (11) into account.

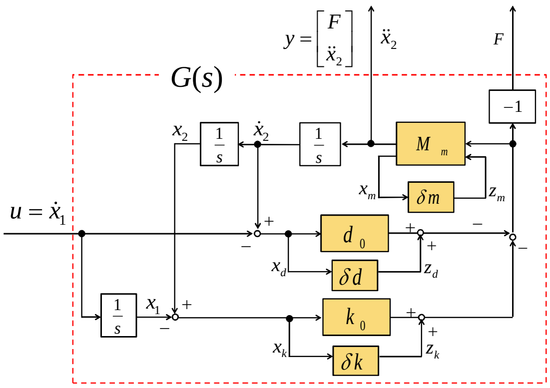

Fig. 3 Block diagram of 1-DOF vibratory system

In this regard, M_m in Fig. 3 is the generalized expression of $1/m$ along the LFT form. That is, equation (7) provides further description of $1/m$ as follows:

$$\frac{1}{m} = \frac{1}{m_0} + \left(-\frac{1}{m_0^2}\right)\delta m \left(1 - \left(-\frac{1}{m_0}\right)\delta m\right)^{-1} \tag{14}$$

Therefore, M_m satisfies below LFT form, and the I/O relationship around M_m is also organized as follows:

$$M_m = \begin{bmatrix} \dfrac{1}{m_0} & -\dfrac{1}{m_0^2} \\ 1 & -\dfrac{1}{m_0} \end{bmatrix} \tag{15}$$

$$\ddot{x}_2 = \frac{1}{m}(-F) \Leftrightarrow \begin{bmatrix} \ddot{x}_2 \\ x_m \end{bmatrix} = M_m \begin{bmatrix} -F \\ z_m \end{bmatrix} \tag{16}$$

In contrast, we can achieve the extractions of deviations δk and δd in Fig. 3 more simply. That is, summarizing the signals around δk and δd provides

$$\begin{cases} k(x_2 - x_1) = (k_0 + \delta k)(x_2 - x_1) \\ \qquad\qquad = k_0(x_2 - x_1) + \delta k(x_2 - x_1) \\ d(\dot{x}_2 - \dot{x}_1) = (d_0 + \delta d)(\dot{x}_2 - \dot{x}_1) \\ \qquad\qquad = d_0(\dot{x}_2 - \dot{x}_1) + \delta d(\dot{x}_2 - \dot{x}_1) \end{cases} . \tag{17}$$

This therefore gives the I/O signals relationship around deviations δk and δd as LFT form:

$$\begin{aligned} & x_k = x_2 - x_1, x_d = \dot{x}_2 - \dot{x}_1 \\ \Leftrightarrow & \begin{cases} \begin{bmatrix} k(x_2 - x_1) \\ x_k \end{bmatrix} = \begin{bmatrix} k_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 - x_1 \\ z_k \end{bmatrix} \\ \begin{bmatrix} d(\dot{x}_2 - \dot{x}_1) \\ x_d \end{bmatrix} = \begin{bmatrix} d_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_2 - \dot{x}_1 \\ z_d \end{bmatrix} \end{cases} \end{aligned} \tag{18}$$

In this way, it is verified that each of the deviations of unknown physical parameters and the organized deviations Δ can be pulled out from the target system $G(s)$ in (13) via LFT.

Now, let $x=[x_k, x_d, x_m]^T$ and $z=[z_k, z_d, z_m]^T$ be I/O signals around Δ. Substituting the condition of (11) ~ (18) into (10) leads the concrete description of $M(s)$ in (6) as follows:

$$
\begin{cases}
\left[\begin{array}{c|c} A_0 & B_0 \\ \hline C_0 & D_0 \end{array}\right] = \left[\begin{array}{c|c} \begin{bmatrix} 0 & 1 & 0 \\ -\frac{k_0}{m_0} & -\frac{d_0}{m_0} & \frac{k_0}{m_0} \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ \frac{d_0}{m_0} \\ 1 \end{bmatrix} \\ \hline \begin{bmatrix} k_0 & d_0 & -k_0 \\ -\frac{k_0}{m_0} & -\frac{d_0}{m_0} & \frac{k_0}{m_0} \end{bmatrix} & \begin{bmatrix} -d_0 \\ \frac{d_0}{m_0} \end{bmatrix} \end{array}\right] \\
B_1 = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{1}{m_0} & -\frac{1}{m_0} & -\frac{1}{m_0^2} \\ 0 & 0 & 0 \end{bmatrix}, C_1 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -k_0 & -d_0 & k_0 \end{bmatrix} \\
D_1 = \begin{bmatrix} 1 & 1 & 0 \\ -\frac{1}{m_0} & -\frac{1}{m_0} & -\frac{1}{m_0^2} \end{bmatrix}, D_2 = \begin{bmatrix} 0 \\ -1 \\ d_0 \end{bmatrix} \\
D_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & -1 & -\frac{1}{m_0} \end{bmatrix}
\end{cases} \tag{19}
$$

Additionally in this example, we choose the dimension of the outputs as r=2 which is less than that of unknown physical parameters; k=3. This means that this paper aims to identify all the physical parameter simultaneously with lower number of sensors.

III. Coordination of I/O Signals in LFT

A. Coordination of I/O Signals in LFT

The I/O signals x and z around Δ play important roles in identifying the parameter deviations Δ corresponding to δp_i. This is because the basic principle of the identification is that the second equation of (6) directly gives the simple identified values as:

$$
\delta p_i = z_i / x_i \tag{20}
$$

However, this may provide inaccurate values because of the noisy signals and the calculation errors of x and z. We therefore developed the method for the accurate identification with some compensation, and clarify the conditions to accomplish the proposed identification.

To begin with, we demonstrate the way to rewrite x and z as functions of y and u with use of (6). Then, simple deformation of (6) provides:

$$
\begin{cases} \hat{y}(s) = M_{12} z(s) \\ x(s) = M_{21} u(s) + M_{22} z(s) \end{cases} \tag{21}
$$

where $\hat{y}(s) \equiv y(s) - M_{11} u(s)$ is equivalent to an error between the real system output $y(s)$ and a calculated output from the nominal system $G_0(s)u(s)$. In the first equation of (21), it is clear that we can obtain z directly from y and u, if a transfer function matrix from $\hat{y}$ to z appropriately exist; that is to say an inverse system of M_{12}. The following subsections mainly discuss the inverse system in two cases, i.e., the case of $r \geq k$ and the case of $r < k$, and respectively design the parameter identifier each case.

B. Parameter Identifier Design (in the Case of r≥k)

Here, we consider the case in which $r \geq k$ in order to design the parameter identifier. In this case, the number of outputs is more than or equal to the number of unknown parameters. First of all, we discuss the inverse system of M_{12}. Now $D_1 \in R^{r \times k}$ in (9) is assumed to be full rank, i.e., rank(D_1)=k because $r \geq k$. D_1 therefore is row full rank. It can be said that this case has rich information to identify the unknown parameters. Following these conditions, first equation of (21) can be solved for $z(s)$, and has the unique solution which satisfies

$$
\begin{cases} z(s) = M_{12}^{+} \hat{y}(s) \\ M_{12}^{+} = \left[\begin{array}{c|c} A_0 - B_1 D_1^{+} C_0 & B_1 D_1^{+} \\ \hline -D_1^{+} C_0 & D_1^{+} \end{array}\right] \end{cases} \tag{22}
$$

where M_{12}^{+} denotes the left inverse system of M_{12}, which obeys $M_{12}^{+} M_{12} = I$. Similarly, D_1^{+} denotes a Moore-Penrose pseudo inverse matrix of D_1, which also obeys $D_1^{+} D_1 = I$. Substituting (22) into (21) therefore leads

$$
\begin{cases} z(s) = M_{12}^{+} \hat{y}(s) \\ x(s) = M_{21} u(s) + M_{21} M_{12}^{+} \hat{y}(s) \end{cases} \tag{23}
$$

This equation indicates that we can obtain x and z by calculation with y and u directly.

And that we can demonstrate the parameter identifier design with (23). For the accurate identification, we consider the other way to design the parameter identifier instead of (20). Let a criterion function be defined in the time domain as follows:

$$
J = \int_0^t \left\| z(\tau) - \Delta x(\tau) \right\|_2^2 d\tau \tag{24}
$$

The criterion function concerns about the least square of the estimated error of deviations based on second equation of (6). In comparison to (20), it is expected that the identifier based on

(24) smooths the signal noises practically. The target parameter deviations to be identified are regarded as Δ=diag[δp] which minimize the criterion function J. So as to obtain the minimizing solution, the equation (5) deforms (24) as follows:

$$J = \int_0^t \sum_{i=1}^{k} (z_i(\tau) - \delta p_i x_i(\tau))^2 d\tau \tag{25}$$

Differentiating (25) partially with δp_i, we obtain the necessary condition for the minimization of J by $dJ/d(\delta p_i)=0$. The following formula therefore represents the desired parameter identifier such that meets with aforementioned conditions:

$$\delta p_i(t) = \frac{\int_0^t z_i(\tau) x_i(\tau) d\tau}{\int_0^t x_i^2(\tau) d\tau} \tag{26}$$

C. Parameter Identifier Design (in the Case of r<k)

In contrast with the previous subsection, we consider the case in which $r < k$ in order to design the parameter identifier. In this case, the outputs are fewer than the unknown parameters. This condition is unfavorable to accomplish the identification. However, this case seems to be more general in comparison to the previous case. Thus, the identification method proposed in this subsection is the main objective of this paper.

As in the previous subsection, let us begin with the discussion of the inverse system of M_{12}. Now $D_1 \in R^{r\times k}$ in (10) is also assumed to be full rank. In contrast to the previous case, as rank(D_1)=r because $r < k$, D_1 is the column full rank matrix. Following this condition, there exists a non-zero z_0 which satisfies $M_{12} z_0 = 0$. That is, first equation of (21) cannot give the unique solution with respect to $z(s)$, but it can be solved with k-dimensional arbitrary vector ζ by

$$\begin{aligned} z(s) &= \hat{z}(s) + z_0(s) \\ &= M_{12}^{+} \hat{y}(s) + \left(I - M_{12}^{+} M_{12}\right)\zeta \end{aligned} \tag{27}$$

where

$$M_{12}^{+} = \left[\begin{array}{c|c} A_0 - B_1 D_1^{+} C_0 & B_1 D_1^{+} \\ \hline -D_1^{+} C_0 & D_1^{+} \end{array}\right] \tag{28}$$

denotes the right inverse system of M_{12}, which obeys $M_{12} M_{12}^{+} = I$. D_1^{+} also denotes a Moore-Penrose pseudo inverse matrix of D_1, which obeys $D_1 D_1^{+} = I$. Now, substituting (27) into the second equation of (21) holds

$$\begin{aligned} x(s) &= \hat{x}(s) + x_0(s) \\ &= \left[M_{21} u(s) + M_{22} M_{12}^{+} \hat{y}(s)\right] + M_{22}\left(I - M_{12}^{+} M_{12}\right)\zeta \end{aligned} \tag{29}$$

Although $\hat{x}$ in (29) can be obtained by calculation with y and u, there is no way to obtain x_0 directly. This indicates that the parameter deviations cannot be identified directly if x_0 remains as going on (29). Hence, the following lemma enables to manage this problem:

Lemma: Let B_1, D_1, and D_3 in (10) satisfy the following condition:

$$\begin{cases} \mathrm{Ker}(B_1) \supset \mathrm{Ker}(D_1^{+} D_1) \\ \mathrm{Ker}(B_1) = \mathrm{Ker}(D_3) \end{cases} \tag{30}$$

Then, it holds that

$$x_0(s) = M_{22}\left(I - M_{12}^{+} M_{12}\right)\zeta = 0 . \tag{31}$$

Proof: In (29), the condition of (10) gives

$$M_{12}^{+} M_{12} = \left[\begin{array}{cc|c} A_0 - B_1 D_1^{+} C_0 & B_1 D_1^{+} C_0 & B_1 D_1^{+} D_1 \\ 0 & A_0 & B_1 \\ \hline -D_1^{+} C_0 & D_1^{+} C_0 & D_1^{+} D_1 \end{array}\right]. \tag{32}$$

Consider a transformation of the state variables in (32) by means of below equivalence transformation:

$$T = \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix} \tag{33a}$$

Then,

$$\begin{aligned} M_{12}^{+} M_{12} &= \left[\begin{array}{cc|c} A_0 - B_1 D_1^{+} C_0 & 0 & -B_1 (I - D_1^{+} D_1) \\ 0 & A_0 & B_1 \\ \hline -D_1^{+} C_0 & 0 & D_1^{+} D_1 \end{array}\right] \\ &= D_1^{+} D_1 + D_1^{+} C_0 \left(sI - A_0 + B_1 D_1^{+} C_0\right)^{-1} B_1 (I - D_1^{+} D_1) \end{aligned} \tag{33b}$$

In this regard, the matrix $D_1^{+} D_1$ appearing on (33) satisfies

$$\left(D_1^{+} D_1\right)^2 = D_1^{+}\ D_1 D_1^{+}\ D_1 = \left(D_1^{+} D_1\right) \tag{34}$$

This therefore belongs to projection matrices [8]. As a property of projection matrices, it obeys

$$\mathrm{Ker}\left(D_1^{+} D_1\right) = \mathrm{Im}\ I - D_1^{+} D_1 \ . \tag{35}$$

That is, there exists a k-dimensional vector ξ corresponding to the k-dimensional arbitrary vector ζ, and they give

$$\xi = (I - D_1^{+} D_1)\zeta \Leftrightarrow \xi \in \mathrm{Ker}\left(D_1^{+} D_1\right). \tag{36}$$

Now, postmultiplying ζ for the second term of (33) gives

$$B_1(I - D_1^+ D_1)\zeta = B_1\xi ; \xi \in \mathrm{Ker}\left(D_1^+ D_1\right). \tag{37}$$

Where, the first equation of (30) indicates that

$$\xi \in \mathrm{Ker}\left(D_1^+ D_1\right) \Leftrightarrow B_1\xi = 0, B_1\left(I - D_1^+ D_1\right)\zeta = 0 . \tag{38}$$

Therefore, the second term of (33) is equal to 0, that is,

$$M_{12}^+ M_{12} = D_1^+ D_1 \tag{39}$$

As with the same scheme of acquiring (38), the following condition is indicated by the second equation of (30) and (38) that

$$\xi \in \mathrm{Ker}\left(D_1^+ D_1\right) \Leftrightarrow D_3\xi = 0, D_3\left(I - D_1^+ D_1\right)\zeta = 0 \tag{40}$$

Therefore by summarizing aforementioned results especially (31), (10), and (38) ~ (40), it follows that

$$\begin{aligned} x_0(s) &= M_{22}\left(I - M_{12}^+ M_{12}\right)\zeta \\ &= \left(D_3 + C_1\left(sI - A_0\right)^{-1} B_1\right)\left(I - D_1^+ D_1\right)\zeta \\ &= D_3\left(I - D_1^+ D_1\right)\zeta + C_1\left(sI - A_0\right)^{-1} B_1\left(I - D_1^+ D_1\right)\zeta \\ &= 0. \end{aligned}$$

Now the condition (30) is further considered with taking the 1-DOF vibratory system described in Subsec.2.B for example. At first, take a proper element $z_1 \in \mathrm{Ker}(B_1)$ in first condition of (30), and equation (19) gives a concrete description of $\mathrm{Ker}(B_1)$ as follows:

$$\begin{aligned} & B_1 z_1 = 0 \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ -1/m_0 & -1/m_0 & -1/m_0^2 \end{bmatrix} \begin{bmatrix} z_{11} \\ z_{12} \\ z_{13} \end{bmatrix} = 0 \\ & \Rightarrow z_1 = z_{12} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z_{13} \begin{bmatrix} -1/m_0 \\ 0 \\ 1 \end{bmatrix} \\ & \therefore \mathrm{Ker}(B_1) = \mathrm{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/m_0 \\ 0 \\ 1 \end{bmatrix} \right\} \end{aligned} \tag{41}$$

As with (41), take a proper element $z_2 \in \mathrm{Ker}(D_1^+ D_1)$, and a concrete description of $\mathrm{Ker}(D_1^+ D_1)$ is also given by

$$\begin{aligned} D_1^+ D_1 z_2 &= D_1^+ \begin{bmatrix} 1 & 1 & 0 \\ -1/m_0 & -1/m_0 & -1/m_0^2 \end{bmatrix} \begin{bmatrix} z_{21} \\ z_{22} \\ z_{23} \end{bmatrix} \\ &= \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0 \\ -m_0 & -m_0^2 \end{bmatrix} \begin{bmatrix} z_{21} + z_{22} \\ -\left(z_{21} + z_{22} + z_{23}/m_0\right)/m_0 \end{bmatrix} \\ &= \begin{bmatrix} 0.5(z_{21} + z_{22}) \\ 0.5(z_{21} + z_{22}) \\ z_{23} \end{bmatrix} = 0 \\ &\Rightarrow z_1 = z_{22} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \therefore \mathrm{Ker}(D_1^+ D_1) = \mathrm{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\} \end{aligned} \tag{42}$$

Therefore, the system in Subsec.2.B satisfies first condition of (30) via (41) and (42). The second condition of (30) can be verified with the same scheme to (41) and (42).

Nevertheless the sizes and the elements of B_1, D_1, and D_3 arbitrarily vary as how the unknown physical parameters exist, the relationship among these matrices' rows do not vary from those of (19) in such cases as identification of coefficients of linear ordinary differential equations. Therefore, it can be said that the condition (30) is widely satisfied among the cases of physical parameter identification.

From the above introduced lemma, $z(s)$ and $x(s)$ are summarized with (27), (29), and (31) as follows:

$$\begin{cases} z(s) = M_{12}^+ \hat{y}(s) + z_0(s) \\ x(s) = \hat{x}(s) = M_{21} u(s) + M_{22} M_{12}^+ \hat{y}(s) \end{cases} \tag{43}$$

The next part is definition of the criterion function. Similarly to x_0, z_0 in (43) cannot be calculated directly. Hence instead of J in (24), let a virtual criterion function $\hat{J}$ be defined as the least square of estimated error in the time domain:

$$\hat{J} = \int_0^t \sum_{i=1}^{k} (\hat{z}_i(\tau) - \delta\hat{p}_i x_i(\tau))^2 d\tau \tag{44}$$

This concerns about virtual estimated values of deviations with $\hat{x} = x$ and $\hat{z}$. The virtual parameter deviation, described as $\hat{\Delta} = \mathrm{diag}[\delta\hat{p}]$, can be obtained by minimization of the criterion function $\hat{J}$. Although $\delta\hat{p}$ does not represent the actual parameter deviations directly, we can obtain the parameter identifier in this case by the following theorem:

Theorem: Let the input vector of the target system $u(t)$ allow $x(t)$ and $z(t)$ to be orthogonal, that is

$$\int_0^t z_j(\tau) x_i(\tau) d\tau = 0, (i \neq j) \tag{45}$$

The parameter identifier for the case $r < k$ holds

$$\delta p_i(t) = \frac{1}{\left(D_1^+ D_1\right)_{ii}} \frac{\int_0^t z_i(\tau) x_i(\tau) d\tau}{\int_0^t x_i^2(\tau) d\tau} \tag{46}$$

Proof: Differentiating (44) partially with $\delta\tilde{p}_i$, we can obtain the necessary condition for the minimization by $d\hat{J}/d(\delta\tilde{p}_i)=0$. The virtual parameter identifier can therefore be formulated such that meets with the necessary condition as follows:

$$\delta\tilde{p}_i=\frac{\int_0^t \hat{z}_i(\tau)x_i(\tau)d\tau}{\int_0^t x_i^2(\tau)d\tau} \tag{47}$$

Now that, substituting the first equation of (21) into $\hat{z}(s)=M_{12}^{+}\hat{y}(s)$ with (39) gives following relationship;

$$\hat{z}(s)=M_{12}^{+}M_{12}z(s)=D_1^{+}D_1z(s)$$
$$\Rightarrow \hat{z}(t)=D_1^{+}D_1z(t). \tag{48}$$

This enables the integrand of the numerator in (47) to deform into a simple form below:

$$\hat{z}_i(t)x_i(t)=\sum_{j=1}^{k}\left(D_1^{+}D_1\right)_{ij}z_jx_i(t) \tag{49}$$

Therefore, the following condition is given by taking the condition of (45) into account and organizing (47), (49), and (26):

$$\delta\tilde{p}_i=\left(D_1^{+}D_1\right)_{ii}\frac{\int_0^t z_i(\tau)x_i(\tau)d\tau}{\int_0^t x_i^2(\tau)d\tau}=\left(D_1^{+}D_1\right)_{ii}\delta p_i$$
$$\Leftrightarrow \delta p_i=\frac{1}{\left(D_1^{+}D_1\right)_{ii}}\delta\tilde{p}_i \tag{50}$$

This formula is equivalent to (46).

IV. Numerical Consideration

A. Simulation Conditions

To evaluate the method proposed in previous section, we carry out a simulation with taking the 1-DOF vibratory system in Subsec.2.B for the example. The simulation aims to identify three physical parameter deviations δk, δd, and δm simultaneously. Particularly, we calculate time responses of the deviations based on (46) in order to accomplish a real-time identification. The input vector in this simulation is selected to sum of five different sine curves as shown in (51):

$$u(t)=\dot{x}_1=\sum_{i=1}^{5}\omega_iA_i\left\{\cos(\omega_it+\phi_i)\right\} \tag{51}$$

This is because the LFT signals satisfy the orthogonal condition of (45) in [0 t] and $u(t)$ have enough independent signal components for the identification. In addition, the initial values of the state variable vector $x_s=\left[x_2\ \ \dot{x}_2\ \ x_1\right]^{\mathrm{T}}$ in (13) are assumed to be 0, and the other simulation conditions such as parameter values are summarized in Table 1.

TABLE I
SPECIFICATION OF PARAMETERS FOR SIMULATION

Symbol	Parameter	Value	Unit
m_0	Nominal mass	2.0	kg
δm	Deviation of mass	0.60	kg
d_0	Nominal damping coefficient	0.20	N·s/m
δd	Deviation of damping coeff.	4.0×10^{-2}	N·s/m
k_0	Nominal spring coefficient	1.0	N/m
δk	Deviation of spring coeff.	0.10	N/m
A_i	Amplitude of input signals	1	m
ω_1,ω_2	Frequency of input signals 1, 2	0.1, 1.0	rad/s
ω_3,ω_4	Frequency of input signals 3, 4	10, 1.0×10^{2}	rad/s
ω_5	Frequency of input signal 5	2.0×10^{2}	rad/s
ϕ_i	Phase of input signals	0	rad
	Sampling interval	1	ms

B. Simulation results

Table 2 shows the results of mean values and rates of identification errors from actual deviation values with respect to the identification of parameter deviations δk, δd, and δm during 10 [s]. Figure 4, 5, and 6 also shows the time responses of the identified value of δk, δd, and δm, respectively.

These results indicate that the identifications for δd and δm are achieved with accuracy. In addition, these results hardly depend on time. Therefore, δd and δm can be effectively identified in real-time. However, Table 2 shows that there is 20 % of the error rate in average on the identification of δk. The time response Fig. 4 also indicates that the identified value of δk has large error on its initial phase. One of the primal reasons of this unfavorable result is that the orthogonal condition of (47) cannot be sufficiently accomplished in the transient phase after the initial input stimulation. The integral I_{21} with respect to the orthogonality between z_2 and x_1, which must be considered in the calculation of δk, holds

$$I_{21}=\int_0^t z_2(\tau)x_1(\tau)d\tau. \tag{52}$$

This I_{21} takes a certain amount of the value in early phase with the input (51). In contrast, because I_{21} converges to 0 in latter phase, the steady state value of δk achieves a better result as shown in Fig. 4. This is due to the property of the input signal component (51) as non-biased periodic function.

Now, in order to manage the influence of the undesirable condition on (52), we employ a time window method. That is, [t*-T t*] is chosen as the integral interval instead of [0 t] in (46), where T is the time window and t* is a time series to conduct the identification. Therefore, the time response with the time window holds

TABLE II
MEAN VALUES OF IDENTIFICATION RESULTS

Parameter	Mean Value	Accurate value	Error Rate [%]
δk	1.204×10^{-1}	0.10	20
δd	4.007×10^{-2}	4.0×10^{-2}	0.17
δm	6.000×10^{-1}	0.60	0.0

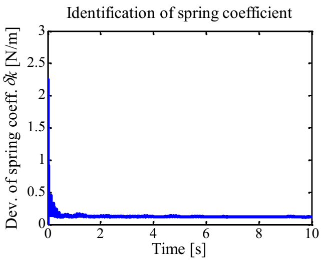

Fig. 4 Identification result of δk

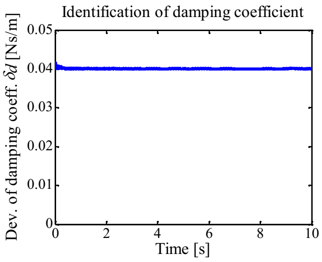

Fig. 5 Identification result of δd

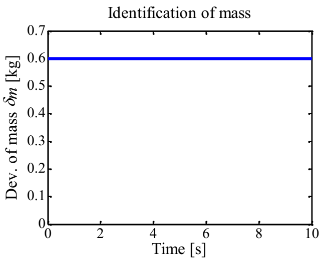

Fig. 6 Identification result of δm

$$\delta p_i(t^*) = \frac{1}{\left(D_1^+ D_1\right)_{ii}} \frac{\int_{*-T}^{*} z_i(\tau) x_i(\tau) d\tau}{\int_{*-T}^{*} x_i^2(\tau) d\tau}. \tag{53}$$

The calculation is conducted with taking the time window as T = 1.0 [s], and the identification interval is t*=[1.0 10].

In this case, the time responses of δd and δm show precision constant values. The time response of δk as shown in Fig. 7 also shows more accurate value and better convergence in comparison to the latter phase of Fig. 4. Table 3 exhibits the time average of identified values and rates of identification errors. Table 3 also indicates that the identifications for δd and δm remain those accuracies. In addition, the time window method improves the identified value of δk.

TABLE III
MEAN VALUES OF IDENTIFICATION RESULTS WITH TIME WINDOW

Parameter	Mean Value	Accurate value	Error Rate [%]
δk	1.052×10^{-1}	0.10	5.2
δd	4.000×10^{-2}	4.0×10^{-2}	0.0
δm	6.000×10^{-1}	0.60	0.0

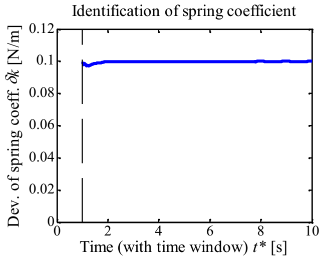

Fig. 7 Identification result of δk with time window

V. CONCLUSION

For physical parameter identification in a linear system, such as the one degree of freedom vibratory system defined in Subsec.2.B, this paper proposes a method for extracting the parameter deviations by means of Linear Fractional Transformation and for identifying these deviations simultaneously and in real time. This study focuses on cases in which the number of outputs is less than the number of unknown parameters. The results of numerical simulations shown in Sec. 4 indicate that in a linear system that has three unknown parameters and only two outputs, the parameters can be identified simultaneously and in real-time.

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