

A Mathematical Framework for Expanding a Railway's Theoretical Capacity

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Abstract—Analytical techniques for measuring and planning railway capacity expansion activities have been considered in this article. A preliminary mathematical framework involving track duplication and section sub divisions is proposed for this task. In railways, these features have a great effect on network performance and for this reason they have been considered. Additional motivations have also arisen from the limitations of prior models that have not included them.

Keywords—Capacity analysis, capacity expansion, railways.

I. INTRODUCTION

THIS article considers the capacity expansion of railways. This is an important topic worldwide and in countries like Australia [26] that have to transport increasing volumes of goods and passengers across large distances by road, rail, air and sea. Railways are considered in this article because of their great importance, and because of limitations in preliminary models and techniques such as those in [3] and [15]. To address this perceived deficiency, two improved analytical methods have been considered. First, an approach which considers where and when to duplicate tracks is proposed. An alternative approach, that sub-divides existing tracks to increase capacity, has then been proposed. In theory, both of these approaches can be integrated. Together they constitute a preliminary mathematical framework for railway capacity expansion.

Articles on railway capacity analysis have increased over the last ten years. In summary, relatively few of the papers have proposed traditional analytical capacity models. One example, however, is [24] which extended the work of [3] and developed a new model that takes into account junctions and other more complex nodes and stations. Interference probabilities between trains are also taken into account. Yaghini et al. [27] also proposed a railway capacity model and applied it to several case studies in Iran. Their model however is based upon a prior binary multi-commodity network design formulation and a space-time representation of the network. Conceptually its nature is very similar to the model of [3]; however, it does additionally provide a saturated schedule at extra computational cost and complexity. Burdett [7] has most recently addressed multi-objective capacity identification and the inclusion of complex train paths to analytical capacity models.

The majority of papers have taken a more case study oriented and empirically based approach. These include [19], [9]-[13], [22], [25]. The approaches developed in those papers are of limited benefit for the problem addressed in this article.

Several recent articles have considered the related topic of infrastructure expansion and investment. In those articles, optimization models were proposed. Lui et al. [23] considered the capacity expansion of railroads and the spatial configuration of yards in a freight network. In particular, they formulated a model for the yard location problem and demonstrated that railroads can make significant savings by reconfiguring their networks. This outcome provides significant motivation for this article which considers general railway networks and their expansion, and proposes more generic models and techniques for expanding line capacity.

Lai and Barkan [16] and [18] considered strategic capacity planning. Lai and Barkan [17] then extended their previous work and proposed a decision support framework for railway capacity planning. Singh et al. [26] developed a mixed integer linear programming (MILP) model for determining capacity requirements and infrastructure improvements for the Hunter Valley Coal Chain. A trade-off was made between the accuracy of simulation and the use of a more generic mathematical model. Due to its size and complexity however, meta-heuristics were necessary to optimize this bulk material supply chain.

Decision support tools have proliferated greatly in recent years. Kontaxi and Ricci [14] have developed an online tool that compares different railway capacity methods. Abril et al. [1] presented an automated tool to perform several different forms of capacity analysis. They reported the presence of six international companies developing railway capacity software.

Train sequencing and scheduling techniques can be used to verify whether a railway network has sufficient capacity to cater for a specified mix of trains, over a given time period, as determined from an independent capacity determination approach. There are many techniques available for doing this. The latest techniques such as [4]-[6], [20], [21], [8] treat this problem as a hybrid machine scheduling (i.e. job-shop) problem. Solving the train scheduling problem however is difficult as these problems are computationally intractable and become considerably harder to solve as the problem size increases. Hence, they are not good techniques for assessing capacity over longer time periods, nor for performing capacity expansion analysis.

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II. METHODOLOGY

The concept of theoretical capacity is utilized in this article. In [3], it is defined as an ideal level that only occurs when critical sections of rail are saturated (i.e. continuously occupied) and train "interaction effects" and "interference delays" that are resolved by proper train scheduling, are ignored. Although theoretical capacity is an overestimation of real "operational" capacity, it is sufficiently accurate for high level planning purposes, for example, that is considered in this article. The purpose of the analytical models in this article is to maximize the theoretical capacity denoted by A .

A. Track Duplications (Static)

Duplicating existing tracks can significantly increase the capacity of a rail network, however as shown in preliminary numerical investigations in [2], choosing the right sections is not entirely transparent, even more so when costs of duplications vary at different locations. Capacity expansion models for doing this are therefore proposed here. The first type of model is "static" and this means that there is no time element to track duplications. The second however considers track duplications over time.

To include track duplication in a static capacity expansion model, the following constraints are necessary.

$$\sum_{i \in I} (\bar{y}_i^s \bar{T}_i^s + \hat{y}_i^s \bar{T}_i^s) \leq (\tau_s + \parallel_s) T \quad \forall s \in S \quad [\text{Section saturation}] \quad (1)$$

$$\sum_{s \in S} (\mathbb{C}_s^{\parallel} \parallel_s) \leq \mathbb{B} \quad [\text{Limitation on spending}] \quad (2)$$

$$0 \leq \parallel_s \leq \parallel_s^{\max} \quad \forall s \in S \quad [\text{Limit on track numbers}] \quad (3)$$

Here: \bar{y}_i^s and \hat{y}_i^s are the number of trains of type i that traverse section s . Their value is determined from the decision variables, \bar{x}_i^c and \hat{x}_i^c , which are the number of trains of type i that traverse corridor c . In particular, $\bar{y}_i^s = \sum_{c \in C | s \in \Omega_c} (\bar{x}_i^c)$ and $\hat{y}_i^s = \sum_{c \in C | s \in \Omega_c} (\hat{x}_i^c)$ $\forall i \in I, \forall s \in S$. Also: \parallel_s is an integer decision variable (i.e. $\parallel_s \in \mathbb{Z}$) for the number of parallel tracks to add on section s . Furthermore \mathbb{C}_s^{\parallel} is the cost of a single additional track (i.e. cost of duplication), \parallel_s^{\max} is an upper bound on track numbers, and there is some total budgetary limit (i.e. \mathbb{B}). Constraint (1) includes the number of newly assigned tracks \parallel_s . The models objective is to maximize the capacity, i.e. $A = \sum_{c \in C} \sum_{i \in I} (\bar{x}_i^c + \hat{x}_i^c)$. It has been assumed that sectional running times are the same on parallel tracks. This assumption must be made otherwise it is necessary to define additional decision variables that describe which parallel track individual trains are assigned to.

From a practical perspective, track duplication cannot be separated from budgetary considerations. This is because the problem would become unbounded, and sections would be duplicated without limit. However, a variant decision making problem is to decide upon what sections should be upgraded in order to achieve a specified level of demand capacity. This variant problem has an alternative objective which is minimization of spending.

B. Track Duplications (Over Time)

Section II. A determines in the preceding section how capacity should be expanded straight away. This is perhaps not entirely useful or realistic. A better approach may be to construct and implement a long term plan of infrastructure expansions that takes into account any intermediate stage capacity requirements, and regular or intermittent budgets and funding. The following questions are, hence, pertinent to infrastructure expansion and modelling activities: i) what is the minimum time required to update the network to a specified level of capacity given a specified budget?, ii) what is the minimum budget required to update the network to a specified level of capacity in a specified time?, and iii) can the network be upgraded to a specified capacity within the given time and with the given budget?. Mathematical models for long term planning are hence investigated to see whether the aforementioned questions can be answered. Those models will be explained in due course. First let $P = \{1, 2, \dots\}$ be the set of planning periods (typically in years) available for expansion activities, where $\bar{P} = |P|$ is the upper bound on the number of periods. Let $\hat{P} \leq \bar{P}$ be the actual number of periods required or designated. Furthermore, let e_p be the expenditure in time period p , and let $\parallel_{s,p}$ be the number of tracks added in time period p in section s . The main assumptions for modelling are as:

- A budget b_p is provided at various planning periods, not necessarily every period (but could be).
- If the budget is not used then it rolls over into the next period. Hence \bar{b}_p is the cumulative budget up to and including period p , i.e. $\bar{b}_p = \sum_{u=1, \dots, p} b_u$.
- There may or may not be intermediate absolute capacity requirements A_p , however there is a definite long term goal for capacity. Let \bar{A} be the absolute capacity required at the end of expansion activities. Hence, $A_p = \bar{A}$ for $p = \bar{P}$. If there is no requirement then A_p is defined as the current "initial" system capacity, say A_0 . It is necessary for $A_p \geq A_{p-1} \quad \forall p | p \geq 1$.
- All track duplications can be constructed within a time period.

Three models can be been formulated:

- Model 1. Given b_p and capacity requirement \bar{A} , create an expansion plan of minimal duration \hat{P} .
- Model 2. Given a planning period P and capacity requirement \bar{A} , create a plan that minimises total spending \mathbb{S} .
- Model 3. Given \mathbb{B}, P, \bar{A} , determine whether a feasible plan can be constructed.

Each of these requires the same set of core constraints:

$$\sum_{i \in I} (\bar{y}_{i,p}^s \bar{T}_i^s + \hat{y}_{i,p}^s \bar{T}_i^s) \leq T \times \tau_{s,p} \quad \forall s \in S, p \in P \quad [\text{Section saturation}] \quad (4)$$

$$\bar{y}_{i,p}^s = \sum_{c \in C | s \in \Omega_c} (\bar{x}_{i,p}^c); \hat{y}_{i,p}^s = \sum_{c \in C | s \in \Omega_c} (\hat{x}_{i,p}^c) \quad [\text{Section usage}] \quad (5)$$

$$\bar{x}_{i,p}^c, \hat{x}_{i,p}^c \geq 0 \quad \forall i \in I, c \in C, p \in P \quad [\text{Positivity}] \quad (6)$$

$$\tau_{s,1} = \tau_{s,0}; \tau_{s,p} = \tau_{s,p-1} + \tau_{s,p-1} \forall s \in S, \forall p \in P | p > 1$$

[Track number counter] (7)

$$0 \leq \sum_{p \in P} \tau_{s,p} \leq \tau_s^{\max} \quad \forall s \in S \quad \text{[Limit on track numbers]} \quad (8)$$

$$A_p = \sum_{c \in C} \sum_{i \in I} (\tilde{x}_{i,p}^c + \tilde{x}_{i,p}^e) \geq A_p \quad \forall p \in P$$

[Intermediate and final requirement] (9)

$$e_p = \sum_{s \in S} (C_{s,p}^{\parallel} \cdot \tau_{s,p}) \quad \forall p \in P \quad \text{[Expenditure]} \quad (10)$$

In summary, the solution of the static planning model for a fixed budget should be quite similar to the solution for Model 1. Model 1, however, determines when spending should occur over time and takes into account the fact that $C_{s,p}^{\parallel}$ is not necessarily static. In Model 2, there is no limit on spending in any time period but some additional upper limit may be imposed. It should be noted that this model is applied under the assumption that $C_{s,p}^{\parallel}$ is not static; otherwise, all spending should occur straight away. Model 3 is also applied under the assumption that $C_{s,p}^{\parallel}$ is not static. Otherwise, the static expansion model would be more appropriate for this task.

C. Section Sub Divisions

In the preceding sections, the expansion of capacity via track duplications was considered. However, that method is quite costly and permanent. Construction time for track duplications may also be prohibitive. As capacity is highly related to travel times on critical bottleneck sections, it is possible to expand system capacity by sub-dividing existing sections of track, for example using signals. The positioning of sub sections is very important. This is clearly demonstrated by preliminary numerical investigations.

The main idea behind partitioning sections is that it allows more trains to run at the same time and to reduce the distance between trains in a safe way. In addition making difficult “slower” sections into proper signalized sections, means that bottleneck issues are reduced as much as possible. The length of the sub section is important; however, it is the travelling time that is most important

The first mathematical model that is proposed for section sub division assumes that each section of rail can be divided into many parts and that the travelling time across each part is linearly proportional to the time to travel across the entire section. Furthermore, there is no profile for the travel times or train speeds across each section. Let n_s be a decision variable for the number of sub sections created on section s . The cost of dividing a section is denoted by C_s^{\ddagger} and it primarily includes the cost of signalization. This model seeks to maximize absolute capacity subject to a specified limit on spending.

When track gradients and curvature vary considerably over the entire length of a section, the previous assumption of a uniform train speed is quite unrealistic. In order to avoid this assumption, and to formulate a suitable mathematical model that determines how many sub sections to have, and where each sub section begins and ends, it is necessary for some type of profile to be provided. The aforementioned profile should

provide the travelling time, gradient, and/or velocity on all parts of a section. From a practical point of view, the best case is when the time to traverse each linear segment in the profile, occurring between adjacent locations, and in both directions, is measured. Otherwise, a theoretical approximation could be used. It should be noted that another mathematical optimization model is not required since the aforementioned “first” model is sufficient. That optimization model does not need to know where to divide sections, because the effect of dividing sections into so many parts is known. For instance, the increase in capacity is at most n times if there are n sub sections. The position of each chosen sub section can be determined later. For that task, however, a separate mathematical model is required and is proposed here. That model must take into account that travelling on an inclined section of rail is efficient in one direction, and inefficient in the opposite direction.

The idea behind the mathematical model for determining the position of the n sub sections is to use the discretization specified by the profile. It is not necessary to further discretize the domain. The following binary variables $\gamma_{u,j}^-, \gamma_{u,j}^+, \gamma_{u,j}$ are defined for each combination of sub section (i.e. j) and profile segment (i.e. u). They signify whether segment u contains the start and end respectively of sub section j or whether segment u is part of sub section j . The constraints of the model must ensure that each sub section starts and ends somewhere. A function that determines the travelling time on each sub section is critical to the success of this model. It is as:

$$\tilde{T}_{i,j}^+ = \sum_{u=1, \dots, U} \left(\gamma_{u,j}^+ \tilde{\tau}_{i,u} - \gamma_{u,j}^- \tilde{\tau}_{i,u-1} + \left(\frac{\gamma_{u,j}^- st_u}{\delta} - \frac{\gamma_{u,j}^+ en_u}{\delta} \right) \tilde{\tau}_{i,u} + \frac{\tilde{\tau}_{i,u}}{\delta} (\gamma_{u,j}^+ \beta_j - \gamma_{u,j}^- \alpha_j) \right) \quad (11)$$

$$\tilde{T}_{i,j}^- = \sum_{u=1, \dots, U} \left(\gamma_{u,j}^- \tilde{\tau}_{i,u} - \gamma_{u,j}^+ \tilde{\tau}_{i,u-1} + \left(\frac{\gamma_{u,j}^+ st_u}{\delta} - \frac{\gamma_{u,j}^- en_u}{\delta} \right) \tilde{\tau}_{i,u} + \frac{\tilde{\tau}_{i,u}}{\delta} (\gamma_{u,j}^- \beta_j - \gamma_{u,j}^+ \alpha_j) \right) \quad (12)$$

where:

$$st_u \gamma_{u,j}^- \leq \alpha_j < en_u + (1 - \gamma_{u,j}^-) BIGNM \quad (13)$$

$$st_u \gamma_{u,j}^+ < \beta_j \leq en_u + (1 - \gamma_{u,j}^+) BIGNM \quad (14)$$

These equations involve the known cumulative travel time from the profile. Above α_j and β_j are the start and end position of the j th sub section. The start and end of the u th segment in the profile is st_u and en_u . The cumulative time to travel to the u th location from the beginning of the profile for trains of type i is denoted by $\tilde{\tau}_{i,u}$ and $\tilde{\tau}_{i,u}$. Finally $\tilde{\tau}_{i,u}$ and $\tilde{\tau}_{i,u}$ are the time to travel across the u th segment by trains of type i . In the profile the spacing between segments is denoted by δ .

D. A General Expansion Model

The two capacity expansion alternatives investigated in previous sections can be combined into a single decision model. The reason for doing this is that the costs and effects of

duplications and sub divisions are very problem specific. There are many real life technical constraints that may need to be added, that will restrict options at different locations within the railway network.

III. CONCLUSIONS

Railway capacity determination and railway capacity expansion are increasingly important topics as railways will become more developed, sophisticated, and have greater demands placed upon them in the future. To help railway capacity planning activities, a mathematical framework involving optimization models has been introduced to expand the theoretical capacity of a railway. The proposed framework is high level and strategic, and this is why increases to theoretical capacity is concentrated upon. This approach provides a valuable reference point to compare other approaches, for example those for determining operational capacity. The results of simulation activities can also be compared to this reference point.

Two capacity expansion possibilities should be considered in such a framework. The first is track duplications, and the second is section sub divisions. Choosing the right sections to duplicate is not entirely transparent, even more so when costs of duplications, and other restrictions vary at different locations. Capacity is highly related to travel times on critical bottleneck sections too. Hence, it is possible to increase capacity by sub-dividing existing sections of track, for example using signals. This alternative is necessary because track duplications are quite costly and permanent and construction times may also be prohibitive. The track sub division approach should utilize a profile of the travelling time, gradient, and/or velocity on all parts of a section if it is available. A model that combines both alternatives is most beneficial.

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