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A Decomposition Method for the Bipartite Separability of Bell Diagonal States

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Abstract—A new decomposition form is introduced in this report to establish a criterion for the bi-partite separability of Bell diagonal states. A such criterion takes a quadratic inequality of the coefficients of a given Bell diagonal states and can be derived via a simple algorithmic calculation of its invariants. In addition, the criterion can be extended to a quantum system of higher dimension.

Keywords—decomposition, bipartite separability, Bell diagonal states.

I. Introduction

QUANTUM entanglement is a characteristic of two or more quantum systems, which reveals the correlations that fail to be explained by classical physics. It plays an important role in the processes of quantum computation, quantum communication and quantum information theory, such as teleportation, dense coding and many quantum protocols. Deciding whether a given quantum state is *separable* is one of the most fundamental problems. Remind that in a bi-partite system $\mathcal{H}_A \otimes \mathcal{H}_B$, a quantum state is separable if it can be written as a *convex combination* of *pure product states*

$$\rho = \sum_{i=1}^{N} p_i |\psi_i^A\rangle\langle\psi_i^A| \otimes |\psi_i^B\rangle\langle\psi_i^B|, \tag{1}$$

where $p_i \geq 0$, $\sum_{i=1}^{N} p_i = 1$, $|\psi_i^A\rangle \in \mathcal{H}_A$ and $|\psi_i^B\rangle \in \mathcal{H}_B$. Otherwise, this state is *entangled*. Note that here a state is a *density operator* or *density matrix* that is *hermitian*, *trace unit* and *positive semidefinite*.

An enormous number of research works have been realized to search for the criteria to answer the separability problem of a state. The earliest criterion, reported by Peres [1], is to use the *partial transpose* of a given density operator. A such criterion provides a sufficient and necessary conditions of deciding the separability of states in lower-dimensional bipartite systems [2], including 2×2 and 2×3 systems. There are other operational criteria for separability, such as concurrence criterionn [3], reduction criterion [4], majorization criterion [5]. Nevertheless, it is very difficult to examine whether some given states of any dimension can be written as a mixture of product states [6]. On the other hand, both entanglement witnesses and positive maps are sufficient and necessary conditions under any dimension system, but these two criteria are non-operational [2]. Based on the above-mentioned criteria, some researchers developed various methods to search for optimal measurements on the given density operators [7], [8].

In 2×2 system, the Bell diagonal states can be characterized by three-parameter set, whose separability are complete studied [6], [9]. The Bell diagonal states in 3×3 system are introduced in [10], then Baumgartner, et al [11] extend their study to $d \times d$ system. In addition to these, more properties of Bell diagonal states were analyzed [12]. Numerous attempts have been made by scholars to write down the decomposition form for Bell diagonal states, but most of them are limited to special cases (Werner state) [15]–[19]. Sanpera, et al. [14] utilize a constructive algorithm to decompose the separable state in either a 2×2 or 2×3 system. The decomposing procedure is examined by Werner state with a non-unique decomposition. Another decomposition method, which are developed by Wootters [3], based on the minimum average entanglement of an ensemble of the eigenstates of a density matrix for Bell diagonal states in a 2×2 system. Although Wootters's method is a successful measure, it is difficult that the physical phenomena to observe when the separability of Bell diagonal states are transformed into entanglement.

In this article, we focus on the separability properties of Bell diagonal states in a $2^p \times 2^p$ system, We propose a criteria (necessary condition) for the bi-partite separability of the Bell diagonal states in a $2^p \times 2^p$ system, and write down a new *separable form* for the Bell diagonal states in a 2×2 system, which is different from the convex combination obtained by previous research work. In order that the any $d \times d$ systems ($2^{p-1} < d \leq 2^p$) could be analysis, they can be embedded to the $2^p \times 2^p$ system. This research work is organized in the following ways. In sec.II we review the relation between the standard basis and the spinor basis (identity matrix and Pauli matrices). In the spinor basis, we could obtain a necessary condition of the separable Bell diagonal states base on the inequality $Tr^2(\rho) \geq Tr(\rho^2)$. Besides, we carry out the proof of the sufficient condition via presented decomposition for the bi-partite separability of the Bell diagonal states in a 2×2 system. This process, based on the definition of density operators (unit trace, hermitian, and positive-semidefinite), is not only different from the method used in [2], [3], [15], [17], but gives us an insight into quantum entanglement. When the separability of Bell diagonal states are transformed into entanglement, implied the local density operators $\rho_k^{\mathcal{I}}$ moved to the outside the Hilbert space $\mathcal{H}_{\mathcal{I}}$ for $\mathcal{I} = A, B$. Then, we would operated Peres PT criterion and compared this result

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with presented. It is known on the condition $p = 1$ that ρ_B is separable iff $1 \leq \sum_{i=1}^{3} |\Omega_{ii}|$. In sec.III we extend this schemes to the condition $p > 1$ and acquire the inequality

II. Bell's Mixture in a Two-Qubit System

The discussion begins with maximally entangled states in the simplest bipartite system, a two-qubit system. The formulation of a density operator in this article employs the *spinor representation* such that one can *recursively* extend the scheme designed in a two-qubit system to that in a multi-qubit system. Including the identity matrix, the Pauli matrices are the following 2×2 matrices

$$I = \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = |0\rangle\langle 0| + |1\rangle\langle 1|;$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |0\rangle\langle 1| + |1\rangle\langle 0|;$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -i|0\rangle\langle 1| + i|1\rangle\langle 0|;$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|. \tag{2}$$

Under this representation, a two-qubit density operator is expressed as

$$\rho = \frac{1}{4}\sum_{i,j=0}^{3} \Omega_{ij}\sigma_i \otimes \sigma_j, \tag{3}$$

here $\Omega_{ij} \in \mathbb{R}$ and $\Omega_{00} = 1$.

By definition, the four maximally entangled two-qubit states(Bell's states) could be expressed as:

$$\begin{aligned}
|\Phi^+\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \\
|\Phi^-\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \\
|\Psi^+\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \\
|\Psi^-\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).
\end{aligned} \tag{4}$$

In the standard basis, the Bell's mixture can be written as:

$$\begin{aligned}
\rho_B = &\lambda_1|\Phi^+\rangle\langle\Phi^+| + \lambda_2|\Phi^-\rangle\langle\Phi^-| \\
&+ \lambda_2|\Psi^+\rangle\langle\Psi^+| + \lambda_4|\Psi^-\rangle\langle\Psi^-|
\end{aligned} \tag{5}$$

where $\lambda_i \;\; (i = 1 \sim 4)$ are the eigenvalues of the ρ_B. With the help of Eq. 2, one may rewrite Eq. 5 in the spinor basis:

$$\rho_B = \frac{1}{4}(I \otimes I + \sum_{i=1}^{3} \Omega_{ii}(-1)^{\epsilon_i}\sigma_i \otimes \sigma_i), \tag{6}$$

here $\epsilon_1 = \epsilon_3 = 0, \epsilon_2 = 1$, and

$$\begin{aligned}
\lambda_1 &= \frac{1}{4}(1 + \Omega_{11} + \Omega_{22} + \Omega_{33}), \\
\lambda_2 &= \frac{1}{4}(1 - \Omega_{11} - \Omega_{22} + \Omega_{33}), \\
\lambda_3 &= \frac{1}{4}(1 + \Omega_{11} - \Omega_{22} - \Omega_{33}), \\
\lambda_4 &= \frac{1}{4}(1 - \Omega_{11} + \Omega_{22} - \Omega_{33}).
\end{aligned} \tag{7}$$

In the following we show the sufficient and necessary conditions for the separability of ρ_B can be expressed by the inequality:

$$1 \leq \sum_{i=1}^{3} |\Omega_{ii}| \tag{8}$$

One can obtain the above inequality based on the method which is different from the PT [2] or Wootters occurrence [3].

First of all, we prove the inequality Eq. 8 is a necessary condition for bipartite separability of ρ_B. Suppose $\rho_B = \sum_{k=1}^{N} p_k\rho_k^A \otimes \rho_k^B$ is separable, $p_k \geq 0$ and $\sum_{k=1}^{N} p_k = 1$. In terms of spinor representation, the density operators ρ_k^A and ρ_k^B are written as

$$\rho_k^A = \frac{1}{2}\sum_{i=0}^{3} \Omega_{k,i}^A\sigma_i \;\; \text{and} \;\; \rho_k^B = \frac{1}{2}\sum_{j=0}^{3} \Omega_{k,j}^B\sigma_j, \tag{9}$$

here $\Omega_{k,i}^A, \Omega_{k,j}^B \in \mathbb{R}$ and $\Omega_{k,0}^A = \Omega_{k,0}^B = 1$ for $0 \leq i, j \leq 3$. The state $\rho_B = \sum_{k=1}^{N} p_k\rho_k^A \otimes \rho_k^B$ is thus rephrased as

$$\rho_B = \frac{1}{2}\sum_{k=1}^{N}\sum_{i,j=0}^{3} p_k\Omega_{k,i}^A\Omega_{k,j}^B\sigma_i \otimes \sigma_j. \tag{10}$$

According to Eqs. 18 and 9, we obtain the following relations, for $0 \leq i, j \leq 3$,

$$\begin{aligned}
&\sum_{k=1}^{N} p_k\Omega_{k,0}^A\Omega_{k,0}^B = 1; \\
&\sum_{k=1}^{N} p_k\Omega_{k,i}^A\Omega_{k,i}^B = \Omega_{ii} \;\; \text{as} \quad i \neq 0; \\
&\sum_{k=1}^{N} p_k\Omega_{k,i}^A\Omega_{k,j}^B = \Omega_{ij} = 0 \;\; \text{as} \quad i \neq j.
\end{aligned} \tag{11}$$

These relations remain true for the general instance in the next section. Since both ρ_k^A and ρ_k^B are positive, the inequalities hold

$$\mathrm{Tr}^2(\rho_k^A) \geq \mathrm{Tr}((\rho_k^A)^2) \;\text{and}\; \mathrm{Tr}^2(\rho_k^B) \geq \mathrm{Tr}((\rho_k^B)^2), \tag{12}$$

which leads to

$$(\Omega_{k,0}^A)^2 \geq \frac{1}{2}\sum_{i=0}^{3}(\Omega_{k,i}^A)^2 \;\text{and}\; (\Omega_{k,0}^B)^2 \geq \frac{1}{2}\sum_{i=0}^{3}(\Omega_{k,i}^B)^2. \tag{13}$$

By multiplying the inequalities of parties A and B of Eq. 13 and using the Cauchy's inequality, one acquires

$$\Omega_{k,0}^A \cdot \Omega_{k,0}^B \geq \frac{1}{2}\sum_{i=0}^{3}|\Omega_{k,i}^A \cdot \Omega_{k,i}^B|. \tag{14}$$

Finally multiplying the weight p_k to both sides of Eq. 14 and summing over the N terms, the following inequality is valid

$$\begin{aligned}
&\sum_{k=1}^{N} p_k\Omega_{k,0}^A\Omega_{k,0}^B \\
&\geq \frac{1}{2}\sum_{k=1}^{N}\sum_{i=0}^{3} p_k|\Omega_{k,i}^A\Omega_{k,i}^B| \geq \frac{1}{2}\sum_{i=0}^{3}|\sum_{k=1}^{N} p_k\Omega_{k,i}^A\Omega_{k,i}^B|.
\end{aligned} \tag{15}$$

Through the relations of Eq. 11, the necessary proof of the inequality Eq. 8 are completed.

In further, we prove the inequality Eq. 8 is also a sufficient condition for bipartite separability of ρ_B. Suppose the inequality Eq. 8 holds for the Bell's mixture ρ_B. We develop a separable form for the Bell's mixture

$$
\begin{aligned}
\rho_B &= \frac{1}{4}\sum_{l=1}^{4}\rho_l^A\otimes\rho_l^B \text{ with}\\
\rho_1^A &= \frac{1}{2}(\sigma_0+(-1)^{\epsilon_1}\sqrt{|\Omega_{11}|}\sigma_1\\
&\quad +(-1)^{\epsilon_2}\sqrt{|\Omega_{22}|}\sigma_2+(-1)^{\epsilon_3}\sqrt{|\Omega_{33}|}\sigma_3),\\
\rho_2^A &= \frac{1}{2}(\sigma_0-(-1)^{\epsilon_1}\sqrt{|\Omega_{11}|}\sigma_1\\
&\quad +(-1)^{\epsilon_2}\sqrt{|\Omega_{22}|}\sigma_2-(-1)^{\epsilon_3}\sqrt{|\Omega_{33}|}\sigma_3),\\
\rho_3^A &= \frac{1}{2}(\sigma_0+(-1)^{\epsilon_1}\sqrt{|\Omega_{11}|}\sigma_1\\
&\quad -(-1)^{\epsilon_2}\sqrt{|\Omega_{22}|}\sigma_2-(-1)^{\epsilon_3}\sqrt{|\Omega_{33}|}\sigma_3),\\
\rho_4^A &= \frac{1}{2}(\sigma_0-(-1)^{\epsilon_1}\sqrt{|\Omega_{11}|}\sigma_1\\
&\quad -(-1)^{\epsilon_2}\sqrt{|\Omega_{22}|}\sigma_2+(-1)^{\epsilon_3}\sqrt{|\Omega_{33}|}\sigma_3),\\
\rho_1^B &= \frac{1}{2}(\sigma_0+\sqrt{|\Omega_{11}|}\sigma_1-\sqrt{|\Omega_{22}|}\sigma_2+\sqrt{|\Omega_{33}|}\sigma_3),\\
\rho_2^B &= \frac{1}{2}(\sigma_0-\sqrt{|\Omega_{11}|}\sigma_1-\sqrt{|\Omega_{22}|}\sigma_2-\sqrt{|\Omega_{33}|}\sigma_3),\\
\rho_3^B &= \frac{1}{2}(\sigma_0+\sqrt{|\Omega_{11}|}\sigma_1+\sqrt{|\Omega_{22}|}\sigma_2-\sqrt{|\Omega_{33}|}\sigma_3), \text{ and}\\
\rho_4^B &= \frac{1}{2}(\sigma_0-\sqrt{|\Omega_{11}|}\sigma_1+\sqrt{|\Omega_{22}|}\sigma_2+\sqrt{|\Omega_{33}|}\sigma_3). \quad (16)
\end{aligned}
$$

where,$(-1)^{\epsilon_i}=sign(\Omega_{ii}),\ i=1,2,3$. We show that each ρ_l^A (ρ_l^B), $1\le l\le 4$ is a density operator if the inequality Eq. 8 is satisfied. Obviously ρ_l^A (ρ_l^B) are hermitian and have unit trace. It is easy to calculate the eigenvalues of each ρ_l^A (ρ_l^B) and there are only two kinds of eigenvalues

$$
\begin{aligned}
\lambda_{l,1}^A=\lambda_{l,1}^B &= \tfrac{1}{2}(\tfrac{2+\sqrt{4-4(1-|\Omega_{11}|-|\Omega_{22}|-|\Omega_{33}|)}}{2});\\
\lambda_{l,2}^A=\lambda_{l,2}^B &= \tfrac{1}{2}(\tfrac{2-\sqrt{4-4(1-|\Omega_{11}|-|\Omega_{22}|-|\Omega_{33}|)}}{2}). \quad (17)
\end{aligned}
$$

These two eigenvalues are positive if $1\ge|\Omega_{11}|+|\Omega_{22}|+|\Omega_{33}|$ and thus ρ_l^A (ρ_l^B) are density operators. Therefore, if the inequality Eq. 8 is satisfied, then ρ_B is separable.

Then, we operated Peres PT criterion:

$$\rho_B^{T_B}=\frac{1}{4}(I\otimes I+\sum_{i=1}^{3}\Omega_{ii}\sigma_i\otimes\sigma_i), \quad (18)$$

the eigenvalues of $\rho_B^{T_B}$ are:

$$
\begin{aligned}
\lambda_1^{T_B} &= \frac{1}{4}(1+\Omega_{11}-\Omega_{22}+\Omega_{33}),\\
\lambda_2^{T_B} &= \frac{1}{4}(1-\Omega_{11}+\Omega_{22}+\Omega_{33}),\\
\lambda_3^{T_B} &= \frac{1}{4}(1+\Omega_{11}+\Omega_{22}-\Omega_{33}),\\
\lambda_4^{T_B} &= \frac{1}{4}(1-\Omega_{11}-\Omega_{22}-\Omega_{33}). \quad (19)
\end{aligned}
$$

When $\lambda_i^{T_B}\ge 0\ \ i=1\cdots 4$,then $\rho_B^{T_B}$ is separable. It should also be added that the conditions of ρ_B is a positive density operator $\lambda_i\ge 0\ \ i=1\cdots 4$. Therefor, one can obtain the same result as Eq. 8 on the grounds that the inequalities both $\lambda_i\ge 0$ and $\lambda_i^{T_B}\ge 0$.

III. Bell Diagonal States in a Bipartite System of Higher Dimension

In this section we show that the proof in the necessary condition can be extended to the more general occasion, a bipartite system $\mathcal{H}_A\otimes\mathcal{H}_B$ of dimension $2^p\times 2^p$. Extending the sufficient condition is difficult because it is not easy to find a separable form as of Eq. 16. Thus we focus on the acquisition of the necessary condition for the bi-partite separability of Bell diagonal states.

Owing to the relations of Eq. 2, a computational basis element can be written as

$$|a\rangle\langle a'|=\frac{1}{2}\sigma_1^{a+a'}(\sigma_0+(-1)^a\sigma_3) \quad (20)$$

for all $a,a'\in Z_2$. In general, for a p-qubit system, we have

$$
\begin{aligned}
&|a_1,a_2,\cdots,a_p\rangle\langle a_1',a_2',\cdots,a_p'|\\
&=\bigotimes_{i=1}^{p}|a_i\rangle\langle a_i'|=\frac{1}{2^p}\bigotimes_{i=1}^{p}\sigma_1^{a_i+a_i'}(\sigma_0+(-1)^{a_i}\sigma_3) \quad (21)
\end{aligned}
$$

for all $a_i,a_i'\in Z_2$ and $1\le i\le p$. Thus, a density operator in a p-qubit system is expressed as

$$\rho=\frac{1}{2^p}\sum_{i_1,i_2,\cdots,i_p=0}^{3}\Omega_{i_1i_2\cdots i_p}\sigma_{i_1}\otimes\sigma_{i_2}\otimes\cdots\sigma_{i_p} \quad (22)$$

with the real coefficients $\Omega_{i_1i_2\cdots i_p}=\mathrm{Tr}\{\rho\cdot\sigma_{i_1}^*\otimes\sigma_{i_2}^*\otimes\cdots\sigma_{i_p}^*\}$.

In the standard basis, the Bell diagonal states in a $2^p\times 2^p$ system can be represented as

$$\rho_B=\Sigma_{\epsilon_a,\epsilon_b\in Z_2^p}\lambda_{\epsilon_a,\epsilon_b}|\psi_{\epsilon_a,\epsilon_b}\rangle\langle\psi_{\epsilon_a,\epsilon_b}| \quad (23)$$

where $|\psi_{\epsilon_a,\epsilon_b}\rangle=\frac{1}{\sqrt{2^p}}\sum_{\alpha\in Z_2^p}(-1)^{\alpha\cdot\epsilon_a}|\alpha+\epsilon_a,\alpha+\epsilon_b\rangle$. Then, $\lambda_{\epsilon_a,\epsilon_b}$ and$|\psi_{\epsilon_a,\epsilon_b}\rangle$ are the eigenvalues and eigenvectors of the ρ_B, respectively. With the help of Eq. 2, one may rewrite Eq. 23 in the spinor basis:

$$
\begin{aligned}
\rho_B=\frac{1}{2^{2p}}[&\sum_{i_1,i_2,\cdots,i_p=0}^{3}(-1)^{\epsilon_{i_1}+\epsilon_{i_2}\cdots+\epsilon_{i_p}}\Omega_{i_1i_2\cdots i_p,i_1i_2\cdots i_p}\\
&(\sigma_{i_1}\otimes\sigma_{i_2}\otimes\cdots\sigma_{i_p})\otimes(\sigma_{i_1}\otimes\sigma_{i_2}\otimes\cdots\sigma_{i_p})], \quad (24)
\end{aligned}
$$

where $\epsilon_{i_m=0}=\epsilon_{i_m=1}=\epsilon_{i_m=3}=0$, $\epsilon_{i_m=2}=1$, $1\le m\le p$, and $\Omega_{i_1i_2\cdots i_p,i_1i_2\cdots i_p}=1$ if $i_1=i_2=\cdots=i_p=0$. We follow the same procedure as in the last section to acquire the necessary condition. Suppose the Bell diagonal states are bi-partite separable

$$
\begin{aligned}
\rho_B &= \sum_{k=1}^{N}p_k\rho_k^A\otimes\rho_k^B \text{ with}\\
\rho_k^A &= \frac{1}{2^p}\{\sum_{i_1,i_2,\cdots,i_p=0}^{3}\Omega_{k,i_1i_2\cdots i_p}^A\sigma_{i_1}\otimes\sigma_{i_2}\otimes\cdots\sigma_{i_p}\} \text{ and}\\
\rho_k^B &= \frac{1}{2^p}\{\sum_{j_1,j_2,\cdots,j_p=0}^{3}\Omega_{k,j_1j_2\cdots j_p}^B\sigma_{j_1}\otimes\sigma_{j_2}\otimes\cdots\sigma_{j_p}\}, \quad (25)
\end{aligned}
$$

here $p_k \geq 0$, $\sum_{k=1}^{N} p_k = 1$, and $\Omega^A_{k,i_1 i_2 \cdots i_p}, \Omega^B_{k,j_1 j_2 \cdots j_p} \in \mathbb{R}$. With Eqs. 24 and 25, we obtain the following relations as of Eq. 11

$$\sum_{k=1}^{N} p_k \Omega^A_{k,00\cdots 0} \Omega^B_{k,00\cdots 0} = 1,$$
$$\sum_{k=1}^{N} p_k \Omega^A_{k,i_1 i_2 \cdots i_p} \Omega^B_{k,i_1 i_2 \cdots i_p} = \Omega_{i_1 i_2 \cdots i_p, i_1 i_2 \cdots i_p} \quad \text{as } i_{1\leq r\leq p} \neq 0, \text{ and}$$
$$\sum_{k=1}^{N} p_k \Omega^A_{k,i_1 i_2 \cdots i_p} \Omega^B_{k,j_1 j_2 \cdots j_p} = 0 \text{ for some } i_r \neq j_r. \quad (26)$$

By virtue of Eq. 12, the coefficients obey the inequalities

$$(\Omega^A_{k,00\cdots 0})^2 \geq \frac{1}{2^p} \sum_{i_1,i_2\cdots,i_p=0}^{3} (\Omega^A_{k,i_1 i_2 \cdots i_p})^2 \text{ and}$$
$$(\Omega^B_{k,00\cdots 0})^2 \geq \frac{1}{2^p} \sum_{j_1,j_2\cdots,j_p=0}^{3} (\Omega^B_{k,j_1 j_2 \cdots j_p})^2. \quad (27)$$

By multiplying the inequalities of parties A and B of Eq. 27 and using the Cauchy's inequality, one acquires

$$\Omega^A_{k,00\cdots 0} \cdot \Omega^B_{k,00\cdots 0} \geq \frac{1}{2^p} \sum_{i_1,i_2,\cdots,i_p=0}^{3} |\Omega^A_{k,i_1 i_2 \cdots i_p} \cdot \Omega^B_{k,i_1 i_2 \cdots i_p}|. \quad (28)$$

Similarly, multiplying the weight p_k to both sides of Eq. 28 and summing over the N terms, one derives the following inequalities

$$\sum_{k=1}^{N} p_k \Omega^A_{k,00\cdots 0} \Omega^B_{k,00\cdots 0}$$
$$\geq \frac{1}{2^p} \sum_{k=1}^{N} \sum_{i_1,i_2,\cdots,i_p=0}^{3} p_k |\Omega^A_{k,i_1 i_2 \cdots i_p} \Omega^B_{k,i_1 i_2 \cdots i_p}|$$
$$\geq \frac{1}{2^p} \sum_{i_1,i_2,\cdots,i_p=0}^{3} |\sum_{k=1}^{N} p_k \Omega^A_{k,i_1 i_2 \cdots i_p} \Omega^B_{k,i_1 i_2 \cdots i_p}|. \quad (29)$$

Complying with the relations of Eq. 26, the inequality of Eq. 29 leads to the required condition

$$1 \geq \frac{1}{2^p} (\sum_{i_1,i_2,\cdots,i_p=0}^{3} |\Omega_{i_1 i_2 \cdots i_p, i_1 i_2 \cdots i_p}|), \quad (30)$$

which is a simple proof of necessary condition for bipartite separability of Bell diagonal states.

IV. Conclusion

In this article, we propose a new scheme to establish a criteria different from the PT and Wootters concurrence for the separability of Bell diagonal states. In a 2×2 system, this scheme provides a sufficient and necessary conditions of the bi-partite separability for the Bell diagonal states and it could gives us an insight into quantum entanglement. When the separability of Bell diagonal states are transformed into entanglement, implied the local density operators $\rho^{\mathcal{I}}_k$ moved to the outside the Hilbert space $\mathcal{H}_{\mathcal{I}}$ for $\mathcal{I} = A, B$. This criteria, in general, in a $2^p \times 2^p$ system is simply a necessary condition for $p \geq 2$ and a sufficient condition in the 2×2 system. However, under the appropriate choices of separable forms, it is possible to modify that such criterion to obtained a sufficient condition of Bell diagonal states of arbitrary dimension.

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