A method for 3D mesh adaptation in FEA

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Abstract—The use of the mechanical simulation (in particular the finite element analysis) requires the management of assumptions in order to analyse a real complex system. In finite element analysis (FEA), two modeling steps require assumptions to be able to carry out the computations and to obtain some results: the building of the physical model and the building of the simulation model. The simplification assumptions made on the analysed system in these two steps can generate two kinds of errors: the physical modeling errors (mathematical model, domain simplifications, materials properties, boundary conditions and loads) and the mesh discretization errors. This paper proposes a mesh adaptive method based on the use of an h-adaptive scheme in combination with an error estimator in order to choose the mesh of the simulation model. This method allows us to choose the mesh of the simulation model in order to control the cost and the quality of the finite element analysis.

Keywords—Finite Element, Discretization Errors, Adaptivity.

I. Introduction

IN this paper, attention is restricted to mesh adaptivity. Traditionally, the most common mesh adaptive strategies for linear problems are used to reach a prescribed accuracy. This goal is best met with an h-adaptive scheme in combination with an error estimator. In an industrial context, the aim of the mechanical simulations in engineering design is not only to obtain greatest quality but more often a compromise between the desired quality and the computation cost (CPU time, storage, software, competence, human cost, computer used). In this paper we propose the use of alternative mesh refinement with an h-adaptive procedure for 3D elastic problems. The alternative mesh refinement criteria allow to obtain the maximum accuracy for a prescribed cost. These adaptive strategies are based on a technique of error in constitutive relation (the process could be used with other error estimators) and an efficient adaptive technique which automatically takes into account the steep gradient areas. This work proposes a 3D method of adaptivity with the latest version of the INRIA automatic mesh generator GAMHIC3D.

II. Estimation of the discretization error

The finite element solution (U_h, σ_h) is an approximation of the true displacement stress pair solution (U, σ). Indeed, it satisfies the kinematical constraints and the elastic constitutive relation but the equilibrium equations are only satisfied in a weak sense. In practice, it is not possible to compute the error $\mathbf{e_h} = \sigma - \sigma_h$. Numerous error estimators for linear problems have been proposed and studied by different authors [1], [2]. All these methods allow us to obtain an approximation $\mathbf{e}$ of $\mathbf{e_h}$.

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So, to measure the discretization errors, we check, following Refs. [7], [3], the concept of error in constitutive relation. The basic principles of the method are briefly mentioned below.

Suppose that $\hat{U}$ is a kinematically-admissible displacement field verifying the kinematical constraints and $\hat{\sigma}$ a statically-admissible stress field verifying the equilibrium equations. In that case, the quantity $\hat{e}$ is the error in constitutive relation associated to the pair $(\hat{U}, \hat{\sigma})$, where K is the elasticity operator of the material (Hooke tensor) and $\varepsilon(\hat{U})$ the strain tensor:

$$\hat{e} = \hat{\sigma} - K\varepsilon(\hat{U}) \tag{1}$$

If $\hat{e}$ is equal to zero, the pair $(\hat{U}, \hat{\sigma})$ is the solution of the mechanical problem. Otherwise, $\hat{e}$ allows us to estimate the quality of $(\hat{U}, \hat{\sigma})$ as an approximate solution of the problem. To measure the error $\hat{e}$, we use the standard energy norm over the whole structure Ω:

$$\mathbf{e} = \| \hat{e} \|_{\Omega} = \| \hat{\sigma} - K\varepsilon(\hat{U}) \|_{\Omega}, \ \| \ . \ \|_{\Omega} = \left[\int_{\Omega} .^{T} K^{-1}.d\Omega \right]^{\frac{1}{2}} \tag{2}$$

To apply this process, a post-processing of the finite element solution (U_h, σ_h) must be carried out in order to build an admissible displacement stress pair $(\hat{U}, \hat{\sigma})$ from the solution (U_h, σ_h). Within the framework of finite element method, the displacement field U_h is kinematically admissible. For the sake of simplicity, we choose $\hat{U} = U_h$.

On the other hand, the calculated stress σ_h is not statically admissible. Therefore, it is necessary to build a stress field $\hat{\sigma}$ that verifies the equilibrium equations. The technique to build admissible stress field which exactly verify the equilibrium equation has become a classic [7], [8]. The essential step is the construction of the force densities $\hat{F}$, as this condition the quality of the resultant $\hat{\sigma}$ and, consequently, the quality of the error estimator in constitutive relation. In this work, a technique of construction of improved statically admissible stress fields for 3D elasticity detailed in [5]. To perform the construction, the correct stress field $\hat{\sigma}$ is computed from σ_h in two steps:

- during the first step the force densities $\hat{F}$ are constructed on the edge of each element; these densities are in equilibrium with the body forces. Moreover, we impose $\hat{\sigma}$ to be linked with σ_h by condition (3), which is called the prolongation condition, on each element:

$$\int_{E} (\hat{\sigma} - \sigma_h)^{T} \varepsilon(\phi_i) \ dE = 0 \tag{3}$$

This condition holds for all basis function ϕ_i associated to the finite element discretization and for each element E of the mesh.

- during the second step, the strictly statically admissible field $\hat{\sigma}$ is constructed element by element, using the densities as boundary conditions, by deriving a simple solution to the equilibrium equations [5].

From the absolute error $\mathbf{e}$, we can define a relative error ϵ and ϵ_E the contribution to the relative error of an element E of the mesh

$$\epsilon = \frac{\| \hat{\sigma} - K\varepsilon(\hat{U}) \|_\Omega}{\| \hat{\sigma} + K\varepsilon(\hat{U}) \|_\Omega}, \quad \epsilon_E = \frac{\| \hat{\sigma} - K\varepsilon(\hat{U}) \|_E}{\| \hat{\sigma} + K\varepsilon(\hat{U}) \|_\Omega} \quad (4)$$

Therefore, we have $\epsilon^2 = \sum_E \epsilon_E^2$.

The global measure ϵ allows us to quantify the global quality of the approximation and the local contributions ϵ_E allows us to localize the errors on the structure.

III. Alternative Mesh Refinement Criteria

The aim of classical adaptive procedure is to offer the user a level of accuracy ϵ_0 at a minimal computational cost. The common MR1 criterion is based on the optimization of the necessary computational cost for the obtainment of a given value of the global error ϵ_0. We use the criterion of optimality introduced by Ref. [7]; a mesh $\mathbf{T}^*$ is optimal with respect to a measure of the error ϵ if:

$$\begin{cases} \epsilon^* = \epsilon_0 \text{ (prescribed accuracy)} \\ N^* = \text{minimum number of elements} \end{cases} \quad (5)$$

This mesh refinement criterion (denoted MR1) naturally leads to a minimization of computation costs. However, with this common approach a prescribed error ϵ_0 can lead to a very large CPU time and even impossible results when we go beyond the capabilities of softwares or computers. Therefore, in order to control at the same time the cost and the quality of the results, we propose an alternative adaptive mesh refinement criterion denoted MR2 which allows us to prescribe a number of elements N_0 while obtaining an adapted mesh with maximum accuracy. Therefore, for the MR2 criterion a mesh $\mathbf{T}^*$ is optimal with respect to a measure of the error ϵ if:

$$\begin{cases} N^* = N_0 \text{ (prescribed number of elements)} \\ \epsilon^* = \text{maximum accuracy} \end{cases} \quad (6)$$

This approach offers transition from adaptive strategies where meeting a prescribed accuracy is the goal towards adaptive strategies where the goal is to obtain the maximum accuracy, with the capabilities of finite element codes and mesh generators used (number of elements). From the MR2 criterion, we can define the MR3 criterion where we impose the CPU simulation time t_0 while obtaining an adapted mesh with maximum accuracy. Therefore, for the MR3 criterion a mesh $\mathbf{T}^*$ is optimal with respect to a measure of the error ϵ if:

$$\begin{cases} t^* = t_0 \text{ (prescribed CPU time)} \\ \epsilon^* = \text{maximum accuracy} \end{cases} \quad (7)$$

Therefore, for the MR3 criterion, the optimized mesh will be dependent on the computer used. For the MR3 criterion, the CPU time concerns the FEA (includes the formation of element stiffness matrices, assembly of global stiffness matrix and solution of linear equations). The most time consuming phase is the solution of the linear problem. It usually contributes more than 90 % of the total CPU time used. All other processes, such as the error estimation and the refinement of meshes consume relatively insignificant proportion of CPU time.

The alternative mesh refinement criteria presented above allow us to control at the same time the cost and the quality of the results, we can adapt the strategy both for qualitative FEA of choice or quantitative FEA of validation. The analyst knows the capabilities of finite element codes and mesh generators used (number of elements), the capabilities of the computer used and allowed time of calculation. Therefore, he can always obtain the best analysis (maximum accuracy) with these variable capabilities.

IV. Computation of Optimal Sizes with the MR2 Criterion

To determine the characteristics of the optimal mesh $\mathbf{T}^*$ while conforming to mesh refinement criterion MR2, the method computes on each element E of the initial mesh $\mathbf{T}$ a coefficient of size modification:

$$r_E = \frac{h_E^*}{h_E} \quad (8)$$

where h_E denotes the size of the elements E of $\mathbf{T}$ and h_E^* the size that must be imposed on the elements of $\mathbf{T}^*$ in the region of E in order to ensure optimality.

The computation of the coefficients r_E for the determination of a size map uses the convergence rate q of the global error ϵ as a function of the element size h_E. In elasticity, if the exact solution is sufficiently smooth, we have $q = p = 1$ for the 4-node tetrahedra and $q = p = 2$ for the 10-node tetrahedra. If the solution includes a singularity of strength α on the structure, we observe numerically that the convergence rate p_E of the elements connected to the singularity is close to α, whereas it displays a value close to p for the other elements. Therefore, to determine the characteristics of the optimal mesh $\mathbf{T}^*$ with the alternative mesh refinement criterion MR2 presented above, the following procedure is used:

- **A first analysis is performed on a regular coarse mesh T.**
- **The global error ϵ and the local contributions ϵ_E are computed for this mesh (error estimator in constitutive relation [3]).**
- **The characteristics of the optimal mesh $\mathbf{T}^*$ are determined:**
  - Detection of singular areas (see IV-A): The method allows us to define the areas where the coefficients p_E of the singularities must be estimated.
  - Computation of the coefficients p_E (see IV-B):

If the area is detected, we evaluate the value of the coefficient p_E; else we use $p_E = p$.

– Determination of size map h_i^* (input data for a 3D mesher):

In order to take into account a more precise definition of the coefficients of size modification that allows the sizes to vary more rapidly in the steep gradient areas and to be an input data for all automatic mesh generators, nodal coefficients h_i^* are introduced. h_i^* are the prescribed sizes on the mesh $\mathbf{T}^*$, computed at the nodes of the mesh $\mathbf{T}$ in order to verify the MR2 criterion.

$\implies$ **Problem of optimization for alternative mesh refinement criterion MR2** (see IV-C):

• **3D mesh generation conforming to the nodal map of mesh sizes h_i^*:**

Once the optimal sizes are computed, the adapted mesh $\mathbf{T}^*$ is built, which requires an automatic mesh generator able to correctly respect a map of mesh sizes. We must provide an adapted mesh of the skin and use a 3D mesher to generate the volumic adapted mesh respecting the size map h_i^* on the mesh $\mathbf{T}$ from the adapted surface mesh:

– Adapted mesh of the skin:

Firstly, the mesh of the skin is generated from an adapted mesh of the edges of the skin respecting the size map h_i^* on the mesh $\mathbf{T}$. Secondly, we use a 2D mesher to generate the adapted surface mesh respecting the size map h_i^* on the mesh $\mathbf{T}$. In this work, we use an adapted version of the 2D mesher GIBI implemented in the finite element code CASTEM. In this case, we have a good agreement with size respect for plane surface or for sufficiently simple curved surfaces.

– Adapted mesh of the volume:

The efficiency of the MR2 criterion requires an automatic mesh generator able to correctly respect a map of mesh sizes. Therefore, we propose to use the possibilities of the mesher GAMHIC3D (generation of controlled tetrahedra)[6]. It is an automatic tetrahedral mesh generator suitable to create volume meshes for complex domains defined by an adapted surface mesh respecting a map of mesh sizes. The aim is then, from the surface mesh respecting a map of mesh sizes defined above, to build a tetrahedral 3D mesh respecting this map of mesh sizes h_i^*. An empty mesh is first constructed, then enriched by field points, and finally optimized. The field points are defined following an algebraic or an advancing-front approach and are connected using a generalized Delaunay type method [6]. This method allows an important improvement of the size respect.

• **A second finite element computation is made with the adapted mesh.**

A. Detection of the steep gradient areas

The detection of the steep gradient zones is enabled by the computation of the local errors $\bar{\epsilon}_E$, defined by:

$$\bar{\epsilon}_E^2 = \frac{|\Omega|}{|E|} \epsilon_E^2 \qquad (9)$$

where ϵ_E denotes the contribution to the relative error of the element E, $|E|$ the volume of the element and $|\Omega|$ the volume of the structure.

Indeed, we can observe that the local errors are larger in the singular areas than in other areas. Thus, a node i of the mesh will be considered as singular if the average $\bar{m}_i$ of local errors $\bar{\epsilon}_E$ for the elements connected to the node and the average $\bar{M}_i$ of local errors $\bar{\epsilon}_E$ on the whole structure satisfy:

$$\bar{m}_i \geq \zeta \bar{M}_i \qquad (10)$$

where ζ is a coefficient. The numerical experiments in 3D lead to $\zeta = 3$ [4].

The method allows us to define the areas where the coefficients p_E of the singularities must be estimated. The method of calculating the regions of stress concentration enables the convergence rate p_E of each element of the mesh to be taken into account

B. Computation of the coefficients p_E

The computation of the convergence rate p_E is based on the computation of the finite element energy $\bar{e}_h$ in the steep gradient areas detected above [4]. In 3D, isolated singularities and singularities on the edges can appear. If the method of detection identifies two nodes at the extremities of a same edge as singular, the edge is considered as singular. Therefore, the coefficients p_E are identified from the computation of the average energy density $\bar{e}_h$ of the coaxial cylinder C with a radius r, built on the edge:

$$\bar{e}_h(r) = \frac{1}{vol(C)} \int_C Tr[\varepsilon(U_h) K \varepsilon(U_h)] dC \qquad (11)$$

The energy density defined above is identified by the least square method with the theoretical value (12) in order to obtain a coefficient $\bar{\alpha}$ close to the theoretical value α given by:

$$e(r) = kr^{2(\alpha-1)} + c \qquad (12)$$

where k and c are coefficients which depend on the mechanical problem.

For the elements connected to the node i, we take $p_E = \bar{\alpha}$. In the case of isolated nodes identified as singular, a similar technique is used with concentric spheres. In practice, this method of identification allows us to account not only singularities (clamping, crack tip, etc.) but also the steep gradient areas which are not mathematically singular. For most static problems in elasticity we find that the region of stress concentration are close to the edge of the structure. Under these conditions, the computation is performed only for the nodes close to the boundary and consequently requires very little CPU time.

C. Problem of optimization and determination of size map

In order to take into account the coefficients p_E defined above and to verify the MR2 criterion, the problem (6) becomes:

$$\text{Minimize} \quad \epsilon^{*^2} = \sum_E r_E^{2\,p_E}\,\epsilon_E^2 \quad \text{with} \quad \sum_E \frac{1}{r_E^3} = N_0 \quad (13)$$

To solve the problem of optimization (13), we introduce the Lagrange multiplier λ, which must satisfy the Lagrangian L defined as:

$$L(\{r_E\}_{E\in T}, \lambda) = \sum_E r_E^{2\,p_E}\,\epsilon_E^2 + \lambda(\sum_E \frac{1}{r_E^3} - N_0) \quad (14)$$

The extremality conditions give:

$$2\,p_E\, r_E^{(2p_E - 1)}\,\epsilon_E^2 + \lambda(-3r_E^{-4}) = 0 \;\text{ and }\; r_E^3 = \left[\frac{3}{2}\,\frac{\lambda}{p_E\,\epsilon_E^2}\right]^{\frac{3}{2p_E+3}} \quad (15)$$

By coupling (15) and (13), λ must satisfy the equation:

$$\sum_E \left[\frac{3}{2}\,\frac{\lambda}{p_E\,\epsilon_E^2}\right]^{-\frac{3}{2p_E+3}} - N_0 = 0 \quad (16)$$

The numerical solving of (16) allows us to compute λ, then to compute the coefficients r_E:

$$r_E = \left[\frac{3}{2}\,\frac{\lambda}{p_E\,\epsilon_E^2}\right]^{\frac{1}{2p_E+3}} \quad (17)$$

In order to take into account a more precise definition of the size coefficients which allows the size to vary more rapidly in the steep gradient areas and to be an input data for all automatic mesh generators, nodal coefficients h_i^* are introduced. h_i^* are the prescribed sizes on the mesh $\mathbf{T}^*$, computed at the nodes of the mesh $\mathbf{T}$. Inside an element of the initial mesh, a hypothesis of the linear distribution of the volume of the elements in the optimized mesh is made. The element number N^* can then be evaluated by:

$$N^* = \frac{1}{|E|} \int_E \frac{h_E^3}{\sum_{i=1}^n h_i^{*^3}\lambda_i} dE \quad (18)$$

and the contribution ϵ_E^* of an element to the global error ϵ by:

$$\epsilon_E^{*^2} = \frac{1}{|E|} \int_E \frac{\left(\sum_{i=1}^n h_i^{*^3}\lambda_i\right)^{\frac{2p_E}{3}}}{h_E^{2p_E}}\epsilon_E^2 dE \quad (19)$$

where n denotes the number of vertices, h_E its size, p_E the computed convergence rate coefficients and λ_i the barycentric coordinates.

Therefore, the problem of optimization (13) takes the following form. It is numerically solved; its solution provides at every nodes of the mesh $\mathbf{T}$, the size h_i^* for building the optimized mesh $\mathbf{T}^*$ by using an automatic mesh generator.

$$\begin{cases} \text{Minimize} \quad \epsilon^{*^2} = \sum_E \frac{1}{|E|} \int_E \frac{\left(\sum_{i=1}^n h_i^{*^3}\lambda_i\right)^{\frac{2p_E}{3}}}{h_E^{2p_E}}\epsilon_E^2 dE \\[2ex] \text{with} \quad \sum_E \frac{1}{|E|} \int_E \frac{h_E^3}{\sum_{i=1}^n h_i^{*^3}\lambda_i} dE = N_0 \end{cases} \quad (20)$$

V. Mesh Adaptivity

In order to introduce adaptive mesh generation criteria presented above in an adaptive procedure, we use the following structures: a flange, a bearing bracket, an air cylinder bracket and a cross-shaped structure. The geometries, the loads and boundary conditions are shown in Fig. 1. For symmetry reasons, only one half of the flange structure is used. The elastic material properties are 210 Gpa for the Young modulus and 0.3 for the Poisson coefficient. The structures will be meshed using 10-node tetrahedra (T10).
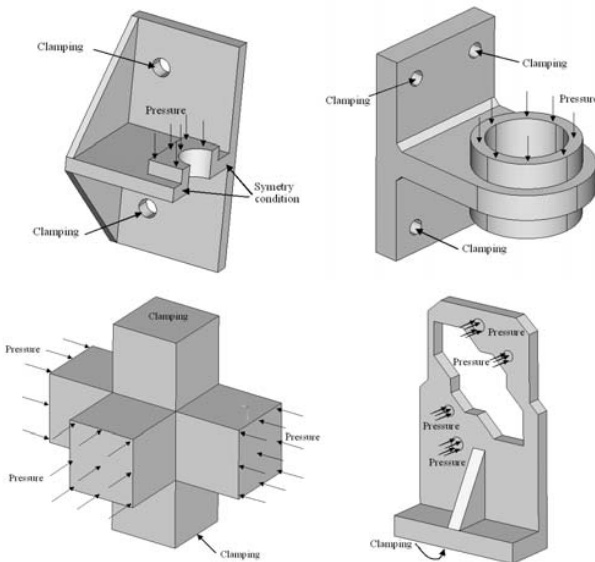

Fig. 1. Geometry, loads and boundary conditions.

Figure 2 shows the ratio $\frac{N_0}{N}$ between prescribed and initial number of elements versus the ratio $\frac{N^*}{N_0}$ between obtained and prescribed number of elements. We can see clearly that the MR2 criterion allows us to be close to the target. The MR2 criterion enables us to accurately control at the same time the cost and the quality of the results.

Figure 3, shows adapted meshes obtained after one adaptive refinement with the MR2 criterion for the structure presented above. For the cross-shaped structure the prescribed number of elements is 32510. We obtain an optimized mesh with 31941 elements and a global error of 7.3 %. For the flange the prescribed number of elements is 38376. We obtain an optimized mesh with 38466 elements and a global error of 17.3 %. For the bearing bracket the prescribed number of elements is 28080. We obtain an optimized mesh with 29276 elements and a global error of 17.38 %. For the air cylinder bracket the prescribed number of elements is 30045. We obtain
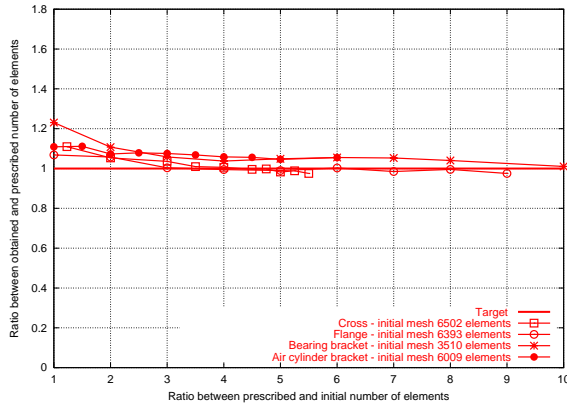
Fig. 2.   MR2 criterion.

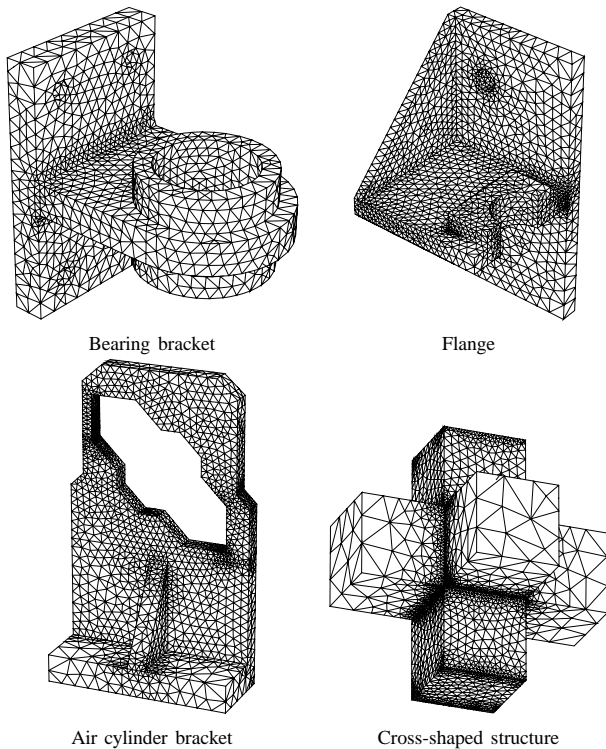an optimized mesh with 31431 elements and a global error of 8.54 %.

Bearing bracket   Flange

Air cylinder bracket   Cross-shaped structure

Fig. 3.   Adapted meshes with the MR2 criterion.

A. Computation of optimal sizes with the MR3 criterion

In order to use the MR3 criterion, we must express the CPU time t as a function of the number of elements N. Results presented in Figure 4 enables us to choose a law in the form $t = cN^d$. The identification and the inversion of this equation give the prescribed number of elements N_0, which in turn allows us to calculate sizes with the MR2 criterion:

$$N_0 = \gamma\, t^{\delta} \qquad (21)$$

γ and δ are two parameters which depend on the type of element, the problem studied, the software and the computer used.
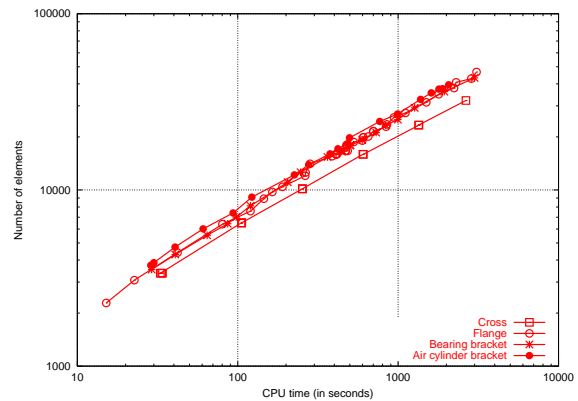

Fig. 4.   Evolution of CPU time (in seconds) with the number of elements.

We have carried out two finite element computations using the MR2 criterion with a prescribed number of elements which fills less than 250 Mb in order to reduce the cost of the identification and to stay in the non-swapping zone of the computer used. However, from Figure 4, we can see after a limit, a decrease of the slopes. This change occurs when the FEA swaps on the disk storage. The swapping increases the CPU time. Therefore, if the parameters γ and δ are defined with a small number of elements, without swapping, it is difficult to concord well with the evolution of CPU time versus the number of elements for a large number of degrees of freedom. A comparison of slopes between swapping zone and non-swapping zone (Figure 4), allows us to define the same multiplicative parameters in order to obtain γ' and δ' in the swapping range from γ and δ defined in the non-swapping range. These coefficients are 0.881 for δ and 1.465 for γ. These parameters depend on the computer used. Therefore, we obtain the following parameters (Table I). Table II shows the results for the MR3 criterion with the structures defined above. We can see clearly that the prescribed CPU time and the obtained CPU time concord well.

TABLE I
IDENTIFIED PARAMETERS FOR THE MR3 CRITERION IN THE SWAPPING RANGE.

	Cross-shaped	Flange	Bearing bracket	Air cylinder bracket
γ'	730.86	697.97	645.41	817.22
δ'	0.482	0.518	0.533	0.504

We have a change of the average band width of the global stiffness matrix between uniform mesh and adaptive mesh refinement. The CPU time depends on the average band width. Therefore, the parameters of the law (21) are identified by using the MR2 criterion. The identification of the law (21) with adaptive mesh generation enables the MR3 criterion to be much more accurate. Fig. 5 shows the comparison between the results of the MR3 criterion where the parameters of (21) are obtained with uniform mesh refinement and the results of the MR3 criterion where the parameters of (21) are obtained

TABLE II
MR3 CRITERION

Prescribed CPU time(s)	Obtained CPU time(s)	Prescribed number of elements	Obtained number of elements	Global Error(%)
Cross-shaped structure - Initial mesh: 6502 elements, $\epsilon = 16.68\,\%$				
500	545	14613	15670	9.37
1000	952	20410	20773	8.27
2000	1941	28506	28193	7.62
3000	2583	34659	34371	7.15
Flange- Initial mesh: 6393 elements, $\epsilon = 29.64\,\%$.				
500	511	17455	18073	21.86
1000	993	24995	25002	19.14
2000	1777	35791	35715	17.16
3000	2833	44156	43800	16.8
Bearing bracket - Initial mesh: 3510 elements, $\epsilon = 30.85\,\%$.				
500	569	17717	18798	19.13
1000	1167	25636	26644	18.4
2000	2220	37093	37459	15.59
2500	3125	41778	41646	15.27
Air cylinder bracket - Initial mesh: 6009 elements, $\epsilon = 20.68\,\%$.				
500	455	19635	20800	10.07
1000	957	29209	30559	9.18
2000	2262	43451	44795	7.67
2500	2834	49377	50210	7.34

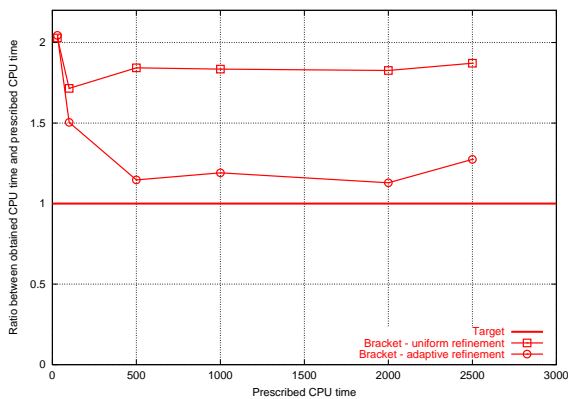with adaptive mesh refinement (MR2 criterion) for the bearing bracket structure.

Fig. 5. Comparison between the MR3 criterion where the parameters γ' and δ' are obtained with uniform refinements or adaptive refinements (MR2 criterion).

VI. CONCLUSION

The use of alternative remeshing strategies based on concepts different than the optimization of the necessary computational cost for the obtainment of a given value of the global error have been revisited and tested. It has been shown how the error in constitutive relation can be used in conjunction with other adaptive mesh refinement criteria like the obtainment of the maximum accuracy with a prescribed number of elements (MR2 criterion) or a prescribed CPU time (MR3 criterion). These new criteria render the approach suitable for coupling with various error estimators and FEA codes. The alternative mesh refinement criteria presented above allow us to control at the same time the cost and the quality of the results. The analyst knows the capabilities of finite element codes and mesh generators used (number of elements), the capabilities of the computer used (number of elements, CPU time) and allowed

time of calculation. Therefore, he can always obtain the best analysis (maximum accuracy) with these variable capabilities.

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