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A constructive proof of the general Brouwer fixed point theorem and related computational results in general non-convex sets

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Abstract—In this paper, by introducing twice continuously differentiable mappings, we develop an interior path following following method, which enables us to give a constructive proof of the general Brouwer fixed point theorem and thus to solve fixed point problems in a class of non-convex sets. Under suitable conditions, a smooth path can be proven to exist. This can lead to an implementable globally convergent algorithm. Several numerical examples are given to illustrate the results of this paper.

Keywords—interior path following method; general Brouwer fixed point theorem; non-convex sets; globally convergent algorithm

I. Introduction

As a powerful mechanism for mathematical analysis, fixed point theory has many applications in areas such as mechanics, physics, transportation, control, economics, and optimization. Fixed point theorems have been extensively studied and generalized in the past years (see [1], [2], [3], [4], [5], [6], [7], [8], etc. and the references therein). In 1976, Kellogg et al. (see [9]) gave a constructive proof of the Brouwer fixed point theorem and hence presented a homotopy method for computing the fixed points of a twice continuously differentiable self-mapping $\Phi(x)$. From then on, this method has become a powerful tool in dealing with fixed point problems (see [10], [11], [12], [13], [14], etc. and the references therein). In 1978, Chow et al. [13] constructed the homotopy

$$(1-\mu)(x-\Phi(x))+\mu(x-x^0) \tag{1}$$

for the bounded closed convex set. This homotopy is used by many authors to compute fixed points and solutions of nonlinear systems. In general, it is difficult to reduce or remove the convexity.

If a bounded, closed subset in R^n is homeomorphic to the unit ball, then any continuous self-mapping $\Phi(x)$ in it has a fixed point. This is the general Brouwer fixed point theorem, which does not require the convexity of the subsets in R^n, certainly it is also very interesting and important to give a constructive proof of it and hence solve fixed point problems numerically in general non-convex subsets. Under the normal cone condition, Yu et al. (see [15]) completed this work in a class of non-convex sets. In this paper, we give a constructive proof of the general Brouwer fixed point theorem and thus solve fixed point problems in more general non-convex sets than those in [15]. To this end, we introduce C^2 mappings $\xi_i(x)\in R^n, i=1,\ldots,m$. Based on the newly introduced mappings, by introducing the idea of Karmarkar's interior point method into the homotopy method, we develop an interior path homotopy following method. Under suitable conditions, we prove that a bounded smooth homotopy path connecting a given point to a fixed point exists. This forms the theoretical base of the interior path following method. Numerically tracking the smooth path can lead to an implementable globally convergent algorithm for solving fixed point problems. In addition, the method proposed in this paper also avoids homeomorphically transforming the bounded closed set to the closed unit ball. It is also important because it is difficult to construct such a homeomorphism in practice. At last, several numerical examples are given to validate the work in this paper.

This paper is organized as follows. Section 2 is the main part, which exhibits a constructive proof of the general Brouwer fixed point theorem in more general non-convex sets than those in [15]. In section 3, we use the reduced predictor-corrector algorithms given by Allgower and Georg [11] to compute some experimental examples, which illustrate the results in this paper.

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II. A constructive proof of the general Brouwer fixed point theorem

In this section, some notations are given as follows: $g(x)=(g_1(x),\ldots,g_m(x))^T\in R^m$, $\nabla g(x)=(\nabla g_1(x),\ldots,\nabla g_m(x))\in R^{n\times m}$, $y\in R^m$, $y^{(0)}\in R^m$, $Y=\mathrm{diag}(y)\in R^{m\times m}$, and $Y^{(0)}=\mathrm{diag}(y^{(0)})\in R^{m\times m}$. The nonnegative and positive orthants of R^m are denoted as R^m_+ and R^m_{++}, respectively. We also denote the active set at x by $B(x)=\{i\in\{1,\ldots,m\}:\ g_i(x)=0\}$. In addition, set $\Omega=\{x\in R^n:\ g_i(x)\le 0,\ i=1,\ldots,m\}$, $\Omega^0=\{x\in R^n:\ g_i(x)<0,\ i=1,\ldots,m\}$, and $\partial\Omega=\Omega\backslash\Omega^0$.

In [15], Yu et al. gave a constructive proof of the general Brouwer fixed point theorem under the following assumptions:
(A_1) Ω^0 is nonempty and Ω is bounded;
(A_2) For any $x\in\partial\Omega$, the matrix $\{\nabla g_i(x):\ i\in B(x)\}$ is of full column rank;
(A_3) (The normal cone condition of Ω) For any $x\in\partial\Omega$, the normal cone of Ω at x only meets Ω at x, i.e., for any $x\in\partial\Omega$,

we have

$$\{x + \sum_{i \in B(x)} y_i \nabla g_i(x) : \ y_i \geq 0 \ for \ i \in B(x)\} \bigcap \Omega = \{x\}.$$

In that paper, the homotopy equation is given as follows

$$H(w, w^{(0)}, \mu) = \begin{pmatrix} (1-\mu)(x - \Phi(x) + \nabla g(x) y) + \mu(x - x^{(0)}) \\ Yg(x) - \mu Y^{(0)} g(x^{(0)}) \end{pmatrix} = 0, \tag{2}$$

where $w = (x, y) \in R^{n+m}$, $w^{(0)} = (x^{(0)}, y^{(0)}) \in \Omega^0 \times R^m_{++}$.

The normal cone condition of a set is a generalization of the convexity. If Ω is a convex set, then it satisfies the normal cone condition. On the other hand, if Ω satisfies the normal cone condition, then the outer normal cone of Ω can not meet Ω^0, but meets Ω at x. In the following, we want to give a constructive proof of the general Brouwer fixed point theorem in more general non-convex sets than those in [15]. To this end, we introduce C^2 mappings $\xi_i(x) \in R^n, \ i = 1, \ldots, m$ and make the following assumptions:

(C_1) Ω^0 is nonempty and Ω is bounded;

(C_2) For any $x \in \partial\Omega$, if

$$\sum_{i \in B(x)} \nabla g_i(x) y_i + \xi_i(x) u_i = 0, \ y_i \geq 0, \ u_i \geq 0,$$

then $y_i = 0$, $u_i = 0$, $\forall i \in B(x)$;

(C_3) For any $x \in \partial\Omega$, we have

$$\{x + \sum_{i \in B(x)} \xi_i(x) u_i : \ u_i \geq 0, \ i \in B(x)\} \cap \Omega = \{x\}.$$

If Ω satisfies assumptions $(A_1) - (A_3)$, let $\xi_i(x) = \nabla g_i(x)$, $i = 1, \ldots, m$, then Ω satisfies assumptions $(C_1) - (C_3)$. Conversely, the conclusion does not hold. This can be illustrated by Examples 1-6 in Section 3. By this fact, we are capable of giving a constructive proof of the general Brouwer fixed point theorem in more general non-convex sets than those in [15].

The following lemma, which plays a key role in this paper, gives an equivalent condition of the existence of fixed points.

Lemma 1 Let $g_i(x), \ i = 1, \ldots, m$ be C^3 functions, let assumptions $(C_1) - (C_3)$ hold, and let $\xi_i(x) \in R^n, \ i = 1, \ldots, m$ be C^2 mappings. Then for any C^2 mapping $\Phi(x) : \ R^n \to R^n$ satisfying $\Phi(\Omega) \subset \Omega$, $x^* \in \Omega$ is a fixed point of $\Phi(x)$ in Ω if and only if there exists $y^* \in R^m$, such that (x^*, y^*) is a solution of the equation

$$\begin{array}{l} x - \Phi(x) + \sum\limits_{i=1}^{m} \xi_i(x) y_i = 0, \\ Yg(x) = 0, \ \ g(x) \leq 0, \ y \geq 0. \end{array} \tag{3}$$

Proof Similar to the analysis of Proposition 2.2 in [15].

To give a constructive proof of the general Brouwer theorem in a broader class of non-convex sets, we construct the following homotopy:

$$H(w, w^{(0)}, \mu) = \begin{pmatrix} (1-\mu)(x - \Phi(x) + (1-\mu)\mu \nabla g(x) y + \xi(x) y) \\ +\mu(x - x^{(0)}) \\ Yg(x) - \mu Y^{(0)} g(x^{(0)}) \end{pmatrix} = 0, \tag{4}$$

where $w = (x, y) \in R^{n+m}$, $w^{(0)} = (x^{(0)}, y^{(0)}) \in \Omega^0 \times R^m_{++}$ and $\xi(x) = (\xi_1(x), \ldots, \xi_m(x))$.

Note that when $\mu = 0$, the homotopy equation (4) turns to (3). When $\mu = 1$, the equation $H(w, w^{(0)}, 1) = 0$ has a unique solution $w = w^{(0)} \in \Omega^0 \times R^m_{++}$.

For a given $w^{(0)}$, rewrite $H(w, w^{(0)}, \mu)$ as $H_{w^{(0)}}(w, \mu)$. The zero-point set of $H_{w^{(0)}}$ is

$$H^{-1}_{w^{(0)}}(0) = \{(w, \mu) \in \Omega \times R^m_+ \times (0, 1] : \ H_{w^{(0)}}(w, \mu) = 0\}.$$

In the following, we recall some basic definitions and results from differential topology, which will be used in our main result of this paper.

The inverse image theorem tells us that, if 0 is a regular value of the map $H_{w^{(0)}}$, then $H^{-1}_{w^{(0)}}(0)$ consists of some smooth curves. And the regularity of $H_{w^{(0)}}$ can be obtained by the following lemma.

Lemma 2 (Transversality Theorem, see [13]). Let Q, N and P be smooth manifolds with dimensions q, m and $\tilde{p}$, respectively. Let $W \subset P$ be a submanifold of codimension p (that is, $\tilde{p} = p+$ dimension of W). Consider a smooth map $\Phi : Q \times N \to P$. If Φ is transversal to W, then for almost all $a \in Q$, $\Phi_a(\cdot) = \Phi(a, \cdot) : N \to P$ is transversal to W. Recall that a smooth map $h : N \to P$ is transversal to W if

$$\{Range(Dh(x))\} + \{T_y W\} = T_y P, \quad whenever \, y = h(x) \in W,$$

where Dh is the Jacobi matrix of h, $T_y W$ and $T_y P$ denote the tangent spaces of W and P at y, respectively.

In this paper, $W = \{0\}$, so the Transversality Theorem is corresponding to the Parameterized Sard's Theorem on smooth manifolds.

Lemma 3 (Parameterized Sard's Theorem). Let $V \subset R^n$, let $U \subset R^m$ be open sets, and let $\Phi : V \times U \to R^k$ be a C^r map, where $r > \max\{0, \ m - k\}$. If $0 \in R^k$ is a regular value of Φ, then for almost all $a \in V$, 0 is a regular value of $\Phi_a \equiv \Phi(a, \cdot)$.

With the preparation of the previous lemmas, we can prove the following main theorem on the existence and boundedness of a smooth path connecting an interior point $x^{(0)}$ in Ω to a fixed point. This implies the global convergence of the interior path following method.

Theorem 1 Let H be defined as in (4), let $g_i(x), \ i = 1, \ldots, m$ be C^3 functions, let assumptions $(C_1) - (C_3)$ hold, and let $\xi_i(x), \ i = 1, \ldots, m$ be C^2 mappings. Then for any C^2 mapping $\Phi(x)$: $R^n \to R^n$ satisfying $\Phi(\Omega) \subset \Omega$,

(1) (existence of the fixed point) $\Phi(x)$ has a fixed point in Ω;

(2) (computation of the fixed point) for almost all $w^{(0)} \in \Omega^0 \times R^m_{++}$, there exists a C^1 curve $(w(s), \mu(s))$ of dimension 1 such that

$$H(w(s), w^{(0)}, \mu(s)) = 0, \quad (w(0), \mu(0)) = (w^{(0)}, 1). \tag{5}$$

And when $\mu(s) \to 0$, $w(s)$ tends to a point $w^* = (x^*, y^*)$. In addition, the component x^* of w^* is a fixed point of $\Phi(x)$ in Ω.

Proof Denoting the Jacobi matrix of $H(w, w^{(0)}, \mu)$ by $DH(w, w^{(0)}, \mu)$, we have

$$DH(w, w^{(0)}, \mu) = \left(\frac{\partial H(w, w^{(0)}, \mu)}{\partial w}, \frac{\partial H(w, w^{(0)}, \mu)}{\partial w^{(0)}}, \frac{\partial H(w, w^{(0)}, \mu)}{\partial \mu}\right). \tag{6}$$

Note that $\forall(w,\mu) \in R^{n+m} \times (0,1]$, we obtain

$$\frac{\partial H(w,w^{(0)},\mu)}{\partial w^{(0)}} = \begin{pmatrix} -\mu I & 0 \\ -\mu Y^{(0)} \nabla g(x^{(0)})^T & -\mu G(x^{(0)}) \end{pmatrix}, \tag{7}$$

where $G(x^{(0)}) = \mathrm{diag}(g(x^{(0)}))$. It is easy to show that $\partial H(w,w^{(0)},\mu)/\partial w^{(0)}$ is nonsingular. Hence $DH(w,w^{(0)},\mu)$ is of full row rank, then 0 is a regular value of $H(w,w^{(0)},\mu)$. By the parameterized Sard's theorem, for almost all $w^{(0)} \in \Omega^0 \times R^m_{++}$, 0 is a regular value of map $H_{w^{(0)}} : \Omega \times R^m_+ \times (0,1] \to R^{n+m}$. By the inverse image theorem, $H^{-1}_{w^{(0)}}(0)$ consists of some smooth curves. Since $H_{w^{(0)}}(w^{(0)},1) = 0$, then there exists a C^1 curve $(w(s),\mu(s))$ (denoted by $\Gamma_{w^{(0)}}$) of dimension 1 such that

$$H(w(s),w^{(0)},\mu(s)) = 0, \quad (w(0),\mu(0)) = (w^{(0)},1).$$

By the classification theorem of one-dimensional smooth manifold, $\Gamma_{w^{(0)}}$ is diffeomorphic either to a unit circle or to the unit interval. Since

$$\frac{\partial H_{w^{(0)}}(w^{(0)},1)}{\partial w} = \begin{pmatrix} I & 0 \\ Y^{(0)} \nabla g(x^{(0)})^T & G(x^{(0)}) \end{pmatrix}$$

is nonsingular, therefore $\Gamma_{w^{(0)}}$ is diffeomorphic to the unit interval $(0,1]$.

Let (w^*,μ^*) be a limit point of $\Gamma_{w^{(0)}}$, then the following cases may occur:
(a) $(w^*,\mu^*) = (x^*,y^*,\mu^*) \in \Omega \times R^m_+ \times \{0\}$,
(b) $(w^*,\mu^*) = (x^*,y^*,\mu^*) \in \Omega^0 \times R^m_{++} \times \{1\}$,
(c) $(w^*,\mu^*) = (x^*,y^*,\mu^*) \in \partial(\Omega \times R^m_+) \times (0,1]$.

From above analysis, the equation $H_{w^{(0)}}(w,1) = 0$ has a unique solution $(w^{(0)},1)$ in $\Omega^0 \times R^m_{++} \times \{1\}$, so case (b) does not occur.

If case (c) holds, there exists a sequence of points $\{(w^{(k)},\mu_k)\}_{k=1}^{\infty} \subset \Gamma_{w^{(0)}}$ such that $\|(w^{(k)},\mu_k)\| \to \infty$. Since Ω and $(0,1]$ are bounded, hence there exists a subsequence of points (denoted also by $\{(w^{(k)},\mu_k)\}_{k=1}^{\infty}$) such that $x^{(k)} \to x^*$, $\|y^{(k)}\| \to \infty$ and $\mu_k \to \mu^*$ as $k \to \infty$. From the second equality of (4), it follows that

$$g(x^{(k)}) = \mu_k (Y^{(k)})^{-1} Y^{(0)} g(x^{(0)}).$$

So the active index set

$$B(x^*) = \{i \in \{1,\dots,m\} : \lim_{k\to\infty} y_i^{(k)} = \infty\}$$

is a nonempty set, i.e., $x^* \in \partial\Omega$, where $y_i^{(k)}$ denotes the ith element of $y^{(k)}$.

From the first equality of (4), we have

$$\begin{aligned}&(1-\mu_k)[x^{(k)} - \Phi(x^{(k)}) + (1-\mu_k)\mu_k \nabla g(x^{(k)}) y^{(k)} \\ &+\xi(x^{(k)}) y^{(k)}] + \mu_k(x^{(k)} - x^{(0)}) = 0.\end{aligned} \tag{8}$$

(1) When $\mu^* = 1$, rewrite (8) as

$$\begin{aligned}&\sum_{i\in B(x^*)} ((1-\mu_k) y_i^{(k)})[(1-\mu_k)\mu_k \nabla g_i(x^{(k)}) + \xi_i(x^{(k)})] \\ &+(x^{(k)} - x^{(0)}) = (1-\mu_k)(\Phi(x^{(k)}) - x^{(0)}) \\ &- \sum_{i\notin B(x^*)} ((1-\mu_k) y_i^{(k)})[(1-\mu_k)\mu_k \nabla g_i(x^{(k)}) + \xi_i(x^{(k)})].\end{aligned} \tag{9}$$

By the fact that $y_i^{(k)}$ is bounded for $i \notin B(x^*)$, and assumption (C_1), when $k \to \infty$, we have

$$\sum_{i\in B(x^*)} \lim_{k\to\infty} ((1-\mu_k) y_i^{(k)}) \xi_i(x^*) + (x^* - x^{(0)}) = 0. \tag{10}$$

Then by assumption (C_2), we have that $\lim_{k\to\infty}(1-\mu_k) y_i^{(k)}$ (denoted by ρ_i^*) exists, furthermore, we get

$$\sum_{i\in B(x^*)} \xi_i(x^*)\rho_i^* + x^* = x^{(0)}, \tag{11}$$

which contradicts assumption (C_3).

(2) When $\mu^* < 1$, rewrite (8) as

$$\begin{aligned}&\sum_{i\in B(x^*)} ((1-\mu_k) y_i^{(k)})[(1-\mu_k)\mu_k \nabla g_i(x^{(k)}) + \xi_i(x^{(k)})] \\ &= -\sum_{i\notin B(x^*)} ((1-\mu_k) y_i^{(k)})[(1-\mu_k)\mu_k \nabla g_i(x^{(k)}) + \xi_i(x^{(k)})] \\ &-(1-\mu_k)(x^{(k)} - \Phi(x^{(k)})) - \mu_k(x^{(k)} - x^{(0)}).\end{aligned} \tag{12}$$

Let $y_I^{(k)}$ be a vector which consists of $y_i^{(k)}, i \in B(x^*)$ and let

$$\alpha_i^{(k)} = \frac{y_i^{(k)}}{\|y_I^{(k)}\|}, \quad i \in B(x^*). \tag{13}$$

Note that $0 \le \alpha_i^{(k)} \le 1$, so there exists a subsequence of $\{\alpha_i^{(k)}\}$, still denoted by $\{\alpha_i^{(k)}\}$, such that $\alpha_i^{(k)} \to \alpha_i^*$ for each $i \in B(x^*)$ as $k \to +\infty$. Furthermore, we denote by α^* the vector consisting of α_i^*, $i \in B(x^*)$, then $\|\alpha^*\| = 1$. Dividing both sides of (12) by $\|y_I^{(k)}\|$, when $k \to +\infty$, we have

$$\sum_{i\in B(x^*)} (1-\mu^*)^2 \mu^* \alpha_i^* \nabla \bar{g}_i(x^*) + (1-\mu^*)\alpha_i^* \xi_i(x^*) = 0, \tag{14}$$

which contradicts assumption (C_2).

From above discussion, we obtain that case (a) is the only possible case. Therefore w^* is a solution of (4), and by Lemma 1, we have x^* is a fixed point of $\Phi(x)$ in Ω.

III. Numerical Analysis

For almost all $w^{(0)} = (x^{(0)},y^{(0)}) \in \Omega^0 \times R^m_{++}$, by Theorem 1, the homotopy generates a C^1 curve $\Gamma_{w^{(0)}}$. Then by differentiating the first equality of (5), we get the following theorem

Theorem 2 The homotopy path $\Gamma_{w^{(0)}}$ is determined by the following initial value problem to the ordinary differential equation

$$DH_{w^{(0)}}(w(s),\mu(s)) \begin{pmatrix} \dot{w}(s) \\ \dot{\mu}(s) \end{pmatrix} = 0, \ (w(0),\mu(0)) = (w^{(0)},1), \tag{15}$$

where s is the arclength of the curve $\Gamma_{w^{(0)}}$.

Next, we discuss how to track the homotopy path $\Gamma_{w^{(0)}}$ numerically in the following remark.

Remark 1 Let $A = DH_{w^{(0)}}(w(s),\mu(s))$, $v = (\dot{w}(s),\dot{\mu}(s))^T$, $y(s) = (w(s),\mu(s))$, then (15) becomes

$$Av = 0, \ y(0) = (w^{(0)},1). \tag{16}$$

By solving the linear system $Av = 0$, we get a solution v, then (15) becomes the following initial value problem

$$\frac{dy}{ds} = v, \ y(0) = (w^{(0)},1). \tag{17}$$

Generally, if we utilize numerical algorithms for initial value problems of ordinary differential equations (for example, Runge-Kutta algorithms) to solve (17) independently, the steplength must be sufficiently small to guarantee that each iterate $(w^{(k)}, \mu_k)$ is close enough to the solution curve. This may increase computational cost greatly. Since ones only try to find a point (w, μ)(when μ is approximately zero) instead of tracking the curve $\Gamma_{w^{(0)}}$ very precisely, hence they would like to combine numerical algorithms for initial value problems of ordinary differential equations with other methods (for example, Newton's methods) to develop more efficient methods, i.e., predictor-corrector methods[11]. In the following, we formulate the implementation of a standard predictor-corrector procedure in detail. Suppose we have obtained a sequence of points $(w^{(i)}, \mu_i), i = 1, \ldots, k$, starting with an initial guess $(w^{(0)}, 1)$. To get the next iterate $(w^{(k+1)}, \mu_{k+1})$, we need to solve the linear system $Av = 0$, which enables us to get a unit tangent vector $v^{(k)}$ at $(w^{(k)}, \mu_k)$. The tangent vector at a point on $\Gamma_{w^{(0)}}$ has two opposite directions, one (the positive direction) makes s increase, another (the negative direction) makes s decrease. Since the negative direction will lead us back to the initial guess, so we must go along the positive direction. The criterion that determines the positive direction is based on a basic theory of the homotopy method, namely, the positive direction at any point keeps the sign of the determinant $\begin{vmatrix} DH_{w^{(0)}}(w,\mu) \\ v^T \end{vmatrix}$ invariant. On the first iterate, the sign is determined by the following lemma.

Lemma 4 If $\Gamma_{w^{(0)}}$ is smooth, then the positive direction $v^{(0)}$ at the initial point $(w^{(0)}, 1)$ satisfies

$$sign \begin{vmatrix} DH_{w^{(0)}}(w^{(0)},1) \\ {v^{(0)}}^T \end{vmatrix} = (-1)^{m+1}.$$

Proof Since $DH_{w^{(0)}}(w^{(0)},1) = \partial H_{w^{(0)}}(w^{(0)},1)/\partial(w,\mu)$

$$= \begin{pmatrix} I & 0 & a^{(0)} \\ Y^{(0)}\nabla g(x^{(0)})^T & G(x^{(0)}) & b^{(0)} \end{pmatrix} = (M_1, M_2), \tag{18}$$

where $a^{(0)} = (\Phi(x^{(0)}) - x^{(0)}) - \xi(x^{(0)})y^{(0)}, b^{(0)} = -Y^{(0)}g(x^{(0)})$, $M_1 \in R^{(n+m)\times(n+m)}, M_2 \in R^{(n+m)\times 1}$. The tangent vector $v^{(0)}$ of $\Gamma_{w^{(0)}}$ at $(w^{(0)}, 1)$ satisfies

$$(M_1, M_2)v^{(0)} = (M_1, M_2)\begin{pmatrix} v_1^{(0)} \\ v_2^{(0)} \end{pmatrix} = 0, \tag{19}$$

where $v_1^{(0)} \in R^{n+m}$, $v_2^{(0)} \in R^1$. By a simple computation, we have $v_1^{(0)} = -M_1^{-1}M_2v_2^{(0)}$, thus

$$\begin{aligned} &\begin{vmatrix} DH_{w^{(0)}}(w^{(0)},1) \\ {v^{(0)}}^T \end{vmatrix} = \begin{vmatrix} M_1 & M_2 \\ {v_1^{(0)}}^T & {v_2^{(0)}}^T \end{vmatrix} \\ &= \begin{vmatrix} M_1 & M_2 \\ -M_2^TM_1^{-T} & 1 \end{vmatrix} v_2^{(0)} \\ &= |M_1|v_2^{(0)}(1 + M_2^TM_1^{-T}M_1^{-1}M_2) \\ &= \begin{vmatrix} I & 0 \\ Y^{(0)}\nabla g(x^{(0)})^T & G(x^{(0)}) \end{vmatrix} v_2^{(0)}(1 + M_2^TM_1^{-T}M_1^{-1}M_2) \\ &= (-1)^m|G(x^{(0)})|v_2^{(0)}(1 + M_2^TM_1^{-T}M_1^{-1}M_2). \end{aligned} \tag{20}$$

Note that $g_i(x^{(0)}) < 0, i = 1, \ldots, m$, $(1 + M_2^TM_1^{-T}M_1^{-1}M_2) > 0$ and $v_2^{(0)}$ should be negative since initially we plan to move along the path $\Gamma_{w^{(0)}}$ by decreasing μ, and hence the sign of $\begin{vmatrix} DH_{w^{(0)}}(w^{(0)},1) \\ {v^{(0)}}^T \end{vmatrix}$ is $(-1)^{m+1}$.

Then, by using the Euler method, for some small steplength $h_k > 0$ (not sufficiently small), we are able to get a predictor point $(\bar{w}^{(k)}, \bar{\mu}_k) = (w^{(k)}, \mu_k) + h_kv^{(k)}$. Here we do not replace the Euler method by more complicated algorithms, for the predictor point need not to be close enough to the curve $\Gamma_{w^{(0)}}$, if only it is located in the convergent region of the Newton's method during the corrector phase. Next, we may make a corrector step. Setting $DH_{w^{(0)}}(w,\mu)^+ = DH_{w^{(0)}}(w,\mu)^T(DH_{w^{(0)}}(w,\mu)DH_{w^{(0)}}(w,\mu)^T)^{-1}$, which is the Moore-Penrose inverse of $DH_{w^{(0)}}(w,\mu)$. The corrector phase then tries to identify a corrector point $(w^{(k+1)}, \mu_{k+1})$ on the path $\Gamma_{w^{(0)}}$. The corrector step is usually carried out by the Newton's method that uses the Moore-Penrose inverse of $DH_{w^{(0)}}(w,\mu)$, starting with $(\bar{w}^{(k)}, \bar{\mu}_k)$ and proceeding until $\|H_{w^{(0)}}(w^{(k+1)}, \mu_{k+1})\|$ is approximately zero. The following pseudocode shows the basic steps of a generic predictor-corrector method.

Algorithm 1 (Euler-Newton method)

Step 0: Give an initial point $(w^{(0)}, 1)$, an initial steplength $h_0 > 0$ and three small positive numbers $\epsilon_1 > 0$, $\epsilon_2 > 0$, $\epsilon_3 > 0$. Set $k = 0$.

Step 1: Compute the direction $\eta^{(k)}$ of the predictor step:

(a) Compute a unit tangent vector $v^{(k)}$.

(b) Determine the direction $\eta^{(k)}$ of the predictor step as follows.

If the sign of the determinant of $\begin{vmatrix} DH_{w^{(0)}}(w^{(k)},\mu_k) \\ {v^{(k)}}^T \end{vmatrix} = (-1)^{m+1}$, then $\eta^{(k)} = v^{(k)}$. If the sign of the determinant of $\begin{vmatrix} DH_{w^{(0)}}(w^{(k)},\mu_k) \\ {v^{(k)}}^T \end{vmatrix} = (-1)^m$, then $\eta^{(k)} = -v^{(k)}$.

Step 2: Compute a corrector point $(w^{(k+1)}, \mu_{k+1})$.

$$(\bar{w}^{(k)}, \bar{\mu}_k) = (w^{(k)}, \mu_k) + h_k\eta^{(k)};$$

$$(w^{(k+1)}, \mu_{k+1}) = (\bar{w}^{(k)}, \bar{\mu}_k) - DH_{w^{(0)}}(\bar{w}^{(k)}, \bar{\mu}_k)^+H_{w^{(0)}}(\bar{w}^{(k)}, \bar{\mu}_k).$$

where $DH_{w^{(0)}}(w,\mu)^+ = DH_{w^{(0)}}(w,\mu)^T(DH_{w^{(0)}}(w,\mu)DH_{w^{(0)}}(w,\mu)^T)^{-1}$ is the Moore-Penrose inverse of $DH_{w^{(0)}}(w,\mu)$.

If $\|H_{w^{(0)}}(w^{(k+1)}, \mu_{k+1})\| \le \epsilon_1$, let $h_{k+1} = \min\{h_0, 2h_k\}$, and go to Step 3;

If $\|H_{w^{(0)}}(w^{(k+1)}, \mu_{k+1})\| \in (\epsilon_1, \epsilon_2)$, $h_{k+1} = h_k$, and go to Step 3;

If $\|H_{w^{(0)}}(w^{(k+1)}, \mu_{k+1})\| \ge \epsilon_2$, $h_{k+1} = \max\{2^{-25}h_0, (h_k/2)\}$, $k = k + 1$; and go to Step 2;

Step 3: If $\mu_{k+1} \le \epsilon_3$, then stop, else $k = k + 1$, and go to Step 1.

By using the homotopy (4) and Algorithm 1, we provide several numerical examples that illustrate the work in this paper. In each example, we set $\epsilon_1 = 1.e-3$, $\epsilon_2 = 1.e-6$ and $h_0 = 0.02$. The behaviors of homotopy paths are shown in Figs. 1-6. Computational results are given in Table 1, where

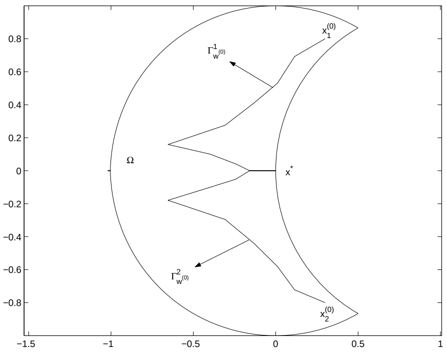

Fig. 1. The homotopy pathways of Example 1

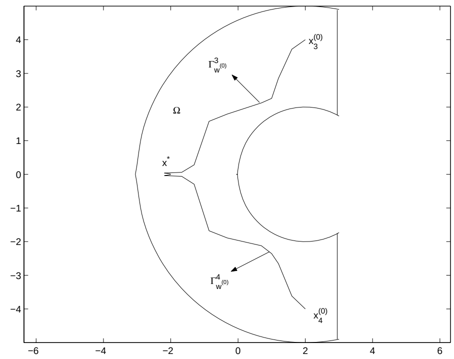

Fig. 2. The homotopy pathways of Example 2

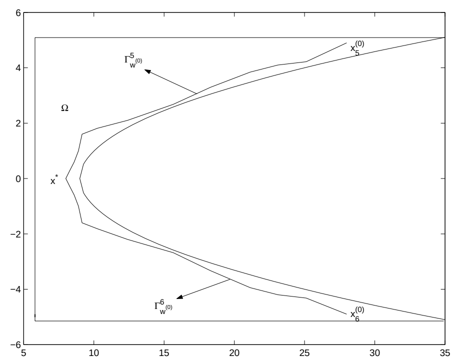

Fig. 3. The homotopy pathways of Example 3

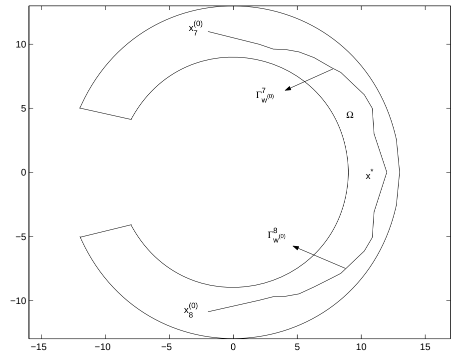

Fig. 4. The homotopy pathways of Example 4

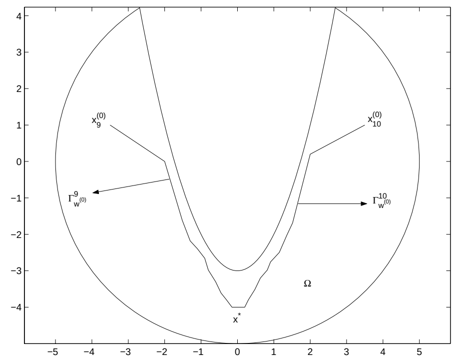

Fig. 5. The homotopy pathways of Example 5

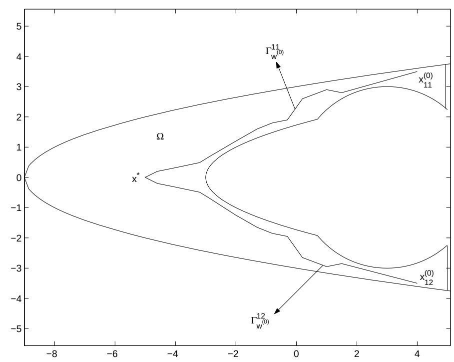

Fig. 6. The homotopy pathways of Example 6

$x^{(0)}$ denotes the initial guess, IT the number of iterations, H the value of $\|H_{w^{(0)}}(w^{(k)}, \mu_k)\|$ when the algorithm stops, and x^* the fixed point.

Example 1 To find a fixed point of self-mapping $\Phi(x) = (-x_1, -x_2)^T$ in $\Omega = \{(x_1, x_2) \in R^2 : \ x_1^2 + x_2^2 \leq 1, \ -(x_1 - 1)^2 - x_2^2 + 1 \leq 0\}$.
Let C^2 mappings $\xi_i(x) = \nabla g_i(x), i = 1, 2$, it is easy to show that the feasible set Ω satisfies assumptions $(C_1) - (C_3)$.

Example 2 To find a fixed point of self-mapping $\Phi(x) = (x_1, -x_2)^T$ in $\Omega = \{(x_1, x_2) \in R^2 : \ -(x_1 - 2)^2 - x_2^2 + 4 \leq 0, \ (x_1 + 1.5)^2 + x_2^2 - 25 \leq 0, \ x_1 - 3.25 \leq 0\}$.
Let C^2 mappings $\xi_1(x) = (10, 0)^T + \nabla g_1(x), \ \xi_i(x) = \nabla g_i(x), i = 2, 3$, it is easy to show that the feasible set Ω satisfies assumptions $(C_1) - (C_3)$.

Example 3 To find a fixed point of self-mapping $\Phi(x) = (x_1, -x_2)^T$ in $\Omega = \{(x_1, x_2) \in R^2 : \ -x_1 - 5 \leq 0, \ x_2 - 5.1 \leq 0, \ -x_2 - 5.1 \leq 0, \ x_1 - 35 \leq 0, \ x_1 - x_2^2 - 9 \leq 0\}$.
Let C^2 mappings $\xi_i(x) = \nabla g_i(x), i = 1, 2, 3, 4, \ \xi_5(x) = (12, 0)^T$, it is easy to show that the feasible set Ω satisfies assumptions $(C_1) - (C_3)$.

Example 4 To find a fixed point of self-mapping $\Phi(x) = (x_1, -x_2)^T$ in $\Omega = \{(x_1, x_2) \in R^2 : \ x_1^2 + x_2^2 \leq 13^2, \ x_1^2 + x_2^2 \geq 9^2, \ g_3(x) \geq 0\}$, where $g_3(x) = x_1 - 8|x_2|$ for all $x_1 \leq -8$ and it is three continuously differentiable in R^2.
Let $\xi_1(x) = \nabla g_1(x), \xi_2(x) = (-16 - 2x_1, -2x_2)^T$ and $\xi_3(x) = \nabla g_3(x)$, it is easy to show that the feasible set Ω

satisfies assumptions $(C_1) - (C_3)$.

Example 5 To find a fixed point of self-mapping $\Phi(x) = (-x_1, x_2)^T$ in $\Omega = \{(x_1, x_2) \in R^2 : \ x_1^2 + x_2^2 \leq 25, \ 3 - x_1^2 + x_2 \leq 0\}$.
Let $\xi_1(x) = \nabla g_1(x)$ and $\xi_2(x) = (0, 8)^T$, it is easy to show that the feasible set Ω satisfies assumptions $(C_1) - (C_3)$.

Example 6 To find a fixed point of self-mapping $\Phi(x) = (x_1, -x_2)^T$ in $\Omega = \{(x_1, x_2) \in R^2 : \quad -x_1 + x_2^2 - 9 \leq 0, -(x_1 - 3)^2 - x_2^2 + 9 \leq 0, \ x_1 - x_2^2 + 3 \leq 0, \ x_1 - 5 \leq 0\}$.
Let $\xi_1(x) = \nabla g_1(x)$, $\xi_2(x) = (10, 0)^T + \nabla g_2(x)$ and $\xi_3(x) = (8, 0)^T$, it is easy to show that the feasible set Ω satisfies assumptions $(C_1) - (C_3)$.

TABLE 1

NUMERICAL RESULTS OF EXAMPLES 1-6

Example	$x^{(0)}$	IT	x^*	$\Phi(x^*)$
3.1	(0.3, 0.8)	16	(0.000001, 0.000003)	(0.000001, 0.000003)
	(0.3, -0.8)	18	(0.000000, -0.000002)	(0.000000, -0.000002)
3.2	(2, 4)	21	(-2.000001, 0.000003)	(-2.000001, 0.000003)
	(2, -4)	24	(-2.000000, 0.000002)	(-2.000000, 0.000002)
3.3	(28, 4.9)	13	(8.000000, 0.000002)	(8.000000, 0.000002)
	(28, -4.9)	15	(8.000000, -0.000001)	(8.000000, -0.000001)
3.4	(-2, 11)	17	(12.000001, 0.000002)	(12.000001, 0.000002)
	(-2, -11)	21	(12.000003, -0.0000001)	(12.000003, -0.0000001)
3.5	(-3.5, 1)	19	(0.000000, -4.000001)	(0.000000, -4.000001)
	(3.5, 1)	18	(0.000001, -4.000000)	(0.000001, -4.000000)
3.6	(4, 3.5)	23	(-5.000001, 0.000000)	(-5.000001, 0.000000)
	(4, -3.5)	21	(-5.000000, -0.000002)	(-5.000000, -0.000002)

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