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A Global Condition for the Triviality of an Almost Split Quaternionic Structure on Split Complex Manifolds

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Abstract**—**Let M be an almost split quaternionic manifold on which its almost split quaternionic structure is defined by a three dimensional subbundle V of $T(M) \otimes T^*(M)$ and $\{F, G, H\}$ be a local basis for V. Suppose that the (global) $(1, 2)$ tensor field defined $[V, V]$ is defined by $[V, V] = [F, F] + [G, G] + [H, H]$, where $[,]$ denotes the Nijenhuis bracket. In ref. [7], for the almost split-hypercomplex structure $\mathscr{H} = J_\alpha, \alpha = 1, 2, 3$, and the Obata connection $\nabla^{\mathscr{H}}$ vanishes if and only if $\mathscr{H}$ is split-hypercomplex.

In this study, we give a prof, in particular, prove that if either M is a split quaternionic Kaehler manifold, or if M is a split-complex manifold with almost split-complex structure F, then the vanishing $[V, V]$ is equivalent to that of all the Nijenhuis brackets of $\{F, G, H\}$. It follows that the bundle V is trivial if and only if $[V, V] = 0$.

Keywords**—**Almost split - hypercomplex structure, Almost split quaternionic structure, Almost split quaternion Kaehler manifold, Obata connection.

I. Introduction

THE split quaternion Kaehler manifolds are new developing a subject. Some of authers are making studies on these topics. In this study, we showed that the vanishing of $[V, V]$ is equivalent to all the Nijenhuis brackets of $\{F, G, H\}$ in split quaternion Kaehler manifolds by the same token the technic used in [7] by of Fatma Özdemir for real quaternion Kaehler manifolds.

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II. Preliminaries

An almost complex structure on a manifold M is a tensor field $J : T(M) \to T(M)$ satisfying the identity $J^2 = -id$. An almost split-hypercomplex structure on a 4n-dimensional semi-Riemannian manifold M with metric tensor g of signature $(-, -, +, +)$ is a triple $S = (F, G, H)$ of F, G *and* H satisfying the following conditions:

$$\begin{aligned} &F^2 = -I, G^2 = H^2 = I, H = FG, \\ &FG + GF = FH + HF = GH + HG = 0 \end{aligned} \tag{1}$$

Where I denotes the identity transformation of $T_x(M)$ [8]. Almost split quaternionic structures have intensively been studied in the literature [8-10].

Definition 1. An almost split quaternion manifold (M, V) is defined to be a 4n-dimensional semi Riemannian manifold M with metric tensor g of signature $(-, -, +, +)$, together with a 3-dimensional bundle V of tensors of type (1,1) over M satisfying the condition (1). If on each coordinate neighborhood U of M, the tensors F, G *and* H satisfy the conditions (1), then the bundle V is called an almost split quaternionic structure on M [6].

The Nijenhuis bracket of two tensor fields A and B of type (1,1) is a tensor field of type (1,2) defined by [6]

$$\begin{aligned} [A, B](X, Y) = &[AX, BY] - A[BX, Y] - B[X, AY] \\ &+ [BX, AY] - B[AX, Y] - A[X, BY] \\ &+ (AB + BA)[X, Y] \end{aligned} \tag{2}$$

In particular, if $A = B$ we have

$$\begin{aligned} [A, A](X, Y) = 2([AX, AY] + A^2[X, Y] \\ - A[AX, Y] - A[X, AY]) \end{aligned} \tag{3}$$

In [8], it has been shown that the vanishing of any two of the Nijenhuis brackets of F, G *and* H is a necessary and sufficient condition for the existence of a torsion free connection ∇' such that $\nabla' F = \nabla' G = \nabla' H = 0$.

Since $[F,F]$, $[G,G]$ and $[H,H]$ are locally defined objects, to obtain a global condition for the triviality of V, we globally define the (1,2) tensor field $[V,V]$ by

$$[V,V] = [F,F] + [G,G] + [H,H] \tag{4}$$

By the Newlander-Nirenberg Theorem an almost complex structure is a complex structure if and only if it is integrable i.e., it has no torsion. Thus, if the tensor fields F, G and H are integrable, then their brackets vanish. In [6], it has been shown that any two of the equations

$$\begin{aligned}[F,F] &= [G,G] = [H,H] \\ &= [F,G] = [F,H] = [G,H] = 0\end{aligned} \tag{5}$$

hold, so do the others. It is then shown that there is a torsion-free connection with respect to which F, G and H are covariantly constant, and it follows that V is a trivial bundle.

V is locally spanned by almost split-hypercomplex structures $S = \{F, G, H\}$ and the structure tensor of S is defined as in [8]

$$T^S = (1/12)\left([F,F] + [G,G] + [H,H]\right). \tag{6}$$

For the almost split-hypercomplex structure $S = \{F, G, H\}$, there exists a unique linear connection ∇^S, which preserves, S, that is, $\nabla^F = 0, \nabla^G = 0, \nabla^H = 0$, where ∇^S is the Obata connection of S [9] and its torsion tensor is T^S. The torsion T^S of ∇^S vanishes if and only if $S = \{F, G, H\}$ is split-hypercomplex [9].

III. Triviality of Almost Split Quternionic Structures on Split Quaternionic Kaehler Manifolds

Firstly, we start by recalling basic definitions. If M admits an almost split quaternionic structure then at each point x in M there is an orthonormal basis of $T(M)$, of the form

$$\{X_i, FX_i, GX_i, HX_i\}, 1 \le i \le n \tag{7}$$

and the set of all such frames at all points $x \in M$ constitute a subbundle of the bundle $O(M)$ of the orthonormal frames denoted by $\boldsymbol{H}(M)$. Such a reduction of the frame bundle is possible if and only if the structure group of the tangent bundle is reducible to $Sp(n)Sp(1)$[6]. The torsion tensor of a connection ∇ is defined by $T(X,Y) = \nabla_X{}^Y - \nabla_Y{}^X - [X,Y]$, and the connection is called "torsion-free" if $T = 0$. A torsion free connection ∇ is compatible with the metric tensor $g = \langle , \rangle$, hence, $Z\langle X,Y\rangle = \langle \nabla_Z{}^X, Y\rangle + \langle X, \nabla_Z{}^Y\rangle$ [2], and this is equivalent to the reducibility of ∇ to $O(M)$. Furthermore, if ∇ is reducible to $\boldsymbol{H}(M)$, the manifold is called "split quaternionic Kaehler manifold"[6].

Our aim is to obtain global conditions for the triviality of V, here, we study the case where M is a split quaternionic Kaehler manifold and use connections, which are reducible to $\boldsymbol{H}(M)$, to obtain our main results.

Theorem 2. Let M be an almost split quaternionic Kaehler manifold. A connection ∇ on $O(M)$ is reducible to a connection on $\boldsymbol{H}(M)$ if and only if the covariant derivatives of the tensor fields of F, G and H satisfy the following conditions:

$$\begin{aligned}&\nabla F = aG + bH, \nabla G = aF + cH, \\ &\nabla H = bF - cG\end{aligned} \tag{8}$$

where a, b and c are 1-forms[5].

Lemma 3. Let M be an 4n dimensional semi-Riemann manifold, ∇ be a torsion-free connection reducible to $\boldsymbol{H}(M)$ and a, b and c be as in Theorem 4. Then

$$\begin{aligned}&[\mathrm{F,F}] = 0 \text{ if and only if } \mathrm{b(X)= -a(FX)}, \\ &[\mathrm{G,G}] = 0 \text{if and only if } \mathrm{c(X)= -a(GX)}, \\ &[\mathrm{H,H}] = 0 \text{ if and only if } \mathrm{c(X)= -b(HX)}.\end{aligned} \tag{9}$$

Proof. In this case, we have $[X,Y] = \nabla_X{}^Y - \nabla_Y{}^X$, and $\nabla F, \nabla G$ and ∇H are given by (8). we give the details of the computation for $[F,F]$:

$$\begin{aligned}\frac{1}{2}[F,F](X,X) &= [FX,FY] - [X,Y] - F[FX,Y] \\ &\quad - F[X,FY] \\ &= \left(\nabla_{FX}{}^{FY} - \nabla_{FY}{}^{FX}\right) \\ &\quad - \left(\nabla_X{}^Y - \nabla_Y{}^X\right) \\ &\quad - F\left(\nabla_{FX}{}^Y - \nabla_Y{}^{FX}\right) \\ &\quad - \left(\nabla_X{}^{FY} - \nabla_{FY}{}^X\right)\end{aligned}$$

$$
\begin{aligned}
&= \left(\nabla_{FX}{}^{F}\right)Y - \left(\nabla_{FY}{}^{F}\right)X \\
&+ F\left(\left(\nabla_{Y}{}^{F}\right)X - \left(\nabla_{X}{}^{F}\right)Y\right)
\end{aligned} \tag{10}
$$

By using the conditions of Theorem 2, we get

$$
\begin{aligned}
\frac{1}{2}\left[F,F\right]\left(X,X\right) &= \left[a\left(FX\right)G\left(Y\right) + b\left(FX\right)H\left(Y\right)\right] \\
&- \left[a\left(FY\right)G\left(X\right) + b\left(FY\right)H\left(X\right)\right] \\
&+ F\left[a\left(Y\right)G\left(X\right) + b\left(Y\right)H\left(X\right)\right] \\
&- F\left[a\left(X\right)G\left(Y\right) + b\left(X\right)H\left(Y\right)\right] \\
&= a\left(FX\right)G\left(Y\right) + b\left(FX\right)H\left(Y\right) \\
&- a\left(FY\right)G\left(X\right) \\
&- b\left(FY\right)H\left(X\right) + a\left(Y\right)H\left(X\right) \\
&- b\left(Y\right)G\left(X\right) \\
&- a\left(X\right)H\left(Y\right) + b\left(X\right)G\left(Y\right) \\
&= \left(-a\left(FY\right) - b\left(Y\right)\right)G\left(X\right) \\
&+ \left(a\left(FX\right)b\left(X\right)\right)G\left(Y\right) \\
&+ \left(a(Y) - b(FY)\right)H\left(X\right) \\
&+ \left(b(FX) - a(X)\right)H\left(Y\right).
\end{aligned} \tag{11}
$$

If X and Y are any linearly independent vector fields, then $\left\{GX, GY, HX, HY\right\}$ is a linearly independent set and their coefficients should vanish. Hence, $b(X) = -a(FX)$. We can perform the similar computations for G and H, but we do not reproduce them here.

Similarly, it can be easily shown that the vanishing of any mixed Nijenhuis bracket is equivalent to the vanishing of all six, but we omit the proof here. We show that for a split quaternionic Kaehler manifold, the vanishing of $\left[V,V\right]$ is equivalent to the vanishing of all Nijenhuis brackets.

Theorem 4. Let M be a split quaternionic Kaehler manifold, ∇ be a torsion-free connection reducible to $\boldsymbol{H}(M)$ and a, b and c be as in Theorem 5. Then, $\left[F,F\right] + \left[G,G\right] + \left[H,H\right] = 0$ if and only if $b(X) = -a(FX)$, $c(X) = -a(GX)$.

Proof. By computing $\left[F,F\right], \left[G,G\right]$ *and* $\left[H,H\right]$ we obtain

$$
\begin{aligned}
&\frac{1}{2}\left(\left[F,F\right] + \left[G,G\right] + \left[H,H\right]\right)\left(X,Y\right) \\
&= \left[-a\left(GY\right) - 2c\left(Y\right) - b\left(HY\right)\right]\left(FX\right) \\
&+ \left[a\left(GX\right) + 2c\left(X\right) + b\left(HX\right)\right]\left(FY\right) \\
&+ \left[-a\left(FY\right) + c\left(HY\right)\right]\left(GX\right) \\
&+ \left[a\left(FX\right) - c\left(HX\right)\right]\left(GY\right) \\
&+ \left[-b\left(FY\right) - c\left(GY\right)\right]\left(HX\right) \\
&+ \left[b\left(FX\right) + c\left(GX\right)\right]\left(HY\right).
\end{aligned} \tag{12}
$$

In dimensions 4n > 4, if X and Y are linearly independent vector fields, then $\left\{FX, FY, GX, GY, HX, HY\right\}$ is a linearly independent set, and the result follows by algebraic calculation. In 4-dimensions, the results can be obtained by direct computation. Thus, $\left[V,V\right] = 0$ implies $b(X) = -a(FX)$ and $c(X) = -a(GX)$. Conversly, if the conditions $b(X) = -a(FX)$ and $c(X) = -a(GX)$ hold, then by using the Lemma 5, we obtain $\left[V,V\right] = 0$.

It then follows from Lemma 3 that all Nijenhuis brackets vanish.

In the next section, we study the implications of $\left[V,V\right] = 0$ on an arbitrary almost split quaternionic manifold, and we also show that $\left[V,V\right] = 0$ implies the vanishing of the individual Nijenhuis brackets, if either of F, G *and* H has no Nijenhuis torsion.

IV. Triviality of Almost Split Quaternionic Structures on Complex Manifolds

We first write down explicitly the action of $\left[V,V\right]$ on pairs of vectors belonging to an orthonormal basis such as given in (7). If M is a 4n dimensional semi-Riemann manifold, then the set $\left\{X_i, FX_i, GX_i, HX_i\right\}$, i =1,2,...,n can be chosen as a local basis for TM. Then, the tensor $\left[V,V\right]$ is determined by its action on the sets S_1 and S_2 given by

$$
\begin{aligned}
S_1 = \{&\left(X_i, FX_i\right), \left(X_i, FX_i\right), \left(X_i, FX_i\right), \\
&\left(X_i, FX_i\right), \left(X_i, FX_i\right), \left(X_i, FX_i\right)\}
\end{aligned} \tag{13}
$$

$i = 1, 2, \ldots, n$

$$S_2 = \{(X_i, X_j), (X_i, FX_j), (X_i, GX_j), (X_i, HX_j), (FX_i, X_j), (FX_i, FX_j), (FX_i, HX_j), (GX_i, X_j), (GX_i, FX_j), (GX_i, GX_j), (GX_i, HX_j), (HX_i, X_j), (HX_i, FX_j), (HX_i, GX_j), (HX_i, HX_j)\} \quad (14)$$

$i, j = 1, 2, \ldots, n; \; i < j$.

For any X any Y, we first compute the actions $[F,F], [G,G]$ *and* $[H,H]$ on the pairs of vectors

$$(X,Y), (X,FY), (X,GY), (X,HY), (FX,Y), (FX,FY), (FX,GY), (FX,HY), (GX,Y), (GX,FY), (GX,GY), (GX,HY), (HX,Y), (HX,FY), (HX,GY), (HX,HY), \quad (15)$$

These actions are given in the tables below.

$[F,F]$	Y	FY	GY	HY
X	A_1	$-FA_1$	A_3	$-FA_1$
FX	$-FA_1$	$-A_1$	$-FA_3$	$-A_3$
GX	A_2	$-FA_2$	A_4	$-FA_4$
HX	$-FA_2$	$-A_2$	$-FA_4$	$-A_4$

$[G,G]$	Y	FY	GY	HY
X	B_1	B_3	$-GB_1$	GB_3
FX	B_2	B_4	$-GB_2$	GB_4
GX	$-GB_1$	$-GB_3$	B_1	$-B_3$
HX	GB_2	GB_4	$-B_2$	B_4

$[H,H]$	Y	FY	GY	HY
X	C_1	C_3	$-HC_3$	$-HC_1$
FX	C_2	C_4	$-HC_4$	$-C_2$
GX	$-HC_2$	$-HC_4$	C_4	C_2
HX	$-HC_1$	$-HC_3$	C_3	C_1

From these tables, we obtain the action of $[V,V]$ on the set given by (15) as:

$$[V,V](X,Y) = A_1 + B_1 + C_1$$
$$[V,V](FX,Y) = -FA_1 + B_2 + C_2$$
$$[V,V](GX,Y) = A_2 - GB_1 - HC_2$$
$$[V,V](HX,Y) = -FA_2 + GB_1 - HC_1$$
$$[V,V](X,FY) = -FA_1 + B_3 + C_3$$
$$[V,V](FX,FY) = -A_1 + B_4 + C_4$$
$$[V,V](GX,FY) = -FA_2 - GB_3 - HC_4$$
$$[V,V](HX,FY) = -A_2 + GB_4 - HC_3$$
$$[V,V](X,GY) = A_3 - GB_1 - HC_3$$

$$
\begin{aligned}
&[V,V](FX,GY) = -FA_3 - GB_2 - HC_4 \\
&[V,V](GX,GY) = A_4 + B_1 + C_4 \\
&[V,V](HX,GY) = -FA_4 - B_2 + C_3 \\
&[V,V](X,HY) = -FA_3 + GB_3 - HC_1 \\
&[V,V](FX,HY) = -A_3 + GB_4 - C_2 \\
&[V,V](GX,HY) = -FA_4 - B_3 + C_2 \\
&[V,V](HX,HY) = -A_4 + B_4 + C_1
\end{aligned} \tag{16}
$$

These linear equations can be solved as

$$
\begin{aligned}
B_1 &= \frac{1}{2}\left[-A_1 - GA_2 - GA_3 + A_3\right] \\
B_2 &= \frac{1}{2}\left[FA_1 + HA_2 - HA_3 + FA_4\right] \\
B_3 &= \frac{1}{2}\left[FA_1 - HA_2 + HA_3 + FA_3\right] \\
B_4 &= \frac{1}{2}\left[A_1 - GA_2 - GA_3 - A_3\right]
\end{aligned} \tag{17}
$$

and

$$
\begin{aligned}
C_1 &= \frac{1}{2}\left[-A_1 + GA_2 + GA_3 - A_3\right] \\
C_2 &= \frac{1}{2}\left[FA_1 - HA_2 + HA_3 - FA_4\right] \\
C_3 &= \frac{1}{2}\left[FA_1 + HA_2 - HA_3 - FA_3\right] \\
C_4 &= \frac{1}{2}\left[A_1 + GA_2 + GA_3 + A_3\right]
\end{aligned} \tag{18}
$$

Thus, the action of $[V,V]$ on the set given by (15) is determined by the action of $[F,F]$ (actually any one of $[F,F]$, $[G,G]$ *and* $[H,H]$ on the same set.

Note that, to see the effect of $[V,V]$ on the set S_1, we set $X = Y = X_i$ in the tables and it can be seen that

$A_1 = A_4 = B_1 = B_4 = C_1 = C_4 = 0,$

$A_2 = -A_3, B_2 = -B_3, C_2 = C_3.$

Then, equations (16) reduce to

$$B_3 + C_3 = 0, A_3 - HC_3 = 0, -FA_3 + CB_3 = 0.$$

Which imply that

$$B_3 = HA_3, C_3 = -HA_3.$$

On the other hand, on the set S_2, we have the full set of equations (17-18) for each i, j. As a result, if in particular $[F,F] = 0$, then $[V,V] = 0$ implies that $[G,G] = [H,H] = 0$. Thus, we have proved the following theorem.

Theorem 5. Let M be a 4n dimensional almost split quaternionic manifold with a local basis $\{F,G,H\}$ for its almost split quaternionic structure and let $[V,V] = [F,F] + [G,G] + [H,H]$. Then if any of the Nijenhuis tensors of F,G and H vanishes and $[V,V] = 0$, then the other two also vanishes.

In particular, from previous section, it follows that V is flat. If M is a split-complex manifold with almost split-complex structure say F, then $[V,V] = 0$ implies that V is a trivial bundle.

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