A Hybrid Method for Determination of Effective Poles Using Clustering Dominant Pole Algorithm

Anuj Abraham, N. Pappa, Daniel Honc, Rahul Sharma

***Abstract*—**In this paper, an analysis of some model order reduction techniques is presented. A new hybrid algorithm for model order reduction of linear time invariant systems is compared with the conventional techniques namely Balanced Truncation, Hankel Norm reduction and Dominant Pole Algorithm (DPA). The proposed hybrid algorithm is known as Clustering Dominant Pole Algorithm (CDPA), is able to compute the full set of dominant poles and its cluster center efficiently. The dominant poles of a transfer function are specific eigenvalues of the state space matrix of the corresponding dynamical system. The effectiveness of this novel technique is shown through the simulation results.

***Keywords*—**Balanced truncation, Clustering, Dominant pole, Hankel norm, Model reduction.

I. INTRODUCTION

MODEL Order Reduction (MOR) has been used extensively in the industrial processes and circuit analysis over the past several years [7]. The need for MOR techniques was fed by the desire to decrease the simulation time required for using computer-generated models in analysis and/or design and to retain accuracy. Numerous methods are available in the literature for order-reduction of linear continuous systems in time domain as well as in frequency domain [3]. However, a common feature in these methods is that the values of the denominator coefficients of the low order system are chosen arbitrarily by some stability preserving. Then the numerator coefficients of the lower order system are determined by minimization of the Integral Square Error (ISE).

The method of truncated balanced realization (TBR) originated from control theory is based on the idea of controllability and observability [1]. Certain states in a dynamical system are hard to control and some others are hard to observe. The balanced realization obtains a dynamical system whose state variables have equal controllability and observability. Then the truncation of the states that are hard to control and observe leads to a reduced order model. It turns out that the TBR method also falls in the linear projection framework and did not allow for an interpretation as an optimal approximation.

The use of the Hankel norm approximation theory has become important, especially in the area of control and system theory. However, there is a serious drawback to applying the standard Hankel norm approximation theory, namely that a balanced state-space model for the system must first be computed. This creates practical difficulties for system models having uncontrollable and unobservable states because the balancing transformations are generally singular for such systems.

The behaviour of a large scale dynamical system can often be described by a relatively small number of its dominant modes. The state space projection on the subspace spanned by the dominant modes, the model equivalent can be obtained. Model approximation has been successfully applied to transfer functions of large scale power systems and electrical circuits for application to stability analysis.

Specific eigenvectors and eigenvalues of the state matrix relate the dominant modes with corresponding dominant poles of the system transfer function [5], [6]. An eigenvalue methods are required that compute the most dominant poles and corresponding modes. The Dominant Pole Algorithm (DPA) works only for stable systems [10]. The proposed hybrid algorithm combines the clustering method and dominant pole algorithm. This method efficiently computes the poles that are more dominant in the system and retains its characteristic behaviour. The denominator polynomial of the reduced order model with respect to original model is determined by forming the clusters of the dominant poles and the coefficients of numerator polynomial with respect to original model are obtained by using the factor division algorithm. The simulation results show the effectiveness of CDPA with the conventional existing techniques available in the literature.

II. CONVENTIONAL MOR TECHNIQUES

A. Balanced Truncation

Balanced truncation guarantees an error bound on the infinity norm of the additive error $\|G - G_{red}\|_{\infty}$ for well-conditioned model reduced problems [1].

Given a state space (*A*,*B*,*C*,*D*) of a system and *k*, the desired reduced order, the following steps will produce a similarity transformation to truncate the original state space system to the k^{th} order reduced model [2], [12].

Step 1. Find the Singular Value Decomposition (SVD) of the controllability and observability grammians.

Step 2. Find the square root of the grammians (left/right eigenvectors).

Step 3. Find the SVD to the above step.

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Step 4. Finally, the left and right transformation for the final k[th] order reduced model is computed.

B. Hankel Norm Approximation

The Hankel norm of a system $G=\left(A,B,C,D\right)\in H_{\infty}$ is defined, as in (1),

$$\left\|G\right\|_{H}^{2}=\sup\frac{\int_{0}^{\infty}y^{2}(t)\,dt}{\int_{0}^{\infty}u^{2}(t)\,dt} \tag{1}$$

where,

$$y(t)=\int_{-\infty}^{0}C\,e^{A(t-s)}\,B\,u(s)$$

The Hankel norm gives how much energy [2] can be transferred from past inputs into future outputs through the system *G(s).* In control theory, eigenvalues define system stability and Hankel Singular Values (HSV) define the "energy" of each state in the system. Its characteristics in terms of stability, frequency, and time responses are preserved by keeping larger energy states of a system based on the Hankel singular values. They can achieve a reduced-order model that preserves the majority of the system characteristics.

Mathematically, given a stable state-space system *(A, B, C, D)* its HSV are defined as in (2),

$$\left\|G\right\|_{H}^{2}=\sqrt{\lambda_{\max}}\,(PQ)=\sigma_{i} \tag{2}$$

where σ_i is the Hankel singular values,

The controllability and observability grammians *P* and *Q* respectively satisfies,

$$AP+PA^{T}=-BB^{T} \tag{3}$$
$$A^{T}Q+QA=-C^{T}C \tag{4}$$

One defines the Hankel operator Γ_G of the system *G(s)* by,

$$\Gamma_{G}:L_{2}(-\infty,0]:(\Gamma_{G}u)(t)=\int_{-\infty}^{0}C\,e^{A(t-s)}B\,u(s)\,ds \tag{5}$$

This method also guarantees an error bound on the infinity norm of the additive error $\left\|G-G_{red}\right\|_{\infty}$ for well-conditioned model reduction problems as in balanced truncation method [2].

$$\left\|G-G_{red}\right\|_{\infty}\leq 2\sum_{k+1}^{n}\sigma_{i} \tag{6}$$

where σ_i are singular values of a given system *G(s).*

C. Dominant Pole Algorithm

DPA computes dominant poles of *G(s)* based on Newton process [13]. A pole λ_i that corresponds to a residue R_i with large magnitude $\left|R_j\right|$ is called a dominant pole. A dominant pole is well observable and controllable in the transfer function [9]. This can also be observed from the corresponding Bode Magnitude plot of *G(s),* where peaks occur at frequencies close to the imaginary parts of the dominant poles of *G(s).*

$$G(s)=C\left(sI-A\right)^{-1}B=\sum_{i=1}^{n}\frac{R_{i}}{s-\lambda_{i}} \tag{7}$$

where, residue $R_i=\left(C\,\mathrm{x}_i\right)\left(y_i\,\mathrm{B}\right)$ and $\mathrm{x}_i,\mathrm{y}_i,\lambda_i$ are eigen triplets (i=1,2,..n)

Consider a pole $\lambda=\alpha+j\beta$, with residue *R* then it is shown that,

$$\lim_{w\to\beta}G\left(jw\right)=\lim_{w\to\beta}\frac{R}{\left[jw-(\alpha+j\beta)\right]}+\sum_{j=1}^{n-1}\frac{C}{\left[jw-\lambda_{j}\right]}=\frac{R}{\alpha}+G_{n-1}\left(j\beta\right) \tag{8}$$

Hence pole λ_j is dominant if $\left|\frac{R_j}{\mathrm{Re}(\lambda_j)}\right|$ is large and causes peak in the bode plot.

III. Clustering Dominant Pole Algorithm

The proposed hybrid algorithm combines the features of clustering method and dominant pole algorithm and effectively matches the original system characteristics.

The poles of transfer function are the $\lambda\in\to C$ for which $\lim_{s\to\lambda}\left|H(s)\right|=\infty$. Consider now the function,

$$G(s)=\frac{1}{H(s)}$$
$$G'(s)=\frac{-H'(s)}{H^{2}(s)} \tag{9}$$

Solve x_k from $\left(s_kE-A\right)x_k=b$ and y_k from $\left(s_kE-A\right)y_k=c$. Then compute the new pole estimate as shown in (10),

$$s_{k+1}=s_{k}-\frac{c\,x_{k}}{y_{k}\,\mathrm{E}\,x_{k}}=\frac{y_{k}\,A\,x_{k}}{y_{k}\,\mathrm{E}\,x_{k}} \tag{10}$$

In this proposed method the dominant poles are grouped into several clusters and then replaced by the corresponding cluster-centers. By Factor division algorithm, the coefficients of the numerator polynomial are determined. Now, consider n^{th} order linear dynamic system described by the transfer function in (11),

$$G\left(s\right)=\frac{N\left(s\right)}{D\left(s\right)}=\frac{e_{0}+e_{1}s+e_{2}s^{2}+\ldots.+e_{n-1}s^{n-1}}{f_{0}+f_{1}s+f_{2}s^{2}+\ldots.+f_{n}s^{n}} \tag{11}$$

where $e_i;0\leq i\leq n-1$ and $f_i;0\leq i\leq n$ are scalar constants.

The corresponding k^{th} *(k<n)* order reduced model is synthesized as,

$$G_k(s) = \frac{N_k(s)}{D_k(s)} = \frac{a_0 + a_1 s + a_2 s^2 + ... + a_{k-1} s^{k-1}}{b_0 + b_1 s + b_2 s^2 + ... + b_k s^k} \quad (12)$$

where $a_i; 0 \le i \le k-1$ and $b_i; 0 \le i \le k$ are scalar constants.

Let *r* real poles in one cluster be (p_1, p_2, p_3…, p_r) then the Inverse Distance Measure (IDM) criterion identifies the cluster center given in (13),

$$p_c = \left\{ \sum_{i=1}^{r} \frac{1}{p_i} \Big/ r \right\}^{-1} \quad (13)$$

where p_c is cluster center from *r* real poles of the original system [11].

The power series of original n^{th} order system can be expanded about *s=0* as,

$$G(s) = C_0 + C_1 s + C_2 s^2 + ... \quad (14)$$

The power series expansion coefficients are determined as:

$$C_0 = e_0 / f_0 \quad (15)$$

$$C_i = \frac{1}{f_0}\left[e_i - \sum_{j-1}^{i} f_j C_{i-j} \right], i > 0 \quad (16)$$

$$e_i = 0, \;\; i > n-1 \quad (17)$$

The reduced k^{th} order model is written as:

$$G_k(s) = \frac{N_k(s)}{D_k(s)} = \frac{\sum_{i=0}^{k-1} a_i s^i}{\sum_{i=0}^{k} b_i s^i} \quad (18)$$

IV. Numerical Example

LTI models can be equivalently defined by their transfer functions *G(s)*. A straightforward model order reduction method is to fit a reduced transfer function $G_r(s)$ that gives a good approximation to *G(s)*.

The famous approach is the Pade approximation [4], i.e., to find a rational reduced transfer function of the form,

$$G_r(s) = \frac{a_0 + a_1 s + a_2 s^2 + ... + a_{n-1} s^{n-1}}{b_0 + b_1 s + b_2 s^2 + ... + b_n s^n} \quad (19)$$

which satisfies,

$$G_r(s) = G(0),$$
$$\frac{d^k}{ds^k} G_r(0) = \frac{d^k}{ds^k} G(0), \text{ k=1,2,3...m+n} \quad (20)$$

where, $\frac{d^k}{ds^k} G(0)$ is also referred to as the *k*-th moment of the transfer function. Therefore, any LTI MOR method that leads to a reduced model satisfying is also called a moment-matching method.

Intuitively, Pade approximation is good for MOR because a rational function is a good fit for LTI transfer functions – the original transfer function *G(s)* is typically a rational function given that there are no repeating poles [8]. In fact, *G(s)* may analytically be written as,

$$G(s) = D^T (sC + H)^{-1} B \quad (21)$$

which is a rational function in s if the matrix $H^{-1}C$ has no repeating eigenvalues.

The algorithms are tested on HiMAT (Highly Maneuverable Aircraft Technology) benchmark example. The state space realization of the HiMAT model has 6 states, with the first four states representing angle of attack (α) and attitude angle (θ) and their rates of change (dα/dt,dθ/dt) and the last two representing elevon and canard control actuator dynamics. Therefore, the model considered has one control inputs as Elevon deflection δe and one measured outputs as angle of attack alpha(α). The continuous transfer function *G(s)* for the model chosen is as given in (22),

$$G(s) = \left[\frac{-5.124s^4 - 1099s^3 - 28390s^2 - 568.5s + 24.08}{s^6 + 64.55s^5 + 1167s^4 + 3729s^3 + 5495s^2 + 1102s + 708.1} \right] (22)$$

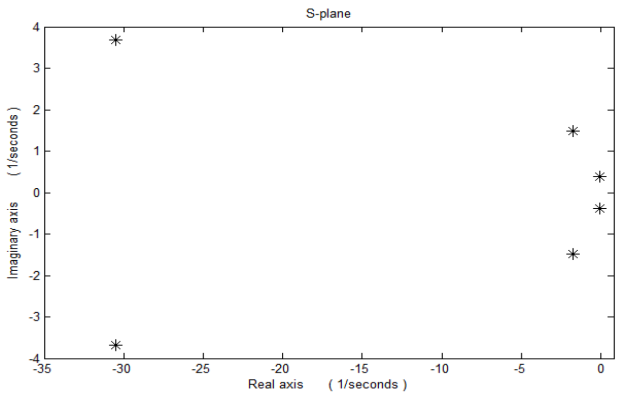

Fig. 1 Pole Spectrum of the 6^{th} order modal equivalent transfer function for HiMAT example

The pole spectrum of the 6^{th} order original transfer function is shown in Fig. 1. It is clear that, the plant poles are located at p_1= -30.4865 + 3.6785i, p_2= -30.4865 - 3.6785i, p_3= -1.7308 + 1.4838i, p_4= -1.7308 - 1.4838i, p_5= -0.0596 + 0.3754i and p_6= -0.0596 - 0.3754i.

From Fig. 1 it is evident that the given original system has all the poles lying on the left side of the s-plane and having conjugate poles. The poles p_1=-30.4865 + 3.6785i and p_2=-30.4865 - 3.6785i, are lying far away from the s-plane origin and takes fast response in decaying with less effect on the system characteristics. The reduced order model equivalent comparison of the proposed hybrid algorithm and conventional order reduction techniques is given in Table I.

The constant coefficients of the denominator polynomial used in the various techniques have an important role to play in stability and performance of an LTI system, while the numerator coefficients also have an influence on the system's response to applied inputs. Model based control schemes are effectively used in industrial application namely cement industry, coal mill industry etc., where accurate model coefficient estimation plays a crucial role in improving the closed loop system performances.

V. Results and Discussions

The optimal Hankel norm approximation gives the following fourth order transfer function model. The transfer function displays a similar frequency response to the reduced order models. The gain characteristics of the reduced models are compared in Fig. 4 with the original system. The HSV of the original system are σ_1=10.46, σ_2=8.22, σ_3= 3.1, σ_4=1.1, σ_5=0, σ_6=0, which means that fourth order model can be good approximants as shown in Fig. 2.

TABLE I
Comparison Of MOR Techniques

Methods	Reduced Models	ISE
Balanced truncation	$\left[\frac{0.51s^3-21.70s^2-0.63s+0.08}{s^4+3.31s^3+5.28s^2+1.03s+0.68}\right]$	3.456
Hankel norm reduction	$\left[\frac{0.62s^3-19.80s^2-0.56s+0.04}{s^4+3.58s^3+5.75s^2+1.12s+0.31}\right]$	1.004
DPA Method	$\left[\frac{0.53s^3-28.08s^2-0.54s+0.02}{s^4+3.34s^3+5.21s^2+1.42s+0.71}\right]$	0.805
Proposed hybrid algorithm	$\left[\frac{0.56s^3-29.07s^2-0.64s+0.03}{s^4+3.47s^3+5.60s^2+1.08s+0.73}\right]$	0.717

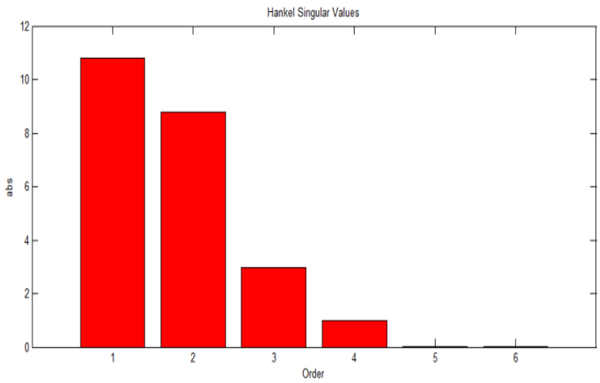

Fig. 2 HSV for a HiMAT example using Hankel norm method

The comparison is made by computing the error index known as ISE values in between the transient parts of the original and reduced order model as given in (23),

$$ISE = \int_0^{\infty} \left[y(t) - y_r(t) \right]^2 dt \tag{23}$$

where $y(t)$ and $y_r(t)$ are the unit step responses of original and reduced order systems respectively.

This error index is calculated for various reduced order models which are obtained in Table I and compared with the other order reduction methods available in the literature. In Fig. 3, the step responses of different reduced order model are compared with the original order system.

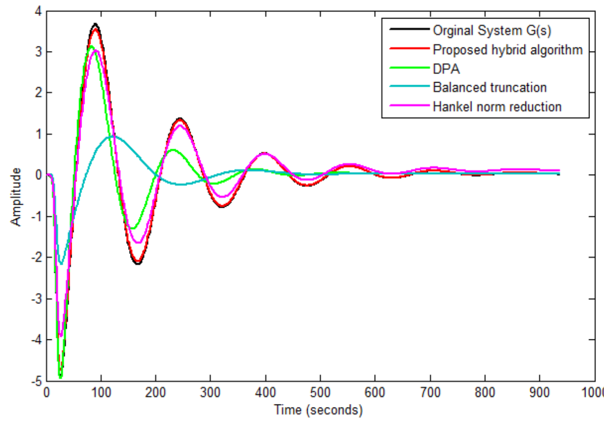

Fig. 3 Step responses of original order and various reduced order systems for HiMAT example

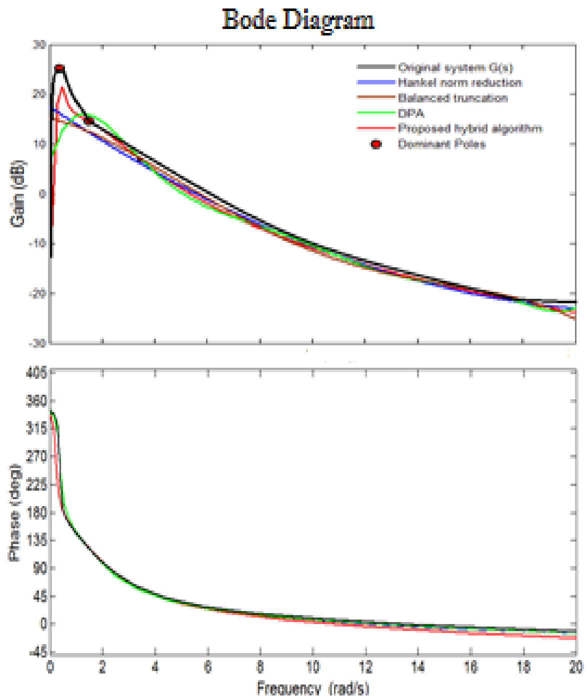

Fig. 4 Comparison of gain and phase characteristics of the original system and four different reduced order systems

The gain and phase characteristics of the original system *G(s)* was compared with the other four different reduced order methods as shown in Fig. 4. It is observed that the essential dynamics of the system lie in the frequency range of 0.5 to 4 radians/second from the frequency response. The magnitude drops in both the very low and the high-frequency ranges. The result shows that the proposed method displays similar characteristics with *G(s).*

VI. Conclusion

In this paper a new hybrid algorithm based model order reduction method is proposed. In CDPA method presented, the denominator of the reduced model is synthesized by using clustering technique in which the dominant poles are grouped into several clusters and replaced by the corresponding cluster centers. The proposed method is simulated with the help of HiMAT benchmark example and it has been observed that it gives better response with existing order reduction techniques. The simulated numerical results shows that the method is simple, efficient to compute dominant poles and matches the original system properties to give minimum ISE error.

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