A Deterministic Dynamic Programming Approach for Optimization Problem with Quadratic Objective Function and Linear Constraints

S. Kavitha, Nirmala P. Ratchagar

Abstract**—**This paper presents the novel deterministic dynamic programming approach for solving optimization problem with quadratic objective function with linear equality and inequality constraints. The proposed method employs backward recursion in which computations proceeds from last stage to first stage in a multi-stage decision problem. A generalized recursive equation which gives the exact solution of an optimization problem is derived in this paper. The method is purely analytical and avoids the usage of initial solution. The feasibility of the proposed method is demonstrated with a practical example. The numerical results show that the proposed method provides global optimum solution with negligible computation time.

Keywords**—**Backward recursion, Dynamic programming, Multi-stage decision problem, Quadratic objective function.

I. Introduction

THE mathematical modeling of various real world applications is formulated as quadratic objective function with a linear set of equality and inequality constraints. For example, planning and scheduling problems, various engineering design problems are formulated as an optimization problem with quadratic objective function. Particularly, in thermal power stations the fuel cost of the generating unit is formulated as a quadratic objective function. Various iterative methods such as lambda iteration method, gradient search method are presented in the literature for the solution of this problem [1]. These methods require initial assumptions and solutions are obtained through the iterative procedure. A branch and bound algorithm [2] has been developed for minimization of linearly constrained quadratic functions.

Dynamic programming is a numerical algorithm based on Bellman's optimality principle that find the control law, which provides the globally minimum value for the given objective function while satisfying the constraints [3]. Bellman's principle of optimality states that "An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision". The advantage of DP compared to other optimization techniques is to be very efficient, not to be influenced by linear or nonlinear nature of the problem and especially to always guarantee that the solution found represents the global optimum [4]. Dynamic programming approach has been applied for solution of various practical optimization problems such as reservoir operational problems [5], agriculture and natural resource problems [6], crew scheduling problems [7], combined economic and emission dispatch problem [8], job scheduling problem [9] etc.

Dynamic programming divides a complex optimization problem into simple sub-problems. The important feature of the dynamic programming approach is structuring of optimization problems into multistage decision problem, which are solved sequentially one stage at a time. Each sub-problem is solved as an optimization problem and its solution helps to define the characteristics of the next sub-problem in the sequence. The recursive optimization procedure of a dynamic programming approach provides a solution of the n-stage problem by solving one stage problem and sequentially including one stage at a time and solving one stage problem. This procedure will be continued till the solution of all the sub-problems is determined.

This paper describes the new analytical solution methodology in which the backward recursive dynamic programming approach is implemented for the solution of quadratic optimization problem. The practical application of the proposed method is demonstrated with suitable example.

II. Problem Formulation

The optimization problem with quadratic objective function, an equality constraint and a set of inequality constraints are given by

$$Min\, f(x) = \sum_{i=1}^{n} \alpha_i x_i^2 + \beta_i x_i + \gamma_i \qquad (1)$$

here the coefficients $\alpha_i, \beta_i\ and\ \gamma_i$ are real numbers subject to equality constraint

$$\sum_{i=1}^{n} x_i = X_T \qquad (2)$$

and inequality constraints

$$x_i^l \le x_i \le x_i^u\ ,\ \ x_i \ge 0 \qquad (3)$$

where X_T is the sum of the decision variables, $x_i^l\ and\ x_i^u$ are

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lower and upper limits of decision *i*th decision variable, respectively.

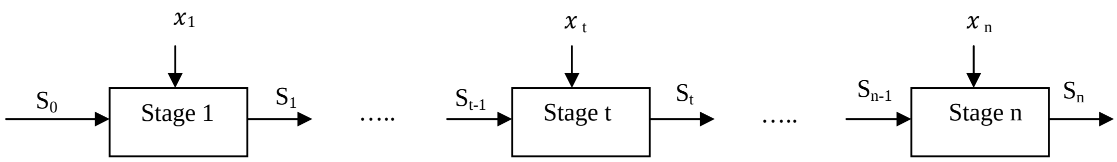

Fig. 1 Multi-stage decision problem

III. Backward Recursive Approach

The optimum solution of n-variable problem is determined using dynamic programming by decomposing the original problem into n-stages. Each stage comprises a single variable sub-problem. Dynamic programming does not provide the computational details for optimizing each stage because the nature of the stage differs depending on the optimization problem. Such details are designed by the problem solver [10]. The computations in dynamic programming are carried out recursively that is the optimum solution of one sub-problem is taken as an input to the next sub-problem. The complete solution for the original problem is obtained by solving the last sub-problem. The sub-problems are linked together by some common constraints. The feasibility of constraints is accounted when each sub-problem is solved.

Recursive equations are derived to solve the problem in sequence. These equations for multi-stage decision problems can be formulated in a forward or backward manner. Dynamic programming literature invariably uses backward recursion because, in general, it may be efficient computationally [11], [12]. A series of recursive equations are solved, each equation depending on the output values of the previous equation.

Consider a multistage problem shown in Fig. 1. The stages 1, 2, … , t , …, n are labeled in an ascending order. For the t-stage, the input is denoted by s_{t-1} and output is denoted by s_t. The objective of a multistage problem is to find optimal values of $x_1, x_2, \dots, x_n$ so as to minimize the objective function subject to satisfying the equality and inequality constraints. The variable which links up two stages is called a state variable. The output of each stage is given by $S_0 = 0$, $S_1 = x_1 = S_2 - x_2$, $S_n = \sum_{i=1}^{n} x_i$.

Consider the last stage as the first sub-problem in the multi-stage decision process. The input variable for the last stage is s_{n-1}. The relation between the state variables and decision variables are given by $S_t = S_{t-1} + x_t, \quad t = 1,2,\dots,n$. The quadratic objective optimization problem is solved by writing recursive equation for the last stage and then proceeding towards the first stage. The objective function of the last stage is given by

$$f_n^*(S_n) = \min(\alpha_n x_n^2 + \beta_n x_n + \gamma_n) \quad (4)$$

Now the last two stages are grouped together as the second sub-problem. The objective function is written as

$$f_{n-1}^*(S_{n-1}) = \min((\alpha_{n-1} x_{n-1}^2 + \beta_{n-1} x_{n-1} + \gamma_{n-1}) + (S_n)) \quad (5)$$

Substituting the objective function of nth stage in (5) gives

$$f_{n-1}^*(S_{n-1}) = \min(\alpha_{n-1} x_{n-1}^2 + \beta_{n-1} x_{n-1} + \gamma_{n-1} + \alpha_n x_n^2 + \beta_n x_n + \gamma_n) \quad (6)$$

The output of the nth stage is given by

$$S_n = x_n + S_{n-1} \quad (7)$$

The output of n-1 stage is given by

$$S_{n-1} = S_{n-2} + x_{n-1} \quad (8)$$

Substituting (8) in (7)

$$x_n = (S_n - S_{n-2}) - x_{n-1} \quad (9)$$

Equation (6) is converted into a single variable problem by substituting (9) in (6), i.e,

$$f_{n-1}^*(S_{n-1}) = \min(\alpha_{n-1} x_{n-1}^2 + \beta_{n-1} x_{n-1} + \gamma_{n-1} + \alpha_n((S_n - S_{n-2}) - x_{n-1})^2 + \beta_n((S_n - S_{n-2}) - x_{n-1}) + \gamma_n) \quad (10)$$

The optimal value of variable x_{n-1} is obtained by differentiating (10) with respect to x_{n-1} and equating to zero,

$$2\alpha_{n-1} x_{n-1} + \beta_{n-1} + 2\alpha_n x_{n-1} - 2\alpha_n (S_n - S_{n-2}) - \beta_n = 0 \quad (11)$$

The expression for x_{n-1} is obtained from (11)

$$x_{n-1} = \frac{2\alpha_n(S_n - S_{n-2}) + \beta_n - \beta_{n-1}}{2\alpha_{n-1} + 2\alpha_n} \quad (12)$$

Substituting (12) in (9) gives the expression for the variable x_n

$$x_n = \frac{2\alpha_{n-1}(S_n - S_{n-2}) - \beta_n + \beta_{n-1}}{2\alpha_{n-1} + 2\alpha_n} \quad (13)$$

By substituting the expressions for x_{n-1} and x_n given in (12), (13) in (6), we get

$$f_{n-1}^*(S_{n-1}) = \min(\alpha_{n-1}\left[\frac{2\alpha_n(S_n - S_{n-2}) + \beta_n - \beta_{n-1}}{2\alpha_{n-1} + 2\alpha_n}\right]^2 + \beta_{n-1}\left[\frac{2\alpha_n(S_n - S_{n-2}) + \beta_n - \beta_{n-1}}{2\alpha_{n-1} + 2\alpha_n}\right] + \gamma_{n-1} + \alpha_n\left[\frac{2\alpha_{n-1}(S_n - S_{n-2}) - \beta_n + \beta_{n-1}}{2\alpha_{n-1} + 2\alpha_n}\right]^2 + \beta_n\left[\frac{2\alpha_{n-1}(S_n - S_{n-2}) - \beta_n + \beta_{n-1}}{2\alpha_{n-1} + 2\alpha_n}\right] + \gamma_n) \quad (14)$$

On simplifying, (14) becomes

$$f_{n-1}^*(S_{n-1}) = \min((S_n - S_{n-2})^2\left(\frac{1}{\left(\frac{1}{\alpha_{n-1}}\right) + \left(\frac{1}{\alpha_n}\right)}\right) + (S_n - S_{n-2})\left(\frac{\left(\frac{\beta_{n-1}}{\alpha_{n-1}}\right) + \left(\frac{\beta_n}{\alpha_n}\right)}{\left(\frac{1}{\alpha_{n-1}}\right) + \left(\frac{1}{\alpha_n}\right)}\right) + (\beta_n - \beta_{n-1})^2\frac{(\alpha_{n-1} + \alpha_n)}{(2\alpha_{n-1} + 2\alpha_n)^2} + (\beta_n - \beta_{n-1})\frac{(\beta_{n-1} - \beta_n)}{(2\alpha_{n-1} + 2\alpha_n)} + \gamma_{n-1} + \gamma_n) \quad (15)$$

The output of the stage *n*-2 is given by

$$S_{n-2} = S_{n-3} + x_{n-2} \quad (16)$$

Substituting $S_n - S_{n-2} = S_n - (S_{n-3} + x_{n-2})$ in (15) gives

$$f_{n-1}^*(S_{n-1}) = \min(((S_n - S_{n-3}) - x_{n-2})^2\left(\frac{1}{\left(\frac{1}{\alpha_{n-1}}\right) + \left(\frac{1}{\alpha_n}\right)}\right) + ((S_n - S_{n-3}) - x_{n-2})\left(\frac{\left(\frac{\beta_{n-1}}{\alpha_{n-1}}\right) + \left(\frac{\beta_n}{\alpha_n}\right)}{\left(\frac{1}{\alpha_{n-1}}\right) + \left(\frac{1}{\alpha_n}\right)}\right) + (\beta_n - \beta_{n-1})^2\frac{(\alpha_{n-1} + \alpha_n)}{(2\alpha_{n-1} + 2\alpha_n)^2} + (\beta_n - \beta_{n-1})\frac{(\beta_{n-1} - \beta_n)}{(2\alpha_{n-1} + 2\alpha_n)} + \gamma_{n-1} + \gamma_n) \quad (17)$$

The last three stages are grouped together as the third sub-problem and the objective function is given by

$$f_{n-2}^*(S_{n-2}) = \min((\alpha_{n-2}x_{n-2}^2 + \beta_{n-2}x_{n-2} + \gamma_{n-2}) + f_{n-1}^*(S_{n-1})) \quad (18)$$

By substituting (17) in (18) gives

$$f_{n-2}^*(S_{n-2}) = \min((\alpha_{n-2}x_{n-2}^2 + \beta_{n-2}x_{n-2} + \gamma_{n-2}) + ((S_n - S_{n-3}) - x_{n-2})^2\left(\frac{1}{\left(\frac{1}{\alpha_{n-1}}\right) + \left(\frac{1}{\alpha_n}\right)}\right) + ((S_n - S_{n-3}) - x_{n-2})\left(\frac{\left(\frac{\beta_{n-1}}{\alpha_{n-1}}\right) + \left(\frac{\beta_n}{\alpha_n}\right)}{\left(\frac{1}{\alpha_{n-1}}\right) + \left(\frac{1}{\alpha_n}\right)}\right) + (\beta_n - \beta_{n-1})^2\frac{(\alpha_{n-1} + \alpha_n)}{(2\alpha_{n-1} + 2\alpha_n)^2} + (\beta_n - \beta_{n-1})\frac{(\beta_{n-1} - \beta_n)}{(2\alpha_{n-1} + 2\alpha_n)} + \gamma_{n-1} + \gamma_n)) \quad (19)$$

The optimal value of variable x_{n-2} is obtained by differentiating (19) with respect to x_{n-2} and equating to zero,

$$2\alpha_{n-2}x_{n-2} + \beta_{n-2} - 2((S_n - S_{n-3}) - x_{n-2})\left(\frac{1}{\left(\frac{1}{\alpha_{n-1}}\right) + \left(\frac{1}{\alpha_n}\right)}\right) - \left(\frac{\left(\frac{\beta_{n-1}}{\alpha_{n-1}}\right) + \left(\frac{\beta_n}{\alpha_n}\right)}{\left(\frac{1}{\alpha_{n-1}}\right) + \left(\frac{1}{\alpha_n}\right)}\right) = 0 \quad (20)$$

From (20), the expression for x_{n-2} is given by

$$x_{n-2} = \frac{2\left(\frac{1}{\left(\frac{1}{\alpha_{n-1}}\right) + \left(\frac{1}{\alpha_n}\right)}\right)(S_n - S_{n-3}) + \left(\frac{\left(\frac{\beta_{n-1}}{\alpha_{n-1}}\right) + \left(\frac{\beta_n}{\alpha_n}\right)}{\left(\frac{1}{\alpha_{n-1}}\right) + \left(\frac{1}{\alpha_n}\right)}\right) - \beta_{n-2}}{2\alpha_{n-2} + 2\left(\frac{1}{\left(\frac{1}{\alpha_{n-1}}\right) + \left(\frac{1}{\alpha_n}\right)}\right)} \quad (21)$$

In general, the optimal value of decision variable of the *i*th sub-problem is expressed from (21)

$$x_i = \frac{2\left(\frac{1}{\sum_{k=i+1}^{n}\left(\frac{1}{\alpha_k}\right)}\right)(S_n - S_{i-1}) + \left(\frac{\sum_{k=i+1}^{n}\left(\frac{\beta_k}{\alpha_k}\right)}{\sum_{k=i+1}^{n}\left(\frac{1}{\alpha_k}\right)}\right) - \beta_i}{2\alpha_i + 2\left(\frac{1}{\sum_{k=i+1}^{n}\left(\frac{1}{\alpha_k}\right)}\right)}$$
$$for\ i = 1,2,\ldots,n-1 \quad (22)$$

Using the generalized recursive given in (22), the solution of sub-problems from stage 1 to stage n-1 can be obtained and finally, the solution for *n*th stage is computed using (7).

IV. Computational Procedure

The procedure for implementing the proposed analytical approach for the solution quadratic objective function is detailed in the following steps:

Step 1. Read the coefficients of the variables, lower and upper bounds of each variable and equality constraint of the given problem.

Step 2. Treat the given optimization problem as a multistage problem. The problem is solved by breaking the original problem into a number of single stage problems. The number of single stage problem is equal to the number of variables.

Step 3. Set S_0=0, determine the optimal value of variable x_1 for the first stage problem using the generalized recursive equation given in (22). The optimal values of the remaining variables are determined using recursive equation by proceeding stage 1 to stage n in a forward manner.

Step 4. During the recursive procedure if any variable violates their maximum or minimum limit then that variable is fixed at the corresponding violated limit and this sub-problem is eliminated from the recursive procedure and this variable value is subtracted from the equality constraint. Now start the recursive procedure for stage 1.

Step 5. Calculate the objective function of the given optimization problem using the optimal solutions obtained through the recursive procedure.

V. Computational Studies

To test the performance of the proposed method, computational studies are performed for the real world optimization problem. The recursive approach is implemented for power generation scheduling problem in a thermal power station. The realistic Indian system data given in [13] is used in this paper. The input-output curve of a generating unit specifies cost of fuel used per hour as the function of the generator power output. In practice [14], the fuel cost of generator i is represented as a quadratic function of power generation P_i,i.e.,

$$C_i(P_i) = a_i P_i^2 + b_i P_i + c_i \tag{23}$$

where $C_i(P_i)$ is the fuel cost in (Rs/h), P_i is the power generated in Megawatt (MW), a_i, b_i, c_i are the fuel coefficients of the *i*th unit. The objective is to minimize total fuel cost of committed generating units, which is expressed as

$$C(P) = Min \sum_{i=1}^{N} C_i\,(P_i) \tag{24}$$

Subject to

i. Power balance constraint: The total power generated by the generating units must supply total load demand P_D:

$$\sum_{i=1}^{N} P_i = P_D \tag{25}$$

ii. Generator capacity constraint: The power generated by each generator is constrained between its minimum and maximum limits, i.e.,

$$P_i^{min} \le P_i \le P_i^{max} \tag{26}$$

where P_i^{min} and P_i^{max} are the minimum and maximum power output of the *i*th generator.

This case study considered the six generating units test system. The fuel cost coefficients and generator capacity constraints are given in Table I. The total load demand expected to meet by the six generating units is P_D=900 MW. The result obtained through the proposed recursive approach is given in Table II. In this case study, the generating output of each generator lies within the operating limits and total generation is equal to the total load demand. The total fuel cost is Rs 45463.4922.

TABLE I
FUEL COST COEFFICIENTS AND OPERATING LIMITS OF GENERATING UNITS

Generating unit	a_i	b_i	c_i	P_i^{min} MW	P_i^{max} MW
G1	0.15247	38.53973	756.79886	10	125
G2	0.10587	46.15916	451.32513	10	150
G3	0.02803	40.39655	1049.32513	40	250
G4	0.03546	38.30553	1243.5311	35	210
G5	0.02111	36.32782	1658.5696	130	325
G6	0.01799	38.27041	1356.6592	125	315

For the same test system, the load demand P_D is increased to 1170MW. During the recursive procedure sixth generating unit violates its maximum limit. Therefore generation of sixth unit is fixed at its maximum limit, i.e., P_6=315 MW. Now the remaining generators have to meet a load demand of 855 MW. With this load demand, recursive procedure is applied for the remaining units. In this case, fifth generator violates the maximum generation limit. Hence the generation of fifth unit is fixed as P_5= 325 MW. The remaining four units should supply the load demand of 530 MW. During the recursive procedure fourth unit violates its maximum limit therefore the generation of fourth unit is fixed to 210MW, and recursive procedure is applied to remaining three generators with a load demand of 320 MW. The total fuel cost is Rs 59095.1804. The optimal generation schedule obtained through the proposed method is given in Table III. The solutions obtained using recursive approach exactly matches the solutions obtained through the lambda iteration method.

TABLE II
OPTIMAL GENERATION FOR LOAD DEMAND 900 MW

Generating Unit	Power Output (MW)
P_1	32.4969
P_2	10.8160
P_3	143.6460
P_4	143.0318
P_5	287.1039
P_6	282.9053
Tot. generation(MW)	900
Fuel cost (Rs/h)	45463.4922

TABLE III
OPTIMAL GENERATION FOR LOAD DEMAND 1170 MW

Generating Unit	Power Output (MW)
P_1	49.3810
P_2	35.1318
P_3	235.4872
P_4	210.0000
P_5	325.0000
P_6	315.0000
Tot. generation(MW)	1170
Fuel cost (Rs/h)	59095.1804

The salient features of the proposed recursive approach are (i) This method avoids the need of initial assumption (ii) the exact recursive formula derived in this paper directly gives the solution without need to perform any iterative procedure and (iii) This approach can be easily implemented for solution of large scale optimization problem with quadratic objective and linear constraints described in Section II.

VI. CONCLUSION

A novel deterministic dynamic programming approach has been developed for the solution of quadratic objective function with linear constraints. The proposed analytical approach directly gives the global optimal solution for the problem without any assumptions. The validity of the approach is verified with a practical power generation scheduling problem. The result of the test system shows the superiority of the proposed method and its potential for solving practical optimization problems. The developed approach is simple and efficient which can be easily implemented for the solution of various optimization problems modeled as quadratic objective function with linear constraints.

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