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A Contribution to the Polynomial Eigen Problem

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Abstract—The relationship between eigenstructure (eigenvalues and eigenvectors) and latent structure (latent roots and latent vectors) is established. In control theory eigenstructure is associated with the state space description of a dynamic multi-variable system and a latent structure is associated with its matrix fraction description. Beginning with block controller and block observer state space forms and moving on to any general state space form, we develop the identities that relate eigenvectors and latent vectors in either direction. Numerical examples illustrate this result. A brief discussion of the potential of these identities in linear control system design follows. Additionally, we present a consequent result: a quick and easy method to solve the polynomial eigenvalue problem for regular matrix polynomials.

Keywords—Eigenvalues/Eigenvectors, Latent values/vectors, Matrix fraction description, State space description.

I. Introduction

THERE exist many approaches for representing linear multi-variable systems. The two approaches considered here are the state space description (SSD) and the matrix fraction description (MFD). In the SSD, the modal decomposition of the state matrix into its eigenstructure is very useful as it defines the stability and the dynamic behaviour of a linear multi-variable system. In general, the speed of response is determined by the eigenvalues whereas the shape of the response is furnished by the eigenvectors. If, through feedback, we are able to assign the eigenvalues to predetermined values and we are able to align the closed loop eigenvectors along predetermined directions, we will be able to control the behaviour of a linear multivariable system in both speed of response and shape of the response, achieving design objectives such as input and output decoupling, reducing sensitivity to perturbations in dynamic structure as well as appropriate stability criteria [1]. Eigenstructure assignment is a design methodology that facilitates control system design by synthesizing a feedback gain matrix that exactly places the closed loop eigenvalues whilst matching the closed loop eigenvectors as closely as possible to a desired set [2].

Such technique can be related to the design objectives as well as the final performance through its clear links with the time domain response [3]. Eigenstructure assignment is an excellent method for incorporating classical specifications on damping, settling time, and mode decoupling into a modern multivariable control framework [1], and has been shown to be a useful tool for flight control design [4], [5].

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Matrix polynomials also play a central role in the mathematical description of the dynamics of multivariable systems. This fact has led to an active research effort in matrix polynomials theory [6]. Multivariable systems described by a MFD can be better decomposed into parallel subsystems for less sensitivity and single designs of lower order controller i.e. a block partial fraction expansion of an MFD [7]. Block Roots (or solvents) can be constructed from latent roots and latent vectors (under some condition of existence). Many papers have considered using solvents for solving some linear algebra problems or control problems, such as: Block partial fraction expansion of a MFD with single and repeated poles [7], [8]; Cascade decomposition and realization of multivariable Systems Via Block-Pole and Block-Zero Placement [9]; State-Feedback decomposition of multivariable Systems Via Block-Pole Placement [10].

Latent structure is, in some ways, analogous to eigenstructure but its use and importance has still to be explored particularly in control theory. It is known and has been verified that latent values of the MFD of a system and eigenvalues of the SSD of the same system are the same, but the link between eigenvectors and latent vectors has not yet been elaborated at my knowledge.

The link between state space representation and polynomial matrices is well established in the book of Rosenbrock [11].

The purpose of our paper is to establish the relationship between the MFD latent vectors and SSD eigenvectors of a linear multivariable system. A consequence of this is that we are able to use it as part of a new formulation for solving the polynomial eigenvalue problem (PEP) through companion matrices. The classical approach to solve polynomial eigenvalue problem is linearization to matrix pencils then determine the eigenvalues of this pencil matrix using classical methods [12]-[15]. At the onset of our studies, we privately postulated that, if we could establish the structural links between latent vectors and eigenvectors, we should then be able to combine design methodologies and get the benefits of both descriptions. Early results are promising and we will report on this work very soon.

The paper is organized as follows: Section II presents some theoretical preliminaries on eigenstructure and latent structure. Section III then focuses on the main results: the relationship between latent structure and eigenstructure, and how to generate one from the other. This is illustrated by a numerical example in Section IV. In Section V, we employ the above result in solving the PEP, via an illustrative example. In Section VI we briefly sum up our conclusions.

II. Preliminaries

Here, we present some established material in order to define symbols and nomenclature and collect.

A. Matrix Fraction Description

A $p \times m$ multivariable linear system may be described by a matrix transfer function $G(s)$ expressed as a matrix fraction description as follows:

$$G(s) = N_r(s)D_r^{-1}(s) \quad or \quad G(s) = D_l^{-1}(s)N_l(s) \tag{1}$$

Where N_r, D_r, N_l, D_l are matrix polynomials.

1) Matrix Polynomials: We will consider r-degree, m^{th} order monic matrix polynomials of the form

$$D(\lambda) = I_m\lambda^r + D_1\lambda^{r-1} + \cdots + D_{r-1}\lambda + D_r \tag{2}$$

Where λ is a complex number and $D_i \in \mathcal{R}^{m\times m}$.

For an $m \times m$ complex matrix X we define the following structures.

A right polynomial matrix is given by:

$$D_R(X) = D_0X^r + D_1X^{r-1} + \cdots + D_{r-1}X + D_r \tag{3}$$

A left polynomial matrix is given by:

$$D_L(X) = X^rD_0 + X^{r-1}D_1 + \cdots + XD_{r-1} + D_r \tag{4}$$

2) Latent Roots and Latent Vectors: A latent root λ_i of $D(\lambda)$ is a complex number satisfying $detD(\lambda) = 0$; A right latent vector $\overline{v}_i \in \mathcal{R}^{m\times 1}$ associated with λ_i satisfies $D(\lambda_i)\overline{v}_i = \theta$ while a left latent vector $\overline{w}_i \in \mathcal{R}^{1\times m}$ is a row vector satisfying $\overline{w}_iD(\lambda_i) = \theta$, where θ denotes the zero row or column vector.

3) Block Roots: A right block root or right solvent of $D(\lambda)$ is an $m \times m$ matrix R such that:

$$R^r + D_1R^{r-1} + \cdots + D_{r-1}R + D_r = 0_m \tag{5}$$

Whilst a left block root or left solvent is an $m \times m$ matrix L satisfying:

$$L^r + L^{r-1}D_1 + \cdots + LD_{r-1} + D_r = 0_m \tag{6}$$

B. State Space Description

Let an m-inputs p-outputs system be described by a state space equation in general form:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \tag{7}$$

Where the state matrix A is $n \times n$, the input matrix B is $n \times m$, and the output matrix C is $p \times n$ and all are real matrices.

1) Eigenvalue/Eigenvector: A complex λ is an eigenvalue of the square $n \times n$ matrix A if there exists a nonzero column vector x in $\mathcal{C}^n$ such that $Ax = x\lambda$ and the vector x is the corresponding right eigenvector. By duality we can define a left eigenvector as a nonzero row vector y in $\mathcal{C}^n$ such that $yA = \lambda y$.

The eigenstructure of the $n \times n$ state matrix A is defined by the n eigenpairs (λ_i , x_i) or (λ_i, y_i) for i=1 to n.

2) Block Controller Form:

Definition 1: A system as described by (7) is said block controllable if:

- the number of states is a multiple of the number of inputs: $\frac{n}{m} = r$ is an integer
- and it is controllable, i.e. the controllability matrix $C = \begin{pmatrix} B & AB & \cdots & A^{r-1}B \end{pmatrix}$ is non-singular.

Remark 1: The integer r is the controllability index.

If the system is block controllable then it can be transformed into a block controller form using the following similarity transformation:

$X_c = T_cX$ where $T_c = \begin{pmatrix} T_{c1} \\ T_{c1}A \\ \vdots \\ T_{c1}A^{r-1} \end{pmatrix}$ and

$$T_{c1} = \begin{pmatrix} 0_m & \cdots & 0_m & I_m \end{pmatrix}\begin{pmatrix} B & AB & \cdots & A^{r-1}B \end{pmatrix}^{-1} \tag{8}$$

to obtain $\begin{cases} \dot{X}_c = A_cX_c + B_cu \\ y = C_cXc \end{cases}$

where $A_c = T_cAT_c^{-1}$, $B_c = T_cB$ and $C_c = CT_c^{-1}$

$$\begin{cases} A_c = \begin{pmatrix} 0_m & I_m & 0_m & \cdots & 0_m \\ 0_m & 0_m & I_m & \cdots & 0_m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_m & 0_m & 0_m & \cdots & I_m \\ -A_r & -A_{r-1} & -A_{r-2} & \cdots & -A_1 \end{pmatrix}; \\ B_c = \begin{pmatrix} 0_m \\ 0_m \\ \vdots \\ 0_m \\ I_m \end{pmatrix}; C_c = \begin{pmatrix} C_r & C_{r-1} & \cdots & C_2 & C_1 \end{pmatrix} \end{cases} \tag{9}$$

3) Block Observer Form:

Definition 2: A system as described by (7) is said block observable if:

- the number of states is a multiple of the number of outputs, i.e. $\frac{n}{p} = r$ is an integer
- and it is observable, i.e. the observability matrix $O = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{r-1} \end{pmatrix}$ is non-singular.

Remark 2: the integer r is the observability index

If the system is block observable it can be transformed into a block observer form using the following similarity transformation $X_o = T_oX$

where $T_o = \begin{pmatrix} T_{o1} & AT_{o1} & \cdots & A^{r-1}T_{o1} \end{pmatrix}$ and

$$T_{o1} = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{r-1} \end{pmatrix}^{-1}\begin{pmatrix} 0_p \\ \vdots \\ 0_p \\ I_p \end{pmatrix} \tag{10}$$

to obtain $\begin{cases} \dot{X}_o = A_oX_o + B_ou \\ y = C_oX_o \end{cases}$

where $A_o = T_o^{-1} A T_o$, $B_o = T_o^{-1} B$ and $C_o = C T_o$

$$\begin{cases} A_o = \begin{pmatrix} 0_p & 0_p & \cdots & 0_p & -A_r \\ I_p & 0_p & \cdots & 0_p & -A_{r-1} \\ 0_p & I_p & \cdots & 0_p & -A_{r-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_p & 0_p & \cdots & I_p & -A_1 \end{pmatrix}; \\ B_o = \begin{pmatrix} B_r \\ B_{r-1} \\ B_{r-2} \\ \vdots \\ B_1 \end{pmatrix}; C_o = \begin{pmatrix} 0_p & 0_p & \cdots & 0_p & I_p \end{pmatrix} \end{cases} \tag{11}$$

4) Conversion from SSD to MFD: From the block controller form we can derive the transfer function in right MFD by:

$$T(s) = N_r(s) D_r^{-1}(s) \tag{12}$$

with $N_r(s) = C_1 s^{r-1} + C_2 s^{r-2} + \cdots + C_{r-1} s + C_r$
and $D_r = I_m s^r + A_1 s^{r-1} + \cdots + A_{r-1} s + A_r$

And from the block observer form we can derive the transfer function in left MFD by:

$$T(s) = D_l^{-1}(s) N_l(s) \tag{13}$$

with $D_l = I_m s^r + A_1 s^{r-1} + \cdots + A_{r-1} s + A_r$
and $N_l(s) = B_1 s^{r-1} + B_2 s^{r-2} + \cdots + B_{r-1} s + B_r$

III. Main Results

The link between latent vectors and eigenvectors of a block controller (observer) form is presented first, then the link for a general matrix, and finally the use of it to solve the PEP.

A. Relation between latent vectors and eigenvectors of the block controller SSD

For an appropriate system, let(λ_i, v_i) be the right eigenstructure of the block controller SSD state matrix A_c, hence $A_c v_i = \lambda_i v_i$, and let $(\lambda_i, \overline{v}_i)$ be a latent pair of the matrix $D_r(\lambda)$ of the corresponding right MFD, hence $D_r(\lambda_i)\overline{v}_i = 0_m$.

Theorem 1: The latent vector $\overline{v}_i$ is obtained from the eigenvector v_i by using the following equation:

$$\overline{v}_i = v_{i1} \tag{14}$$

where v_{i1} is composed of the first m components of v_i.

Proof

$D_r(\lambda_i)\overline{v}_i = 0_m \Rightarrow$
$\left[I_m \lambda_i^r + A_1 \lambda_i^{r-1} + \cdots + A_{r-1}\lambda_i + A_r \right] \overline{v}_i = 0_m$
So $\overline{v}_i$ satisfies

$$\lambda_i^r \overline{v}_i + \lambda_i^{r-1} A_1 \overline{v}_i + \cdots + \lambda_i A_{r-1} \overline{v}_i + A_r \overline{v}_i = 0_m \tag{15}$$

On the other hand $A_c v_i = \lambda_i v_i$ leads to:

$$\begin{pmatrix} 0_m & I_m & 0_m & \cdots & 0_m \\ 0_m & 0_m & I_m & \cdots & 0_m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_m & 0_m & 0_m & \cdots & I_m \\ -A_r & -A_{r-1} & -A_{r-2} & \cdots & -A_1 \end{pmatrix} \begin{pmatrix} v_{i1} \\ v_{i2} \\ \vdots \\ v_{ir-1} \\ v_{ir} \end{pmatrix} = \lambda_i \begin{pmatrix} v_{i1} \\ v_{i2} \\ \vdots \\ v_{ir-1} \\ v_{ir} \end{pmatrix} \tag{16}$$

Where v_{ij} with j=1 to r are the block elements of the eigenvector v_i of dimension m. Hence the following set of equations may be obtained:

$$\begin{cases} v_{i2} = \lambda_i v_{i1} \\ v_{i3} = \lambda_i v_{i2} = \lambda_i^2 v_{i1} \\ \vdots \\ v_{ir} = \lambda_i v_{ir-1} = \lambda_i^{r-1} v_{i1} \\ -A_r v_{i1} - A_{r-1} v_{i2} - \cdots - A_1 v_{ir} = \lambda_i v_{ir} \end{cases} \tag{17}$$

The last equation can be rewritten as:

$$-A_r v_{i1} - \lambda_i A_{r-1} v_{i1} - \cdots - \lambda_i^{r-1} A_1 v_{i1} = \lambda_i^r v_{i1}$$

or

$$\lambda_i^r v_{i1} + \lambda_i^{r-1} A_1 v_{i1} + \cdots + \lambda_i A_{r-1} v_{i1} + A_r v_{i1} = 0_m \tag{18}$$

Comparing (15) and (18), we conclude that $\overline{v}_i = v_{i1}$ or

$$\overline{v}_i = \begin{pmatrix} I_m & 0_m & \cdots & 0_m \end{pmatrix} \begin{pmatrix} v_{i1} \\ v_{i2} \\ \vdots \\ v_{ir} \end{pmatrix} \tag{19}$$

QED.

So the latent vector $\overline{v}_i$ of $D_r(\lambda)$ is constituted from the first m components of the eigenvector of A_c corresponding to the same latent root/eigenvalue λ_i.

Conversely we can state the consequent result as a corollary.

Corollary 1: The eigenvector v_i of A_c can be constructed from the latent vector $\overline{v}_i$ using

$$v_i = \begin{pmatrix} \overline{v}_i \\ \lambda_i \overline{v}_i \\ \lambda_i^2 \overline{v}_i \\ \vdots \\ \lambda_i^{r-1} \overline{v}_i \end{pmatrix} \tag{20}$$

Proof The result is obtained from (17) and the fact that $\overline{v}_i = v_{i1}$.

B. Relation between latent vectors and eigenvectors of the block observer SSD

By duality we can show the relationship between left eigenvectors and left latent vectors.

For the same system as in precedent section, let(λ_i, w_i) be an eigenvalue and its corresponding left eigenvector of the state matrix A_o, hence $w_i A_o = \lambda_i w_i$, and let $(\lambda_i, \overline{w}_i)$ be a latent pair of $D_l(\lambda)$ of the corresponding left MFD, hence $\overline{w}_i D_l(\lambda_i) = 0_p$.

Theorem 2: The latent vector $\overline{w}_i$ is obtained from the eigenvector w_i by using the following equation:

$$\overline{w}_i = w_{i1} \tag{21}$$

where w_{i1} is composed of the first p components of the eigenvector w_i

Proof

$\overline{w}_i D_l(\lambda_i) = 0_p \Rightarrow \overline{w}_i[I_p \lambda_i^r + A_1 \lambda_i^{r-1} + \cdots + A_{r-1}\lambda_i + A_r] = 0_p$.

So $\overline{w}_i$ satisfies

$$\lambda_i^r \overline{w}_i + \lambda_i^{r-1}\overline{w}_i A_1 + \cdots + \lambda_i \overline{w}_i A_{r-1} + \overline{w}_i A_r = 0_p \tag{22}$$

On the other hand $w_i A_o = \lambda_i w_i$ leads to:

$$\begin{pmatrix} w_{i1} \\ w_{i2} \\ w_{i3} \\ \vdots \\ w_{ir} \end{pmatrix}^T \begin{pmatrix} 0_p & 0_p & \cdots & 0_p & -A_r \\ I_p & 0_p & \cdots & 0_p & -A_{r-1} \\ 0_p & I_P & \cdots & 0_p & -A_{r-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_p & 0_p & \cdots & I_p & -A_1 \end{pmatrix} = \lambda_i \begin{pmatrix} w_{i1} \\ w_{i2} \\ w_{i3} \\ \vdots \\ w_{ir} \end{pmatrix}^T \tag{23}$$

Where w_{ij} with j=1 to r are the block elements of the eigenvector w_i of dimension p. Hence the following set of equations may be obtained:

$$\begin{cases} w_{i2} = \lambda_i w_{i1} \\ w_{i3} = \lambda_i w_{i2} = \lambda_i^2 w_{i1} \\ \vdots \\ w_{ir} = \lambda_i w_{ir-1} = \lambda_i^{r-1} w_{i1} \\ -w_{i1}A_r - w_{i2}A_{r-1} - \cdots - w_{ir}A_1 = \lambda_i w_{ir} \end{cases} \tag{24}$$

The last equation can be rewritten as:

$$-w_{i1}A_r - \lambda_i w_{i1} A_{r-1} - \cdots - \lambda_i^{r-1} w_{i1} A_1 = \lambda_i^r w_{i1}$$

or

$$\lambda_i^r w_{i1} + \lambda_i^{r-1} w_{i1} A_1 + \cdots + \lambda_i w_{i1} A_{r-1} + w_{i1} A_r = 0_p \tag{25}$$

Comparing (22) and (25), we conclude that $\overline{w}_i = w_{i1}$ or

$$\overline{w}_i = \begin{pmatrix} w_{i1} & w_{i2} & \cdots & w_{ir} \end{pmatrix} \begin{pmatrix} I_p \\ 0_p \\ \vdots \\ 0_p \end{pmatrix} \tag{26}$$

QED.

So the latent vector $\overline{w}_i$ of $D_l(\lambda)$ is composed of the first p components of the eigenvector w_i of A_o corresponding to the same latent root/eigenvalue λ_i.

Conversely, we can state a consequent result as a corollary.

Corollary 2: the eigenvector w_i of A_o can be constructed from the latent vector $\overline{w}_i$ as:

$$w_i = \begin{pmatrix} \overline{w}_i & \lambda_i \overline{w}_i & \cdots & \lambda_i^{r-1}\overline{w}_i \end{pmatrix} \tag{27}$$

Proof The result is straight forward from the fact that $\overline{w}_i = w_{i1}$ and from (24).

C. Relationship between latent vectors and eigenvectors of a general SSD

If (λ_i, x_i) is an eigenvalue and right eigenvector of the general state matrix A, and let $(\lambda_i, \overline{v}_i)$ be a right latent pair of the matrix $D_r(\lambda)$ of the corresponding right MFD then the following theorem states the relationship:

Theorem 3: The latent vector $\overline{v}_i$ can be obtained from the eigenvector x_i by using the following equation:

$$\overline{v}_i = T_{c1} x_i \tag{28}$$

where T_{c1} is given by (8)

Proof: Recalling that the block controller SSD (9) is generated from a general SSD via the similarity transformation $A_c = T_c A T_c^{-1}$, we have $A_c T_c = T_c A$. If (λ_i, x_i) is a right eigenpair of the general state matrix A, then $Ax_i = \lambda_i x_i$.

Hence $A_c T_c x_i = T_c A x_i = \lambda_i T_c x_i$. Thus $(\lambda_i, T_c x_i)$ is an eigenpair of A_c and

$$v_i = T_c x_i \tag{29}$$

It follows from (19) that:

$\overline{v}_i = \begin{pmatrix} I_m & 0_m & \cdots & 0_m \end{pmatrix} T_c x_i$.

Since $T_c = \begin{pmatrix} T_{c1} \\ T_{c1}A \\ \vdots \\ T_{c1}A^{r-1} \end{pmatrix}$

Then we have:

$$\overline{v}_i = T_{c1} x_i = \begin{pmatrix} 0_m \\ \vdots \\ 0_m \\ I_m \end{pmatrix}^T \begin{pmatrix} B \\ AB \\ \vdots \\ A^{r-1}B \end{pmatrix}^{-T} x_i \tag{30}$$

QED.

The reverse identity, determining eigenvectors from latent vectors is established in a corollary:

Corollary 3: The eigenvector x_i is obtained from its corresponding latent vector $\overline{v}_i$ by using the following equation:

$$x_i = T_c^{-1} \begin{pmatrix} \overline{v}_i \\ \lambda_i \overline{v}_i \\ \vdots \\ \lambda_i^{r-1} \overline{v}_i \end{pmatrix} \tag{31}$$

Proof: From $v_i = T_c x_i$ we have $x_i = T_c^{-1} v_i$ Then using (20) we obtain the precedent result.

We now establish the dual results:

If (λ_i, y_i) is an eigenvalue and left eigenvector of the general state matrix A, and let $(\lambda_i, \overline{w}_i)$ be a left latent pair of the matrix $D_r(\lambda)$ of the corresponding left MFD then the following theorem states the relationship:

Theorem 4: The latent vector $\overline{w}_i$ can be obtained from the eigenvector y_i by using the following equation:

$$\overline{w}_i = y_i T_{o1} \tag{32}$$

where T_{o1} is given by (10)

Proof: From $A_o = T_o^{-1} A T_o$ we have $AT_o = T_o A_o$. If (λ_i, y_i) is a left eigenpair (eigenvalue and left eigenvector) of the general state matrix A, then $y_i A = \lambda_i y_i$.

Hence $y_i A T_o = y_i T_o A_o = \lambda_i y_i T_o$. Thus $(\lambda_i, y_i T_o)$ is an eigenpair of A_o and

$$w_i = y_i T_o \tag{33}$$

It also follows that $\overline{w}_i = y_i T_o \begin{pmatrix} I_p \\ 0_p \\ \vdots \\ 0_p \end{pmatrix}$

Since $T_o = \begin{pmatrix} T_{o1} & AT_{o1} & \cdots & A^{r-1}T_{o1} \end{pmatrix}$ we have:

$$\overline{w}_i = y_i T_{o1} = y_i \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{r-1} \end{pmatrix}^{-1} \begin{pmatrix} 0_p \\ \vdots \\ 0_p \\ I_p \end{pmatrix} \tag{34}$$

QED.

Again, the reverse identity, determining eigenvectors from latent vectors is given by the following corollary:

Corollary 4: The eigenvector y_i is obtained from its corresponding latent vector $\overline{w}_i$ by using the following equation:

$$y_i = \begin{pmatrix} \overline{w}_i & \lambda_i \overline{w}_i & \cdots & \lambda_i^{r-1} \overline{w}_i \end{pmatrix} T_o^{-1} \tag{35}$$

Proof: From $w_i = y_i T_o$ we have $y_i = w_i T_o^{-1}$ and using (27) we obtain the precedent result.

To summarize:

- If we have the eigenstructure of a system: (λ_i, x_i) or (λ_i, y_i) then we can determine the latent vectors $\overline{v}_i$ using (30) or $\overline{w}_i$ using (34)
- If we have the latent structure of a system: $(\lambda_i, \overline{v}_i)$ or $(\lambda_i, \overline{w}_i)$ then we can determine the eigenvectors of A: x_i using (31) or y_i using (35)

IV. Numerical Example

Consider a general SSD of a system:

$$\begin{cases} \dot{X} = \begin{pmatrix} 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} X + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} u \\ y = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} x \end{cases}$$

A. Obtaining latent vectors from eigenvectors using the block controller form

Using $T_{c1} = \begin{pmatrix} -0.25 & 0.25 & 0.25 & -0.25 \\ 0.25 & 0.75 & -0.25 & -0.75 \end{pmatrix}$ we obtain the following block controller form:

$$\begin{cases} \dot{X}_c = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & 2 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} X_c + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} u \\ y = \begin{pmatrix} -1 & -1 & 0 & 2 \\ -3 & 1 & 2 & 0 \end{pmatrix} x \end{cases}$$

Then the corresponding right MFD is:

$$\begin{cases} D_r(s) = I_2 s^2 + \begin{pmatrix} -2 & -1 \\ -1 & 0 \end{pmatrix} s + \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \\ N_r(s) = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} s + \begin{pmatrix} -1 & -1 \\ -3 & 1 \end{pmatrix} \end{cases}$$

We can check that:

- The latent roots are: 0,1,-1 and 2 with corresponding latent vectors:

$\overline{v}_i : \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$

- Eigenvalues of A: $\lambda_i = \{0, 1, -1, 2\}$
- Right eigenvectors x_i are the columns of $V = \begin{pmatrix} 1 & 1 & 6 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -3 & 0 \end{pmatrix}$
- The latent vectors $\overline{v}_i$ can be obtained from the eigenvectors of A using (30):

$$\overline{v}_1 = T_{c1} x_1 = T_{c1} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -0.25 \\ 0.25 \end{pmatrix};$$

$$\overline{v}_2 = T_{c1} x_2 = T_{c1} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix};$$

$$\overline{v}_2 = T_{c1} x_2 = T_{c1} \begin{pmatrix} 6 \\ -2 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix};$$

$$\overline{v}_4 = T_{c1} x_4 = T_{c1} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix};$$

It can be verified that $\overline{v}_i$'s are indeed latent vectors of $D_r(s)$.

B. Obtaining eigenvectors from latent vectors using block observer form

Using $T_o = \begin{pmatrix} -0.25 & -0.75 & -0.25 & -0.75 \\ 0.25 & -0.25 & 0.75 & 0.25 \\ 0.25 & 0.75 & 0.25 & 1.75 \\ -0.25 & 0.25 & 0.25 & -0.25 \end{pmatrix}$ we obtain the following block observer form:

$$\begin{cases} \dot{X}_o = \begin{pmatrix} 0 & 0 & -2 & 0 \\ 0 & 0 & -2 & 1 \\ 1 & 0 & 0.5 & 2.5 \\ 0 & 0 & 1.5 & 1.5 \end{pmatrix} X_o + \begin{pmatrix} -4 & -2 \\ -2 & 0 \\ 0 & 2 \\ 2 & 0 \end{pmatrix} u \\ y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} x \end{cases}$$

Then the corresponding left MFD is:

$$\begin{cases} D_l(s) = I_2 s^2 + \begin{pmatrix} -0.5 & -2.5 \\ -1.5 & -1.5 \end{pmatrix} s + \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix} \\ N_l(s) = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} s + \begin{pmatrix} -4 & -2 \\ -2 & 0 \end{pmatrix} \end{cases}$$

We can check the following:

- For the same set of eigenvalues $\lambda_i = \{0, 1, -1, 2\}$ the left eigenvectors of the system are given by the rows of $W = V^{-1} = \begin{pmatrix} 1 & -1 & 1 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 1 \end{pmatrix}$

- The left latent vectors of the system are: $\overline{w}_i = \left\{ \begin{array}{c} \begin{pmatrix} -1 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & -5 \end{pmatrix} \\ \begin{pmatrix} 1 & -1 \end{pmatrix} \\ \begin{pmatrix} -1 & -5 \end{pmatrix} \end{array} \right\}$

- The left eigenvectors y_i of A are obtained using (35) where $w_i = \begin{pmatrix} \overline{w_i} & \lambda_i \overline{w}_i \end{pmatrix}$:

$\overline{w}_1 = \begin{pmatrix} -1 & 1 \end{pmatrix} \Rightarrow w_1 = \begin{pmatrix} -1 & 1 & 0 & 0 \end{pmatrix} \Rightarrow y_1 = \begin{pmatrix} 1 & -1 & 1 & 3 \end{pmatrix}$

$\overline{w}_2 = \begin{pmatrix} 1 & -5 \end{pmatrix} \Rightarrow w_2 = \begin{pmatrix} 1 & -5 & 1 & -5 \end{pmatrix} \Rightarrow y_2 = \begin{pmatrix} 0 & 4 & -4 & -4 \end{pmatrix}$

$\overline{w}_3 = \begin{pmatrix} 1 & -1 \end{pmatrix} \Rightarrow w_3 = \begin{pmatrix} 1 & -1 & -1 & 1 \end{pmatrix} \Rightarrow y_3 = \begin{pmatrix} 0 & 0 & 0 & -4 \end{pmatrix}$

$\overline{w}_4 = \begin{pmatrix} -1 & -5 \end{pmatrix} \Rightarrow w_4 = \begin{pmatrix} -1 & -5 & -2 & -10 \end{pmatrix} \Rightarrow y_4 = \begin{pmatrix} 0 & 0 & -6 & -2 \end{pmatrix}$

It can be verified that the y_i's are the left eigenvectors of the matrix A.

C. Comments

The important identities of (30), (31), (34), and (35) open up some interesting possibilities. The link between an SSD eigenstructure and that of an MFD opens the way for a direct way of combining traditional eigenstructure assignment objectives with polynomial methods, moving beyond traditional fixed state or output feedback gains, but enabling dynamic compensators to be incorporated into the controller structure is a very natural way. We have developed an approach that allows us to design dynamic compensators and pre-compensators that place block poles and block zeros. This enables latent structure and, hence, eigenstructure assignment possible, not only for poles but for zeros as well. The approach can therefore be used to improve the behaviour of MIMO systems and resolving control problems such as: sensitivity, robustness, decoupling, and disturbance rejection. The employment of dynamic compensation affords additional degrees of freedom in the design process, enabling a designer to achieve closer matches to a closed loop time domain specification than otherwise afforded using simple gain output feedback of conventional eigenstructure assignment. In the meantime, our more immediate contribution is to the polynomial eigenvalue problem (PEP). We now share our results on this.

V. Polynomial Eigenvalue Problem

A. Introduction

Let an r^{th} degree nxn regular matrix polynomial be given by:

$$P(\lambda) = P_r\lambda^r + P_{r-1}\lambda^{r-1} + ... + P_1\lambda + P_0 \tag{36}$$

where P_i are nxn real matrices and either P_0 or P_r is non singular. If it is the case then $P(\lambda)$ can be rewritten such that $P_0 = I_n$ and $P(\lambda)$ will be monic.

The Polynomial eigenvalue problem (PEP) consists of computing the eigenvalues and eigenvectors of a polynomial matrix (called in this paper latent values and latent roots)

To solve the PEP, in ([15]), matrices in companion forms (Controller) are proposed to determine the eigenvalues of a matrix polynomial but without referring to eigenvectors either right or left.

The idea here is to construct from the polynomial matrix a block controller form or a block observer form matrix and then compute the eigenvalues and right or left eigenvectors of these "normal" matrices. Then using the relationship established between these eigenvectors and latent vectors, we can directly obtain the latent values and vectors, either right or left, of the polynomial matrix.

B. Algorithm

Let an r^{th} degree n^{th} order monic matrix polynomial be rewritten as:

$$P(\lambda) = I_n\lambda^r + P_{r-1}\lambda^{r-1} + ... + P_1\lambda + P_0 \tag{37}$$

step1: Construct the block controller form state matrix A_c as in (9):

$$Ac = \begin{pmatrix} 0_n & I_n & 0_n & \cdots & 0_n \\ 0_n & 0_n & I_n & \cdots & 0_n \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0_n & 0_n & 0_n & \cdots & I_n \\ -P_0 & -P_1 & -P_2 & \cdots & -P_{r-1} \end{pmatrix}$$

or the block observer form state matrix A_o as in (11):

$$Ao = \begin{pmatrix} 0_n & 0_n & \cdots & -P_0 \\ I_n & 0_n & \cdots & -P_1 \\ 0_n & I_n & \cdots & -P_2 \\ \vdots & \vdots & \cdots & \vdots \\ 0_n & \cdots & I_n & -P_{r-1} \end{pmatrix}$$

step2: Compute the eigenvalues and right (left) eigenvectors of the block controller (observer) matrix which gives the eigenvalues of $P(\lambda)$.

step3: Using(14) or (21) compute the right or the left latent vectors.

C. Illustrative Example

Consider $P(\lambda) = I_2\lambda^3 + P_2\lambda^2 + P_1\lambda + P_0$ where:

$P_2 = \begin{pmatrix} 0 & 1 \\ 0 & 5 \end{pmatrix}$

$P_1 = \begin{pmatrix} -1 & 5 \\ 0 & 6 \end{pmatrix}$

$P_0 = \begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix}$

1) Right Latent Structure: The latent roots are $\{0, 1, -1, -2, -3\}$ with 0 being a double root.

The right latent vectors corresponding to these latent roots are respectively:

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ -12 \end{pmatrix} \right\}$$

Remark 3: using the MATLAB function `polyeig(P0,P1,P2,P3)` (P_3 is equal to I_2) we obtain the following:

$$\text{latent values}=\begin{pmatrix} 0 \\ 1.0000 \\ 0.0000 \\ -1.0000 \\ -3.0000 \\ -2.0000 \end{pmatrix}$$

$$\text{latent row vectors}=\begin{pmatrix} -1.0000 & 0 \\ 1.0000 & -0.0000 \\ -1.0000 & -0.0000 \\ -1.0000 & -0.0000 \\ -0.0830 & 0.9965 \\ 0.3162 & -0.9487 \end{pmatrix}$$

These all have the correct directions.

2) Left Latent Structure: The left latent vectors associated with the precedent latent roots are:

$$\{\begin{pmatrix} 0 & 1 \end{pmatrix}, \begin{pmatrix} -6 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \end{pmatrix}\}$$

Remark 4: by using the same MATLAB function `polyeig(P0',P1',P2',P3')` to compute the left latent vectors we obtain the following:

$$\text{latent values}=\begin{pmatrix} -3.0000 \\ -2.0000 \\ 0.0000 \\ -0.0000 \\ -1.0000 \\ 1.0000 \end{pmatrix}$$

$$\text{latent row vectors}=\begin{pmatrix} -0.0000 & -1.0000 \\ -0.0000 & 1.0000 \\ 0.0000 & -1.0000 \\ 0.0000 & 1.0000 \\ -1.0000 & 0.0000 \\ 0.7682 & -0.6402 \end{pmatrix}$$

Again all these have the correct directions.

3) Latent Roots and Right Latent Vectors: First we construct the block controller form of this Polynomial matrix:

$$Ac = \begin{pmatrix} 0_2 & I_2 & 0_2 \\ 0_2 & 0_2 & I_2 \\ -P_0 & -P_1 & -P_2 \end{pmatrix}$$

Its eigenvalues and right eigenvectors are the following (computed using MATLAB by using the function `eig(Ac)`):

eigenvalues=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1.0000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2.0000 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3.0000 \end{pmatrix}$$

eigenvectors=

$$\begin{pmatrix} 1.0 & 0.5774 & 1.0 & -0.5774 & -0.0690 & -0.0087 \\ 0 & -0.0 & -0.0 & 0 & 0.2070 & 0.1045 \\ 0 & -0.5774 & -0.0 & -0.5774 & 0.1380 & 0.0261 \\ 0 & 0 & 0 & 0 & -0.4140 & -0.3134 \\ 0 & 0.5774 & 0 & -0.5774 & -0.2760 & -0.0783 \\ 0 & 0 & 0 & 0 & 0.8281 & 0.9402 \end{pmatrix}$$

The latent values are the eigenvalues. The right latent vectors are computed using (14):

$$\text{latent row vectors} = \begin{pmatrix} 1.0000 & 0 \\ 0.5774 & -0.0000 \\ 1.0000 & -0.0000 \\ -0.5774 & 0.0000 \\ -0.0690 & 0.2070 \\ -0.0087 & 0.1045 \end{pmatrix}$$

It is easy to verify that the right latent vectors directions are correct, through appropriate scaling.

4) Latent Roots and Left Latent Vectors: The block observer form of $P(\lambda)$ is the following:

$$Ao = \begin{pmatrix} 0_2 & 0_2 & -P_0 \\ I_2 & 0_2 & -P_1 \\ 0_2 & I_2 & -P_2 \end{pmatrix}$$

Its eigenvalues and left eigenvectors are computed by using the function `eig(Ao.')` of MATLAB:

eigenvalues=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3.0000 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2.0000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1.0000 \end{pmatrix}$$

eigenvectors=

$$\begin{pmatrix} 0 & -0.0 & 0 & 0 & 0.4435 & -0.5774 \\ 1.0 & -1.0 & -0.1048 & -0.2182 & -0.3696 & 0 \\ 0 & 0 & 0 & 0 & 0.4435 & 0.5774 \\ 0 & 0 & 0.3145 & 0.4364 & -0.3696 & 0 \\ 0 & 0 & 0 & 0 & 0.4435 & -0.5774 \\ 0 & -0.0 & -0.9435 & -0.8729 & -0.3696 & 0 \end{pmatrix}$$

The latent values are the eigenvalues. The left latent vectors are computed using (21):

$$\text{latent vectors} = \begin{pmatrix} 0 & 1.0000 \\ -0.0000 & -1.0000 \\ 0.0000 & -0.1048 \\ 0.0000 & -0.2182 \\ 0.4435 & -0.3696 \\ -0.5774 & 0 \end{pmatrix}$$

Again, it is easy to verify that the left latent vectors directions are correct, through appropriate scaling.

VI. Conclusion

The importance of both MFD and SSD in control theory is well known. The MFD provides a very natural way of expressing desired zero/pole positions, whereas the eigenstructure of the SSD is a natural way of describing a desired multivariable system time response. At the onset of this work, we privately postulated that, if we could establish the structural links between them, then we would be able to combine design methodologies and get the benefits of both descriptions.

In this paper we have achieved this very important initial result and have been able to utilize it in proposing a new algorithm for solving the polynomial eigenvalue problem for any regular matrix polynomial. The proposed algorithm is easier and requires less computing time and memory to determine the eigenvalues and eigenvectors of block controller/observer form state matrix than pencil matrix (see [15]). The proposed method is similar to the one used by the

Control system tool box of Matlab, but the latter uses two matrices.

Our current efforts to combine polynomial methods and eigenstructure assignment methods support our belief.

REFERENCES

[1] K. M. Sobel and E.J. Lallman, "Eigenstructure assignment for the control of highly augmented aircraft," *J. of Guidance, Control, and Dynamics*, 12(3), 1989, pp. 18–324.

[2] T. Clarke, S.J. Griffin and J. Ensor, "Output feedback eigenstructure assignment using a new reduced orthogonality condition," *Int. J. Control*, 76(4), 2002, pp. 1–13.

[3] A.N. Andry and E.Y. Shapiro, "Eigenstructure assignment for linear systems," *IEEE Trans. on Aerospace and Electronic Systems*,19(5), 1983, pp. 711–727.

[4] K.M. Sobel, E.Y. Shapiro and A.N. Andry, "Eigenstructure assignment," *Int. J. Control*, 59 (1), 1994, pp. 13–37.

[5] J. Farineau, "Lateral Electric Flight Control Laws of Civil Aircraft based on Eigenstructure Assignment Technique," Paper 89-3594 in *AIAA Guidance Navigation and Control Conference*, Boston, MA, pp.1–15, August 1989.

[6] K. Hariche and E.D. Denman, "On Solvents and Lagrange Interpolating Lambda-matrices," *App. Math. And Comp.*,25(4), 1988, pp. 321-332.

[7] J. Leyva-Ramos, "A New Method for Block Partial Fraction Expansion of Matrix Fraction Descriptions," *IEEE Trans.Auto. Cont.*, 36(12), 1991, pp.1482–1485.

[8] L.S. Shieh, F.R. Chang and B.C. Mcinnis, "The Block Partial Fraction Expansion of a Matrix Fraction Description with Repeated Block Poles," *IEEE Trans. Auto. Cont.*, AC-31(3), 1986, pp. 236–239.

[9] L.S. Shieh, M.M. Mehio and R.E. Yates, "Cascade Decomposition and Realization of Multivariable Systems Via Block-Pole and Block-Zero Placement,", *IEEE Trans. Auto. Cont.*, AC-30(11), 1985, pp.1109–1112.

[10] L.S. Shieh, Y.T. Tsay and R.E. Yates, "State-Feedback Decomposition of Multivariable Systems Via Block-Pole Placement", *IEEE Trans. Auto. Cont.*, AC-28(8), 1983, pp.850–852.

[11] H.H. Rosenbrock, *State-space and Multivariable Theory*, Nelson UK, 1970.

[12] P. Van Dooren and P. Dewilde, "The Eigenstructure of an Arbitrary Polynomial Matrix: Cdmputational Aspects",Lin. Alg. App., Vol.50, 1983, pp 545–579.

[13] D.S. Mackey, N. Mackey, C. Mehl and V. Mehrmann,"Structured Polynomial Eigenvalue Problems: Good Vibrations from good Linearizations", SIAM J. Matrix Anal. App., Vol. 28, No. 4, 2006, pp. 10291051.

[14] N.J. Higham, D.S. Mackey and F. Tisseur, "The Conditiong of Linearization of Matrix Polynomials", SIAM J. Matrix Anal. App., Vol. 28, No. 4, 2006, pp. 10051028.

[15] A. Edelman and H. Murakami, "Polynomial Roots from Companion Matrix Eigenvalues", Math. Comp., Vol. 64, 1994, pp. 763-776.